Constructive and Predicative Locale Theory in Univalent Foundations

by

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Abstract

We develop locale theory constructively and predicatively in univalent foundations (also known as homotopy type theory or HoTT/UF), with a particular focus on the theory of spectral and Stone locales. The *constructivity* of our foundational setting means that we do not use any classical principles such as the axiom of choice (or any of its weaker forms), the law of excluded middle, or any form of the limited principle of omniscience. In the context of univalent foundations, *predicativity* refers specifically to the development of mathematics without the use of Voevodsky's propositional resizing axioms. The traditional approach to the predicative development of point-free topology is to work with presentations of locales known as formal topologies. Here, we take a different approach: we work directly with frames and locales, keeping careful track of the universes involved and adopting certain size assumptions to ensure that the theory is amenable to predicative development. Although it initially appears that many fundamental constructions of locale theory rely on impredicativity, we show that these can be circumvented under rather natural size assumptions. Our development here is inspired by de Jong and Escardó's constructive and predicative development of domain theory in univalent foundations.

We first lay the groundwork for the predicative development of locale theory. We then orient this towards an investigation of the theory of spectral and Stone locales, using the univalence axiom and the set replacement principle to ensure a predicatively well-behaved notion of spectral locale. We also develop Stone duality in the context of spectral locales, showing that there is a categorical equivalence between the type of large, locally small, and small-complete spectral locales and the type of small distributive lattices. Moreover, we exhibit the category of Stone locales as a coreflective subcategory of the category of spectral locales and spectral maps, using the construction known as the *patch locale* of a spectral locale (the localic manifestation of the so-called constructible topology). Finally, we investigate the topology of algebraic DCPOs and Scott domains in this constructive and predicative framework for locale theory. We develop the Scott locale of a Scott domain, show that it forms a spectral locale, and then proceed to investigate its patch. Using this, we obtain a topological characterization of de Jong's notion of sharp element: we establish a correspondence between the sharp elements of a Scott domain and the points of the patch of its Scott locale.

Our development is completely formalized and has been machine-checked using the Agda proof assistant.

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Chapter 1

Introduction

The aim of this thesis is to develop point-free topology constructively and predicatively in univalent foundations, with a particular focus on the theory of *spectral* and *Stone locales*.

Point-free topology [Joh82; Joh83; PP11] is an approach to topology where the fundamental object of study is taken to be a *locale* rather than a topological space. The latter is a set of *points* equipped with a topology, which is a class of subsets of the set of points required to be closed under finite intersection and arbitrary union. A locale, in contrast, is a notion of space that is characterized solely by an abstract lattice of opens, without any stipulation that this lattice consist of the subsets of some set of points. The lattice-theoretic structure abstracting the behaviour of the class of open sets is called a *frame*.

The point-free approach to topology arises from the remarkable insight that non-trivial topological questions can be entertained solely through the algebraic study of frames. The origin of this idea can be traced back to the work of Stone [Sto34; Sto36] on what is now called *Stone duality*.

The primary motivation for adopting a point-free approach to topology is that it makes it possible to develop topology in a constructive foundational setting. It is well known that some crucial theorems of point-set topology are fundamentally classical. The prime example is the Tychonoff theorem, which not only uses the axiom of choice in its proof, but is in fact known to be *equivalent* to it [Kel50]. Although this shows that there can be no hope of proving the Tychonoff theorem constructively in the point-set setting, it turns out to be constructively provable [Coq03; Vic05] if one adopts the point of view that locales are the subject matter of topology. Point-free topology thus holds a particular significance in the foundational setting of constructive mathematics, in addition to being also useful in classical mathematics.

This thesis is concerned with the development of locale theory in the foundational setting of univalent type theory, with a particular focus on the theory of spectral and Stone locales. Univalent type theory (also called homotopy type theory or HoTT/UF)

is a recent refinement of Martin-Löf Type Theory [Mar75; Mar82; Mar84], emanating from the view of types as ∞-groupoids [HS98; War06; Awo06; Voe06]. In addition to having applications in synthetic homotopy theory and higher topos theory, univalent type theory also serves as a *practical* foundation for mathematics, and turns out to be particularly useful for the development of constructive mathematics.

Constructive mathematics The discipline of *constructive mathematics* has its roots in the ideas of L.E.J. Brouwer [Bro08], who rejected the use of *classical principles* in accordance with his epistemological view of mathematics [Bee24, §3.1]. Brouwer's vision of constructive mathematics was an *anti-classical* one: he worked with axioms contradicting the law of excluded middle.

Later on, Bishop [Bis67] developed a neutral form of constructive mathematics. In the Bishop school of constructive mathematics, classical principles are *taboos*: they are neither provable nor disprovable. This view of mathematics is strictly more general than classical mathematics, as the possibility of adopting classical axioms is always left open. Bishop argued that constructive mathematics can also be *practical*, buttressing this view through his constructive development of a significant part of 20^{th}-century analysis (as explained in [Bee24, §3.3]).

In this thesis, we adopt a similar neutral approach based on univalent foundations, where constructivity means the development of mathematics without the law of excluded middle, the limited principle of omniscience [Bis67] (including its weaker forms), or the axiom of choice (including countable choice).

Type theory Type theory, in the sense of Martin-Löf Type Theory [Mar75; Mar82; Mar84], is a family of foundational systems for mathematics. It was originally developed [Mar75] to clarify the language of Bishop-style constructive mathematics, through a *deepening* of the so-called Curry-Howard correspondence [How69]. This enables the view of constructive proofs in mathematics as the terms of a suitable programming language with a rich type system.

One practical advantage of the type-theoretical view of mathematics is the importation of techniques from the discipline of programming language theory to mathematical foundations. Turning mathematical foundations into a problem in programming languages gives rise to the development of computer implementations of type theory, called *proof assistants*. Some noteworthy examples of type-theoretical proof assistants are: NuPRL [CAB+86], Coq [CH88], Agda [Nor09], and Lean [AdMKU+]. One of the major contributions of this thesis is the development of a locale theory library in the Agda proof assistant, as part of Escardó's open-source TypeTopology library [Ec23].

Homotopical view of types Homotopy type theory [Uni13] is a recent refinement of Martin-Löf Type Theory, whose origins can be traced back to Hofmann and Streicher's groupoid interpretation of type theory [HS98]. Exploiting this interpretation in order to develop a new form of type theory turned into a research programme of its own around the late 2000s and early 2010s, starting with the works of Voevodsky [Voe06] as well as Awodey and Warren [AW09]. Two main novelties that spring from the homotopical interpretation of type theory are *the univalence axiom* and *higher inductive types*.

Historically, the treatment of extensionality principles as postulates in Martin-Löf Type Theory has been a deficiency of it as a foundational system. Voevodsky's formulation of the univalence axiom identified a single extensionality axiom that implies *all* other extensionality axioms, the prime examples being functional and propositional extensionality. This paved the way for the invention of a type theory in which such axioms can be given computational rules through the assignment of a computational interpretation to the the univalence axiom.

The problem of finding such a type theory remained open until the invention of Cubical Type Theory [CCHM18] in 2018. In Cubical Type Theory, the univalence axiom is a theorem rather than a postulate. This is achieved through a fundamentally new design: Cubical Type Theory has a built-in interval object along with primitive operations for manipulating higher dimensional paths. Cubical Type Theory has attracted widespread interest in the HoTT/UF community, and multiple computer implementations of it have been developed. Among them, CUBICAL AGDA [VMA19] has become the leading implementation, with its standard library [Com24] being one of the major libraries of formalized univalent mathematics at the time of writing.

Predicative mathematics The term *predicative mathematics* is used by various researchers to refer to closely related, though not identical, concepts. In the context of univalent foundations, predicativity refers to the development of mathematics without the use of *propositional resizing axioms* [Uni13, Axiom 3.5.5], which were originally proposed as rules by Voevodsky [Voe11]. The avoidance of these axioms amounts to the restriction that propositions involving large quantifications over a universe are not admitted as propositions in that universe. It has been noted [Uni13, §3 Notes] that the propositional resizing axioms bear a close resemblance to Russell's notorious axiom of reducibility [Rus08, pp. 241–244], originating in the context of the theory of types [Rus03, Appendix B]. Our use of the term predicativity is thus in the same sense as in the tradition of Russell.

In the context of our use of univalent type theory as a constructive foundation, we have a concrete motivation for taking a predicative approach to mathematics: finding computational rules for the propositional resizing axioms is an open problem at the time of writing [CCHM18, p. 27]. Our commitment to predicative mathematics is thus a further form of our commitment to constructive mathematics, as we would like to refrain from the use of postulates without a well-understood computational behaviour. Furthermore, it has also been shown [Uem19] that there are models of univalent type theory in which propositional resizing fails.

1.1 Overview and summary of contributions

In Section 1.1.1, we provide a chapter-wise outline of the thesis, and explain the contributions presented in each chapter. In Section 1.1.2, we list the publications containing parts of the work presented here.

Almost all of the results presented in this thesis have been fully formalized using the AGDA proof assistant [NDCA+07; Nor09], as part of Escardó's TYPETOPOLOGY [Ec23] library. All main results have been formalized, and the unformalized exceptions mostly

relate to tangential developments or examples. For salient definitions, lemmas, theorems, and examples, we provide links to the corresponding constructions in the HTML rendering of our Agda formalization. For this, we use the hyperlinked Agda icon  (pronounced *chicken*).

1.1.1 Thesis outline and list of contributions

Chapter 2: Univalent foundations We provide a standard presentation of univalent foundations, mostly following the HoTT Book [Uni13] and Escardó's tutorial introduction [Esc19]. Our exposition has also benefited from the presentations of de Jong [dJon23, §2] and Knapp [Kna18, §I.2]. This chapter contains no new results and is merely a summary of the type-theoretical foundation in which we work.

Chapter 3: Basics of point-free topology The main contribution of this chapter is to develop the groundwork for locale theory in the context of constructive and predicative univalent foundations, working explicitly with universes. The univalent study of frames (Definition 3.1.1) and nuclei on them (Section 3.2) started in the author's MSc thesis [Tos20] as well as his work with Coquand [CT21] on formal topology. The main novelty of the predicative approach here is the use of small locale bases (Section 3.5) and the use of the posetal adjoint functor theorem (Section 3.7). As we explain in Section 7.2, our predicative approach to locales is closely related to formal topology, though it differs in certain key aspects.

Parts of the work presented here have been published in [TE23] and parts of it are in the process of publication in [AET24], although the account given in this chapter contains several improvements and simplifications.

Chapter 4: Spectral and Stone locales We present a predicative treatment of the theory of spectral and Stone locales in this chapter. It is here that we start to make use of the foundational novelties of univalent type theory. For instance, we give a definition of spectrality in Definition 4.2.1 that makes use of the univalence axiom. Furthermore, in Theorem 4.2.11, we prove the logical equivalence of various characterizations of the notion of spectral locale using the the set replacement principle. This relies on Lemma 4.2.9, which entails a strengthening of de Jong and Escardó's characterization of algebraic DCPOs [dJon23, Lemma 4.8.3].

In Section 4.5, we undertake a brief detour into the predicative treatment of zero-dimensional and regular locales in our setting, and pin down the predicatively well-behaved notions of zero-dimensional and regular locales. This finds application in Section 4.6, where we develop the theory of Stone locales (i.e. locales that are compact and zero-dimensional).

After this, we proceed to establish Stone duality for spectral locales in Section 4.7. We show that a frame is spectral if and only if it is isomorphic to the frame of *small* ideals over a *small* distributive lattice, which justifies that our definition of spectrality from Definition 4.2.1 captures the correct category of spectral locales. We then extend these two maps to functors and show that they form a categorical equivalence.

A preliminary version of our work in this chapter has been published in [TE23]. Theorems 4.2.11 and 4.6.9 characterizing spectral and Stone locales have appeared in a

more recent version of our work [AET24]. These provide a substantive simplification of our predicative account of spectral locales. In particular, it is remarked at the end of [TE23] that the formalization of [TE23, Lemma 7.11] was work in progress at the time of writing. The development in [AET24, Theorem in §4.1] allows this proof to be completed.

Chapter 5: The patch locale We contribute a constructive and predicative construction of the patch locale of a spectral locale along with a proof of its universal property.

The construction of the patch locale is given in Section 5.3. Although this is based on earlier work by Escardó [Esc98; Esc99; Esc01], it contains some new ingredients, since the development in *op. cit.* uses various impredicative constructions that are not available in our predicative setting. These new ingredients are the following:

1. The patch locale is defined by the frame of Scott continuous nuclei in [Esc99]. In order to prove that this is indeed a frame, one starts with the frame of all nuclei, and then exhibits the Scott continuous nuclei as a subframe. This procedure, however, does not seem to be possible in the context of our work, as it is not clear whether all nuclei can be shown to form a frame predicatively. We thus construct the frame of Scott continuous nuclei *directly*, which requires predicative reformulations of all standard facts about it inherited from the frame of all nuclei.
2. Escardó's description of the patch locale makes use of the notion of *open nucleus*, which does not exist a priori in a predicative setting. This is solved by our use of the posetal adjoint functor theorem from Section 3.7.
3. Escardó's proofs from [Esc99; Esc01] make use of the notion of *perfect map*, which again does not exist a priori in a predicative setting. This is because the right adjoint of a frame homomorphism is not a priori available in a predicative context.

The proof of the universal property is given in Section 5.4.3. The predicative proof that this construction satisfies its universal property is new as Escardó's proof relies on the existence of the frame of all nuclei, which does not seem possible to construct in an impredicative setting (see Section 5.2).

A preliminary version of the results presented here have first been published in [TE23]. Our more recent work [AET24, §8] contains a new and simplified proof of the universal property of the patch locale, which has been developed in collaboration with Igor Arrieta.

Chapter 6: Topology of Scott domains In this chapter, we focus our investigation of point-free topology on the topology of algebraic DCPOs and Scott domains. In Section 6.2, we contribute a predicative construction of the Scott locale of an algebraic DCPO and show that it is predicatively well-behaved i.e. that it is large, locally small, and small-complete. For the special case of Scott domains, we also construct a small strong base for the Scott topology in Section 6.2.5.

We then study the topology of Scott domains using the framework we built in Chapter 4. In Section 6.2.5, we give a constructive and predicative proof of the fact that the Scott topology of a Scott domain is always a *spectral locale*, which yields a rich class of examples of spectral locales that can be predicatively constructed.

In addition to these, we also study the patch locale of the spectral Scott locale of a Scott domain, using the results developed in Chapter 5. We show that the *sharp elements* [dJon21a] of a Scott domain coincide with the *spectral points* of its Scott locale, which in turn coincide with the points of the patch of the Scott locale in consideration. This is an illustrative application of the universal property of the patch locale from Section 5.4.3.

Results presented in this chapter have not been published, but are planned to be submitted for publication in the not-too-distant future. Furthermore, we also believe that the equivalence between sharp elements and the points of patch presented here will allow us to clarify, in a constructive setting, the relationship between the patch locale of the Scott locale of a Scott domain and the *Lawson topology* on the domain in consideration. The extension of our work in this direction is work in progress at the time of writing.

1.1.2 Publications

Parts of the work presented in this thesis have been published in [TE23]. They have also been accepted for publication in [AET24], which is in the process of being finalized at the time of writing.

[AET24] Igor Arrieta, Martín H. Escardó, and Ayberk Tosun.
"The Patch Topology in Univalent Foundations". Preprint accepted for publication in MSCS special issue on Advances in HoTT. Feb. 2024.
URL: https://arxiv.org/abs/2402.03134 (cit. on pp. 4–6).

[TE23] Ayberk Tosun and Martín H. Escardó.
"Patch Locale of a Spectral Locale in Univalent Type Theory".
In: *Proceedings of MFPS XXXVIII*. Vol. 1.
Electronic Notes in Theoretical Informatics and Computer Science.
Feb. 2023. DOI: 10.46298/entics.10808.
URL: https://entics.episciences.org/10808 (cit. on pp. 4–6).

Chapter 2

Foundations

Our development takes place in the foundational setting of univalent type theory, which is Martin-Löf Type Theory [Mar75; Mar82] extended with features that enable the development of *univalent mathematics*. In this chapter, we present the fundamental ideas of univalent foundations [Uni13] and discuss the details of our foundational setting.

Type theory differs fundamentally from set theory as a mathematical foundation. In set theory, one works with the untyped notion of *set*, out of which all mathematical constructions are built. The rules governing the construction of sets are given by the *axioms* of set theory. In type theory, on the other hand, one works with *types* which come equipped with rules governing the construction of their inhabitants. These rules are used to carry out the mathematical constructions of interest. This means that mathematical constructions in type theory do not exist in isolation, but *only as the inhabitants* of some type. In other words, all mathematical constructions carried out in type theory are fundamentally typed.

This typed approach to mathematical foundations yields a useful foundation for constructive mathematics. For example, the view of equality as a type in itself, which is a hallmark feature of Martin-Löf Type Theory, paves the way to the formulation of the *the univalence axiom* and *higher inductive types* — concepts that cannot quite be formulated in set theory (as explained by Escardó [Esc18]).

The organization of this chapter is as follows.

- Section 2.1: We present the basics of Martin-Löf Type Theory, which is the starting point of the univalent type theory that we use.
- Section 2.2: We discuss the universe structure of our type theory.
- Section 2.3: We present W-types, which give a formal account of *inductively generated* types.
- Section 2.4: We explain the Curry-Howard view of propositions as types in Martin-Löf Type Theory.
- Section 2.5: We present the technical details of the identity type and explain its

homotopical interpretation, which underlies the key features of univalent type theory.

Section 2.6: We discuss the notion of decidability for a type and present Hedberg's theorem.

Section 2.7: We discuss function extensionality as well as invertible and left-cancellable maps, and explain the problem that their naïve formulations are not properties.

Section 2.8: We present the notions of embedding and equivalence, which refine the notions of left-cancellable and invertible maps to the finer-grained setting of higher groupoids.

Section 2.9: We explain the stratification of types with respect to their homotopy levels, which lies at the heart of univalent foundations.

Section 2.10: We present the the univalence axiom.

Section 2.11: We present the notion of higher inductive type.

Section 2.12: We revisit the naïve form of the propositions-as-types principle from Section 2.4 and discuss its refinement in univalent type theory.

Section 2.13: We present the theory of small and locally small types, which underpins the notion of *predicativity*, and its lack thereof, in our foundational setting. We then explain the propositional resizing axioms, embodying what we mean by *impredicativity*, and discuss our motivations for avoiding them.

Section 2.14 Finally, we discuss the the set replacement principle in univalent type theory and its presentation as a higher inductive type. Besides propositional truncations, the set replacement principle is the only other higher inductive type that we use in this thesis.

2.1 Basics of Martin-Löf Type Theory

We give a high-level summary of the basics of Martin-Löf Type Theory and refrain from formally presenting it in full detail. For a complete formal presentation, we refer the reader to [Uni13, Appendix A].

Type theory is a formal system based on *judgements*. In the type-theoretical development of mathematics, we are concerned with the two basic judgements:

1. The *typing judgement*, written $a : A$, expressing that term a has type A.
2. The *equality judgement*, written $a \equiv b : A$, expressing that terms a and b are equal inhabitants of type A.

Types in type theory are defined by specifying what constitutes evidence for their typing, equality, and formation judgements (as explained by Martin-Löf [Mar83]). More specifically, this consists in giving their

- *formation rules*, explaining how to form the type;
- *introduction rules*, explaining how to build inhabitants of the type;
- *elimination rules*, explaining how to use the inhabitants of the type;
- *computation rules*, specifying the computational behaviour of the eliminator on the inhabitants of the type.

2.1.1 Basic types

The starting point of our foundational setup is Martin-Löf Type Theory with the following types:

1. The *empty type* $\mathbf{0}$.
2. The *unit type* $\mathbf{1}$.
3. The type $\mathbb{N}$ of *natural numbers*.
4. The type former $\mathsf{List}(\text{-})$, giving the type $\mathsf{List}(A)$ of *lists*, or *words*, over every type A.
5. The type former $(\text{-}) \times (\text{-})$, giving the *product type* $A \times B$, for every pair of types A and B.
6. The type former $(\text{-}) + (\text{-})$, giving the *sum type* $A + B$, for every pair of types A and B.
7. The type former $\Sigma(\text{-}, \text{-})$, giving the *dependent sum type* $\Sigma(A, B)$, for every type A and every A-indexed family of types $(B_a)_{a:A}$.
8. The type former $\Pi(\text{-}, \text{-})$, giving the *dependent product type* $\Pi(A, B)$, for every type A and every A-indexed family of types $(B_a)_{a:A}$.
9. The type former $\mathsf{Id}_{(\text{-})}(\text{-}, \text{-})$, giving the type $\mathsf{Id}_A(a, b)$ of *identifications* or *equalities* between every pair of inhabitants $a, b : A$, for every type A.

Remark 2.1.1. Types (3) and (4) above are examples of *inductive types*, the formal account of which is given using the general mechanism of W-types. These types are the only instances of W-types that we are interested in so we prefer to present them directly. Nevertheless, we discuss their representation using W-types in Section 2.3.

Notation

Given a type A and an A-indexed family of types $(B_a)_{a:A}$, we denote the dependent product type and the dependent sum type by, respectively,

$$\Pi_{(x:X)} Y_x \quad \text{and} \quad \Sigma_{(x:X)} Y_x.$$

The dependent product type is sometimes called the *dependent function type* or the Π-type. If the type family $(B_a)_{a:A}$ in consideration is constant, we abbreviate $\Pi_{(a:A)} B_a$ to $A \to B$, and call this the *function type*, or the *non-dependent function type* if additional clarity is required.

Given some $a : A$ and $b : B(a)$, the inhabitant of $\Sigma_{(a:A)} B_a$ is denoted by the tuple notation (a, b). Inhabitants of the Π-type are constructed using *λ-abstraction*, which are denoted $\lambda x.t$ or $x \mapsto t$.

The unique constructor of the unit type is denoted $\star$. The two constructors of $\mathbb{N}$ are denoted zero and succ. The constructors of the sum type $A + B$ are denoted $\mathsf{inl}(a)$ and $\mathsf{inr}(b)$. For the special case $\mathbf{2} :\equiv \mathbf{1} + \mathbf{1}$, we define the more suggestive notation of $0 :\equiv \mathsf{inl}(\star)$ and $1 :\equiv \mathsf{inr}(\star)$. We denote the empty list by ε, and for the prepending of an element x onto a list s, we write xs. We use the same notation for the concatenation of lists s and t and write st.

For identity types, we often use the more suggestive notation $a =_A b$ for $\mathsf{Id}_A(a, b)$. If the type is clear from the context, we further abbreviate this to $a = b$. Given some

inhabitant $a : A$, the constructor of its identity with itself is denoted $\mathrm{refl}_a : a =_A a$. Moreover, we carefully distinguish between judgemental equality and the identity type, refraining from using the symbols (-) ≡ (-) and (-) = (-) in place of each other. The former always refers to judgemental equality and the latter always to the identity type. Finally, we note that we use (-) :≡ (-) for the introduction of new definitional equalities, following the convention of [Uni13]. When defining new functions, we often abbreviate $f :\equiv \lambda x.t$ to $f(x) :\equiv t$, following the standard convention.

2.1.2 Axioms in type theory

An *axiom* in the context of type theory is an inhabitant of a particular type that is declared to exist, independently of the type's introduction and elimination rules. Working with axioms in type theory is problematic for at least two reasons:

1. Extending one's foundational system with axioms is potentially dangerous, as it poses the risk of introducing inconsistency into the system.
2. The *computational content* of axioms in type theory are not specified, as they are not built using introduction and elimination principles which all have specified computational behaviour. This means that the computational interpretation of a proof using an axiom is unclear, which violates the main desideratum of the type-theoretic view of mathematics: to elucidate the computational content of constructive proofs.

In our development, we will use several principles that we will call axioms, though our use of this term will be slightly different. In this thesis, we axiomatize our foundational setting instead of committing to a concrete implementation of type theory. Therefore, an "axiom" for us will simply be an assumption about our foundational setting, and our use of this term does not imply that we are committing to the view of their inhabitants as uninterpreted constants.

We will circumvent the above problems by only using axioms that follow either from the univalence axiom or the existence of certain well-behaved higher inductive types. Both of these are known to be consistent and to admit computational interpretations. The axioms we use are all theorems in, for instance, Cubical Type Theory [CCHM18].

2.2 Universes

We explained in the introduction to this chapter that all terms of Martin-Löf Type Theory are inherently typed. This invites the question of how to assign types to types themselves. Such types that govern the structure of types are called *universes*. For the purposes of our predicative development of locale theory, universes are of paramount importance, as they provide a notion of *size* for types. This underpins the *predicativity* of our foundational setting.

2.2.1 Universe operations

The universe structure in our type-theoretical setting closely follows the type system of the Agda proof assistant [NDCA+07; Nor09] and the presentation of Escardó [Esc19,

§2.4]. Universes form a hierarchy of increasingly larger universes. We do not assume that this hierarchy is, or is not, cumulative. This hierarchy is formally given by the following operations:

- **Ground universe**. There is a basic universe $\mathcal{U}_0$, called the *ground universe*, which forms the base level of the universe hierarchy.
- **Successor operation.** There is an operation $(\text{-})^+$ that gives the *successor universe*, denoted $\mathcal{U}^+$, of every universe $\mathcal{U}$.
- **Maximum operation.** There is an operation $(\text{-}) \sqcup (\text{-})$ giving the least upper bound $\mathcal{U} \sqcup \mathcal{V}$ of every pair of universes $\mathcal{U}$ and $\mathcal{V}$.

We reserve the calligraphic letters $\mathcal{U}, \mathcal{V}, \mathcal{W}, \ldots$ for variables ranging over universes. Furthermore, we note that we sometimes use the term *$\mathcal{U}$-type* for a type inhabiting universe $\mathcal{U}$, when additional clarity is required.

Remark 2.2.1. In Section 2.1, we used the notation $(B_a)_{a:A}$ for an A-indexed family of types, without explaining its precise representation in type theory. After having introduced universes, we can make this precise: a family $(B_a)_{a:A}$ of types living in universe $\mathcal{U}$ is simply a function $B : A \to \mathcal{U}$.

2.2.2 Universes rules for basic types

Under this universe structure, we assume that the types given in Section 2.1 are assigned universes as follows:

1. The empty type $\mathbf{0}$ and the unit type $\mathbf{1}$ have a copy in every universe. When additional clarity is needed, we denote their universes by using subscripts and write $\mathbf{0}_{\mathcal{U}}$ and $\mathbf{1}_{\mathcal{U}}$.
2. The type $\mathbb{N}$ has a copy in every universe $\mathcal{U}$.
3. For every type $X : \mathcal{U}$, the type $\mathsf{List}(X)$ inhabits the universe $\mathcal{U}$.
4. For every type $X : \mathcal{U}$ and every pair of inhabitants $x, y : X$, the identity type $x =_X y$ inhabits the universe $\mathcal{U}$.
5. For every pair of types $X : \mathcal{U}$ and $Y : \mathcal{V}$, the sum type $X + Y$ and the product type $X \times Y$ both inhabit the universe $\mathcal{U} \sqcup \mathcal{V}$.
6. For every pair of types $X : \mathcal{U}$ and type family $Y : X \to \mathcal{V}$, the dependent product $\Pi_{(x:X)} Y(x)$ and the dependent sum $\Sigma_{(x:X)} Y(x)$ inhabit the universe $\mathcal{U} \sqcup \mathcal{V}$.

If the inductively defined types above were defined using W-types, as discussed in Remark 2.1.1, it would be possible to obtain these universe rules as a special case of the universe rule for W-types (Definition 2.3.2). However, we prefer to present these directly and refrain from involving the generality of W-types.

2.3 W-types

We mentioned in Remark 2.1.1 that some of the basic types we gave in Section 2.1.1 are examples of *inductively generated types*. The formal account of the general notion of an inductively generated type is given by the mechanism of W-types [NPS90, §I.15], which we briefly summarize in this section.

2.3.1 Definition of W-types

Definition 2.3.1. The W-type former $\mathsf{W}_{(\text{-}),(\text{-})}$ gives a type $\mathsf{W}_{A,B}$, for every type $A : \mathcal{U}$ and every A-indexed family of types $B : A \to \mathcal{V}$. This type has a single constructor $\mathsf{sup} : \prod_{(a:A)} \big((B(a) \to \mathsf{W}_{A,B}) \to \mathsf{W}_{A,B}\big)$, and has the following induction principle: for every type family $Y : \mathsf{W}_{A,B} \to \mathcal{T}$, if it is the case that, for every $a : A$ and $f : B(a) \to \mathsf{W}_{A,B}$, the type

$$\big(\prod_{(b:B(a))} Y(f(b))\big) \Rightarrow Y(\mathsf{sup}(a, f)),$$

is inhabited, then we have $\prod_{(w:\mathsf{W}_{A,B})} Y(w)$.

The witness of the above induction principle is denoted wrec.

Definition 2.3.2 (Universe rule for W-types)**.** W-types are assigned universes by the following rule: given some type $A : \mathcal{U}$ and a type family $B : A \to \mathcal{V}$, the type $\mathsf{W}_{A,B}$ inhabits the universe $\mathcal{U} \sqcup \mathcal{V}$.

The key idea in the above definition of W-type is that every inductively generated type has two components:

- a type of *constructor names*,
- a type family of *subtree selectors* for each constructor.

This is exactly what the W-type former encodes: the type A is the type of constructor names, whereas the type family $B : A \to \mathcal{V}$ gives the type of subtree labels for each constructor $a : A$. The introduction principle sup then takes (1) a constructor $a : A$, and (2) a function encoding how to construct a smaller inhabitant of $\mathsf{W}_{A,B}$, for each $b : B(a)$, and returns an inhabitant of $\mathsf{W}_{A,B}$.

2.3.2 Examples of W-types

Let us look at an example to make things a bit more concrete.

Example 2.3.3. The type of natural numbers is an illustrative example of a W-type. We show how the copy of the type $\mathbb{N}$ can be constructed in a given universe $\mathcal{U}$. Since the type $\mathbb{N}$ has two constructors, we pick $A :\equiv \mathbf{2}_{\mathcal{U}}$. The two constructors have types $\mathsf{zero} : \mathbb{N}$ and $\mathsf{succ} : \mathbb{N} \to \mathbb{N}$. The former has zero arguments whereas the latter takes one subtree. To encode these, we define the type family $B : A \to \mathcal{U}$ as follows:

$$\begin{aligned} B(0) &:\equiv \mathbf{0}_{\mathcal{U}} \\ B(1) &:\equiv \mathbf{1}_{\mathcal{U}} \end{aligned}$$

Using the sup constructor of $\mathsf{W}_{A,B}$, we then encode the two constructors as

$$\begin{array}{lcl} \mathsf{zero}' & : & \mathsf{W}_{A,B} \\ \mathsf{zero}' & :\equiv & \mathsf{sup}(0, \mathbf{0}\text{-elim}) \end{array} \qquad \begin{array}{lcl} \mathsf{succ}' & : & \mathsf{W}_{A,B} \to \mathsf{W}_{A,B} \\ \mathsf{succ}'(w) & :\equiv & \mathsf{sup}(1, \star \mapsto w) \end{array}$$

Example 2.3.4. For every type $X : \mathcal{U}$, the type $\mathsf{List}(X)$ can be encoded as a W-type. We define $A_X :\equiv \mathbf{1} + X$, as we have one constructor for the empty list and X-many cons constructors. We define the type family $B : A_X \to \mathcal{U}$ as

$$\begin{aligned} B(0) &:\equiv \mathbf{0}_{\mathcal{U}} \\ B(1) &:\equiv \mathbf{1}_{\mathcal{U}} \end{aligned}$$

The empty and cons constructors are then encoded as

$$\begin{array}{lcllcl} \mathsf{empty} & : & \mathsf{W}_{A_X,B} & \mathsf{cons} & : & X \to \mathsf{W}_{A_X,B} \to \mathsf{W}_{A_X,B} \\ \mathsf{empty} & :\equiv & \mathsf{sup}(0, \mathbf{0}\text{-elim}) & \mathsf{cons}(x, w) & :\equiv & \mathsf{sup}(\mathsf{inr}(x), \star \mapsto w) \end{array}$$

The type family $X \mapsto \mathsf{W}_{A_X,B}$ is then equivalent to the type family $\mathsf{List} : \mathcal{U} \to \mathcal{U}$.

We also note here that there is a generalization of W-types, called *indexed W-types*, allowing the encoding of *inductively defined families of types* in a similar manner. We do not present indexed W-types here as we will not need them in our development.

2.4 The Curry-Howard interpretation of logic

Set theory is a foundational system with two layers: one starts with the logical layer of first-order logic, and then uses it to formulate the axioms of a theory of sets, such as Zermelo-Fraenkel set theory. In contrast, logic has no special status in type theory; it is obtained by assigning a logical reading to a certain fragment of the types.

This idea springs from the so-called *Curry-Howard interpretation of logic* [How69] by which a type A is viewed as a proposition, and the typing judgement $a : A$ is read as expressing that term a constitutes a proof for proposition A. Under this interpretation, the following logical readings are assigned to types:

- The empty type $\mathbf{0}$ is interpreted as falsum $\bot$.
- The unit type $\mathbf{1}$ is interpreted as truth $\top$.
- The product type $A \times B$ is interpreted as logical conjunction $A \wedge B$.
- The sum type $A + B$ is interpreted as logical disjunction $A \vee B$.
- The function type $A \to B$ is interpreted as logical implication $A \Rightarrow B$.
- A type family $B : X \to \mathcal{U}$ is interpreted as a predicate on type X.
- Negation of a proposition A is accordingly given by the type $A \to \mathbf{0}$.
- The dependent product type $\Pi_{(a:A)} B(a)$ is interpreted as the universal quantifier $\forall a : A.\ B(a)$.
- The dependent sum type $\Sigma_{(a:A)} B(a)$ is interpreted as the existential quantifier $\exists a : A.\ B(a)$.
- If there exist functions $f : X \to Y$ and $g : Y \to X$ between types X and Y, they are viewed as *logically equivalent* propositions.

Under this logical reading of types, to assert that proposition A holds is to assert that type A is inhabited. We will hereafter use this interpretation tacitly, and will not explicitly assert types to be inhabited if it is clear that we are taking their view as propositions. For example, we will simply say "B whenever A" instead of "B is inhabited

whenever A is inhabited".

Example 2.4.1. The logical statement $\forall n : \mathbb{N}.\ n = \mathsf{zero} \vee \exists m : \mathbb{N}.\ n = \mathsf{succ}(m)$, expressing that every natural number is either zero or the successor of some natural number, is interpreted under the Curry-Howard principle as

$$\Pi_{(n:\mathbb{N})}\ (n = \mathsf{zero}) + \big(\Sigma_{(m:\mathbb{N})} n = \mathsf{succ}(m)\big).$$

In some type theories, there is a designated universe for those types viewed as propositions. The prime example is the Calculus of Constructions [CH88] (CoC for short). In CoC, there is a universe Prop of types that are interpreted as propositions. However, this notion of being a proposition is a judgement rather than an internal statement in CoC.

One of the salient novelties of univalent type theory is that it makes it possible to work with an internal notion of propositional type, which allows one to *define the type of propositional types* inside the type theory itself. This gives rise to a refined interpretation of the Curry-Howard principle. In Section 2.12, we will discuss this refined view of logic in univalent type theory, and will contrast it with the naïve form that we presented in this section.

2.5 The identity type

We briefly mentioned the identity type in Section 2.1. In this section, we revisit it to focus on its technical details and to discuss its *homotopical interpretation*.

2.5.1 Formal definition

Definition 2.5.1 (The identity type)**.** The *identity type* of a type X is the inductive type family $(\text{-}) =_X (\text{-})$ generated by the single constructor:

$$\mathsf{refl}_{(\text{-})} : \Pi_{(x:X)}\ x =_X x.$$

Its elimination principle, often called the **J**-*rule*, states that, for every family

$$C : \Pi_{(x,y:X)}\ (x =_X y \to \mathcal{U}),$$

and every function $c : \Pi_{(x:X)}\ C(x, x, \mathsf{refl}_x)$, we have an inhabitant

$$f : \Pi_{(x,y:X)}\ \Pi_{(p:x=_X y)}\ C(x, y, p)$$

satisfying the computation rule $f(x, x, \mathsf{refl}_x) \equiv c(x)$.

Given a type X and type family C as in the above definition, the witness of this elimination principle is denoted

$$\mathbf{J}_X^C : \big(\Pi_{(x:X)}\ C(x, x, \mathsf{refl}_x)\big) \to \Pi_{(x,y:X)}\ \Pi_{(p:x=_X y)}\ C(x, y, p).$$

In practice, it turns out to be rather cumbersome to work directly with this elimination principle. The standard practice in type theory is to explicitly prove that the identity type satisfies some basic properties using the **J**-rule, and then work only with these properties. We address this next.

2.5.2 Transport and action on paths

We now prove two immediate corollaries of the elimination principle **J**: the so-called *transport* and *action on paths* lemmas. In practice, working with these corollaries turns out to be much more convenient than working directly with the **J**-rule.

Lemma 2.5.2 (Transport). *For every type $X : \mathcal{U}$ and type family $Y : X \to \mathcal{V}$, we have an inhabitant*

$$\mathsf{transport}^Y : \Pi_{(x,x':X)}\, x = x' \to Y(x) \to Y(x'),$$

satisfying the computation rule $\mathsf{transport}^Y(x, x, \mathsf{refl}_x) \equiv \mathsf{id}$*, for every* $x : X$.

Proof. Consider a type $X : \mathcal{U}$ and type family $Y : X \to \mathcal{V}$. Define

$$C(x_1, x_2, p) :\equiv Y(x_1) \to Y(x_2),$$
$$c(x) :\equiv \mathsf{id}.$$

The function $\mathsf{transport}^Y$ is then defined as $\mathsf{transport}^Y(x, x', p, y) :\equiv \mathbf{J}^X_C(c, x, x', p, y)$. □

The transport lemma is sometimes called the type-theoretic principle of the *indiscernibility of identicals*. It says that, if two terms of a type are equal, then no predicate (i.e. type family) can distinguish them.

As a further corollary of the transport lemma, we obtain the following.

Lemma 2.5.3. *For every function $f : X \to Y$, we have an inhabitant*

$$\mathsf{ap}_f : x =_X x' \to f(x) =_Y f(x').$$

Proof. Let $f : X \to Y$. Define the type family $Z(x_1, x_2) :\equiv f(x_1) =_Y f(x_2)$. We then use Lemma 2.5.2 to define $\mathsf{ap}_f(x, x', p) :\equiv \mathsf{transport}^{Z(x,-)}(x, x', p, \mathsf{refl}_{f(x)})$. □

The term ap is an abbreviation of the term ***a**ction on **p**aths*, springing from the homotopical interpretation of the identity type. We will discuss in the next section.

Lemma 2.5.4. *For every type X, the identity type $(\text{-}) =_X (\text{-})$ is symmetric and transitive.*

Proof. Let $X : \mathcal{U}$ be a type.

For symmetry, consider two inhabitants $x, y : X$ and let $p : x =_X y$. We need to construct an inhabitant of type $y =_X x$. We define the type family $P(z) :\equiv z =_X x$

and appeal to transport. It is easy to see that we have an inhabitant

$$\mathsf{transport}^P(x, y, p, \mathsf{refl}_x) : y =_X x.$$

For transitivity, let $x, y, z : X$, and consider proofs $p : x =_X y$ and $q : y =_X z$. We need to construct an inhabitant of type $x =_X z$. We define $P(w) :\equiv x =_X w$. The transport lemma then gives us

$$\mathsf{transport}^P(y, z) : y =_X z \to x =_X y \to x =_X z,$$

using which we can easily construct an inhabitant of the type $x =_X z$ as

$$\mathsf{transport}^P(y, z, q, p) : x =_X z.$$

□

2.5.3 Identity type of Σ-types

Consider two pairs (x_1, y_1) and (x_2, y_2) inhabiting the dependent sum type $\Sigma_{(x:X)} Y(x)$ for some $X : \mathcal{U}$ and type family $Y : X \to \mathcal{V}$. It is natural to expect that the identity type

$$(x_1, y_1) = (x_2, y_2)$$

be inhabited if and only if the identity types between the two components are inhabited. This indeed turns out to be the case in type theory, although some care has to be taken due to the dependency in $Y(x)$.

Lemma 2.5.5. *Let $X : \mathcal{U}$ and let $Y : X \to \mathcal{V}$. Consider two pairs $(x_1, y_1), (x_2, y_2)$, inhabiting the type $\Sigma_{(x:X)} Y(x)$. We have that the identity type $(x_1, y_1) = (x_2, y_2)$ is logically equivalent to the type*

$$\Sigma_{(p : x_1 =_X x_2)} \mathsf{transport}^Y(x_1, x_2, p, y_1) = y_2.$$

Proof. Let $X : \mathcal{U}$ and $Y : X \to \mathcal{V}$. Consider two inhabitants $(x_1, y_1), (x_2, y_2)$ of type $\Sigma_{(x:X)} Y(x)$.

($\Rightarrow$) By path induction, it suffices to show that the implication holds for $\mathsf{refl}_{(x_1, y_1)}$. The pair $(\mathsf{refl}_{x_1}, \mathsf{refl}_{y_1})$ then inhabits the type

$$\Sigma_{(p : x_1 =_X x_1)} \mathsf{transport}^Y(x_1, x_1, p, y_1) = y_1.$$

($\Leftarrow$) By the induction principle of the Σ-type and path induction, we need to show the implication for $\mathsf{refl}_{x_1} : x_1 = x_1$ and $\mathsf{refl}_{y_1} : \mathsf{transport}^Y(x_1, x_1, \mathsf{refl}_{x_1}, y_1) = y_1$. The constructor $\mathsf{refl}_{(x_1, y_1)}$ inhabits the type $(x_1, y_1) = (x_1, y_1)$. □

We will later show that the above logical equivalence can be extended to a full *equivalence of types*. As pinning down the right notion of type equivalence will require some discussion, however, we postpone this fact for now.

2.5.4 Homotopical interpretation of the identity type

The key insight that gives rise to univalent mathematics arises from the following semantic interpretation of types:

1. a type X is viewed as a space (up to homotopy equivalence), and under this view,
2. the identity type $x =_X y$ can be interpreted as the type of *paths* from point x to point y in space X.

In this context,

- transitivity of equality corresponds to *path composition*,
- symmetry of equality corresponds to *path inversion*,
- the **J**-rule is interpreted as a form of *path induction*.

To highlight this view, we denote the inhabitant of $y =_X x$, obtained from some $p : x =_X y$ by symmetry (Lemma 2.5.4), by p^{-1}. Similarly, given two proofs of identity $p : x =_X y$ and $q : y =_X z$, we denote the inhabitant of $x =_X z$, obtained by transitivity of identity (Lemma 2.5.4), by

$$p \bullet q : x =_X z.$$

In algebraic topology, spaces are classified with respect to the complexity of their homotopy groups. The above observation, that the identity type can be read as the type of paths, paves the way to the importation of this technique into type theory, giving rise to the idea of classifying types with respect to the complexity of their identity types. However, some caution needs to be exercised to ensure that one's type-theoretical setting is suitable for such an interpretation.

Uniqueness of identity types

Up until this point, the only means of constructing inhabitants of the identity type that we have seen is the refl constructor from Definition 2.5.1. Therefore, there appears to be just one way to construct a path: using refl_x to construct a proof $x =_X x$. In other words, the proof structure of identity types appears, prima facie, to be trivial.

The statement that there is at most one way to prove equality is known as the **K** axiom [Str93].

Definition 2.5.6 (The **K** axiom)**.** Given a type X, define

$$\mathbf{K}(X) \quad :\equiv \quad \Pi_{(x,y:X)} \Pi_{(p,q:x=_X y)} p = q.$$

*The **K** axiom* then says that every type X satisfies $\mathbf{K}(X)$.

The **K** axiom is known to be *independent* of, but consistent with, the basic type theory that we presented in Section 2.1. We therefore have the option to adopt it or not. Adopting this axiom results in a type theory in which a rich homotopical interpretation of the identity type is not possible, since this axiom asserts that all types have *trivial path types*.

This is the bifurcation point at which the vision of univalent type theory emerges: by refraining from adopting the **K** axiom, we keep open the option of *adopting certain other axioms that contradict it* by providing new ways of constructing inhabitants of the identity type. These are:

1. The *univalence axiom*, which allows us to construct nontrivial inhabitants of the identity type $A =_{\mathcal{U}} B$, between types $A, B : \mathcal{U}$.
2. *Higher inductive types*, which are types that come equipped with identity constructors other than refl.

In Example 2.10.5, we will look at a concrete counterexample to the **K** axiom, constructed using the univalence axiom.

First look at homotopy levels

In a type-theoretical setting where nontrivial identity types are possible, types can be sensibly classified with respect to the complexity of their identity types. This leads to the hierarchy of *homotopy levels* of types [Uni13, §7.1].

Types with the simplest possible homotopy structure are the contractible types.

Definition 2.5.7 (Contractible type [Uni13, Definition 3.11.1])**.** A type X is called *contractible* (also called *singleton*) if it has exactly one inhabitant, i.e. the type

$$\mathsf{is\text{-}contractible}(X) \quad :\equiv \quad \Sigma_{(c:X)} \Pi_{(x:X)}\, x =_X c$$

is inhabited.

This unique inhabitant is sometimes called the *centre of contraction* of type X.

After contractible types, we have the propositional types, which are types that have *at most* one element:

Definition 2.5.8. A type X is called a *proposition* (also called *subsingleton*) if it has at most one inhabitant, i.e. it satisfies

$$\mathsf{is\text{-}prop}(X) \quad :\equiv \quad \Pi_{(x,y:X)}\, x =_X y.$$

We define the type of propositional types as

$$\Omega_{\mathcal{U}} \quad :\equiv \quad \Sigma_{(X:\mathcal{U})} \mathsf{is\text{-}prop}(X).$$

Throughout the HoTT/UF literature, this is sometimes denoted $\mathsf{hProp}_{\mathcal{U}}$.

Example 2.5.9. The unit type $\mathbf{1}$ is contractible, with centre $\star : \mathbf{1}$. It follows easily from the elimination principle of $\mathbf{1}$ that every inhabitant of it is equal to $\star$. The empty type $\mathbf{0}$ is a proposition but is not contractible since it has no inhabitants.

Example 2.5.10. For every type X, the types $X \to \mathbf{1}$ and $\mathbf{0} \to X$ are both contractible (assuming function extensionality). The former has the function $!_1(x) :\equiv \star$ as its centre of contraction: it is easy to see that every $f : X \to \mathbf{1}$ is equal to $!_1$, since

$f(x) = \star = !_1(x)$. As to the latter, there is a function $\mathbf{0} \to X$ given by the induction principle of the type $\mathbf{0}$, which is vacuously equal to every other such map.

Example 2.5.11. Every contractible type is a proposition, since every pair of inhabitants are equal to the centre of contraction and thus to each other.

Lemma 2.5.12. *A type X is contractible if and only if it is an inhabited proposition.*

The next homotopy level is that of *sets*: types with trivial identity types.

Definition 2.5.13. A type X is called a *set* (or, more explicitly, *homotopy set*) if it satisfies $\mathbf{K}(X)$. In other words, a set is a type X with the property that, for every pair of inhabitants $x, y : X$, the identity type $x =_X y$ is a proposition.

Later on, we will discuss the general definition of this hierarchy, defining the generalized notion of an *n-type* [Uni13, Definition 7.1.1]. For now, however, this preliminary explanation of the core ideas of univalent type theory will be sufficient to start developing the basics of our foundational setting.

2.5.5 Structure vs. property

If a type X is a proposition in the sense of Definition 2.5.8, we think of it as expressing a *property*. The idea is that propositional types are naturally *proof irrelevant*, as the propositionality of a type amounts to the assertion that there cannot be two distinct proofs of it. If a type X is not a proposition, this means that it bears nontrivial proof *structure*. A careful distinction between structure and property has to be maintained in univalent type theory.

Example 2.5.14. We translated the logical statement from Example 2.4.1 into type theory as

$$\Pi_{(n:\mathbb{N})} \, (n = \mathsf{zero}) + \left(\Sigma_{(m:\mathbb{N})} n = \mathsf{succ}(m)\right), \tag{$\dagger$}$$

following the Curry-Howard view of propositions as types. For every $n : \mathbb{N}$, the type

$$(n = \mathsf{zero}) + \left(\Sigma_{(m:\mathbb{N})} n = \mathsf{succ}(m)\right),$$

is a *property* of the type of natural numbers. This implies that the type ($\dagger$) is also a property.

Let us now look at an example of a type, obtained through the Curry-Howard principle, that is *not* a proposition.

Example 2.5.15. Let $\alpha : \mathbb{N} \to \mathbb{N}$ and consider the statement $\exists_{(i:\mathbb{N})} \alpha(i) = \mathsf{zero}$, expressing that α has a root. Translating this into type theory through Curry-Howard principle gives us

$$\Sigma_{(i:\mathbb{N})} (\alpha(i) = \mathsf{zero}).$$

This is, however, structure and *not* property, since a sequence $\alpha : \mathbb{N} \to \mathbb{N}$ can have distinct roots. It should therefore be thought of as the *type of roots* of α rather than a property expressing that α has a root.

This example justifies our use of the adjective "naïve" for the propositions-as-types principle that we presented in Section 2.4. In a context where we are carefully stratifying types with respect to their homotopy levels, it is misleading that translations of logical statements yield types that are not propositions. In Section 2.12, we will revisit this problem and explain how univalent type theory provides us with the means to circumvent it.

In our development, we will use the propositional extensionality axiom, stating that logically equivalent properties are *equal*.

Definition 2.5.16 (Axiom: Propositional extensionality)**.** A universe $\mathcal{U}$ is said to satisfy *propositional extensionality* if, for every pair of propositions $P, Q : \Omega_{\mathcal{U}}$ with $P \Rightarrow Q$ and $Q \Rightarrow P$, we have an inhabitant of the identity type $P = Q$. The *propositional extensionality axiom* states that all universes satisfy propositional extensionality.

2.6 Decidable types and sets

In this section, we discuss the notion of a decidable type, and then present Hedberg's Theorem [Hed98; KECA17], which states that all decidable types are sets in the sense of Definition 2.5.13.

2.6.1 Decidability of a type

In logic, the *law of excluded middle* asserts that every proposition is either true or false. If we apply the naïve propositions-as-types translation to this, we obtain

$$\Pi_{(A:\mathcal{U})} (A + \neg A).$$

There is no reason for the above type to be inhabited in general as we are working in a constructive foundational setting. Those types that satisfy this are called *decidable types*.

Definition 2.6.1. A type X is said to be *decidable* if we have an inhabitant of the sum type $X + \neg X$.

Definition 2.6.2. A type X is said to have *decidable equality* (or to be *discrete*) if the identity type $x =_X y$ is decidable, for every pair of inhabitants $x, y : X$.

Example 2.6.3. Every proposition P (in the sense of Definition 2.5.8) has decidable equality, as the identity type $p =_P q$ is a decidable type by virtue of being always inhabited.

Lemma 2.6.4. *A proposition $P : \Omega_{\mathcal{U}}$ is decidable if and only if $P = \top_{\mathcal{U}}$ or $P = \bot_{\mathcal{U}}$, assuming propositional and functional extensionality.*

Proof. The fact that $\top_{\mathcal{U}}$ and $\bot_{\mathcal{U}}$ are decidable propositions is immediate. Let $P : \Omega_{\mathcal{U}}$ be a decidable proposition. If P is inhabited, it is obvious that $\top \Rightarrow P$. One can thus easily construct a logical equivalence $P \Leftrightarrow \top$, and use propositional extensionality to conclude that the underlying types are equal. If P is not inhabited, which is to say we have an inhabitant of $\neg P$, we have $P \Rightarrow \bot$ by construction. Again, by propositional extensionality, we have that the underlying types of the propositions in consideration are equal. Moreover, since being proposition is a proposition (by Lemma 2.6.13, assuming function extensionality), we may appeal to Lemma 2.5.5 to conclude that $P = \top$, in the former case, and that $P = \bot$, in the latter case. □

Definition 2.6.5 (Naïve LEM)**.** The naïve law of excluded middle states that every type is decidable.

This axiom is independent of, but consistent with, the base type theory from Section 2.1.1. It is, however, inconsistent with the univalence axiom as shown in [Uni13, Corollary 3.2.7].

Example 2.6.6. The type $\mathbb{N}$ has decidable equality, witnessed by the type family:

$$\begin{aligned} \mathsf{nateq}(\mathsf{zero}, \mathsf{zero}) &:\equiv \mathbf{1} \\ \mathsf{nateq}(\mathsf{succ}(m), \mathsf{zero}) &:\equiv \mathbf{0} \\ \mathsf{nateq}(\mathsf{zero}, \mathsf{succ}(n)) &:\equiv \mathbf{0} \\ \mathsf{nateq}(\mathsf{succ}(m), \mathsf{succ}(n)) &:\equiv \mathsf{nateq}(m, n) \end{aligned}$$

It is easy to see that $\mathsf{nateq}(m, n)$ is either $\mathbf{1}$ or $\mathbf{0}$. Furthermore, it can be shown by straightforward induction that $(m =_{\mathbb{N}} n) = \mathsf{nateq}(m, n)$.

2.6.2 Sethood of decidable types

There is a well-known theorem in type theory, due to Michael Hedberg [Hed98], stating that every type with decidable equality is a set. This is perhaps the most useful proof technique for establishing that certain types are sets. Here, we will present Hedberg's Theorem through a recent reformulation of it due to Kraus, Escardó, Coquand, and Altenkirch [KECA17]. To be more specific, our presentation here follows the AGDA development of Escardó [Esc12].

Lemma 2.6.7. *For every type X, inhabitant $x : X$, and X-indexed family*

$$\varphi : \Pi_{(y:X)}\, x =_X y \to x =_X y$$

of endofunctions on the identity type, we have the equality

$$p = \varphi_x(\mathsf{refl}_x)^{-1} \bullet \varphi_y(p),$$

for every $y : X$ and every proof of identity $p : x =_X y$.

Proof. Straightforward path induction. □

Definition 2.6.8 (Weakly constant function)**.** A function $f : X \to Y$ is called *weakly constant* if $f(x) =_Y f(x')$, for every pair of inhabitants $x, x' : X$.

Lemma 2.6.9 (Hedberg's Lemma)**.** *For every type X and every $x : X$, if there is a family of weakly constant endofunctions*

$$\varphi : \Pi_{(y:X)} (x =_X y \to x =_X y),$$

then the identity type $x =_X y$ is a proposition, for every $y : X$.

Proof. Let X be a type and consider an inhabitant $x : X$. Suppose that, for every $y : X$, there is a weakly constant endofunction

$$\varphi_y : x =_X y \to x =_X y.$$

To see that the type $x =_X y$ is a proposition for every $y : X$, consider some $y : X$ and let $p, q : x =_X y$.

$$\begin{aligned} p &= \varphi(\mathrm{refl}_x)^{-1} \bullet \varphi(p) && \text{[Lemma 2.6.7]} \\ &= \varphi(\mathrm{refl}_x)^{-1} \bullet \varphi(q) && \text{[φ is weakly constant]} \\ &= q && \text{[Lemma 2.6.7]} \end{aligned}$$

□

Theorem 2.6.10 (Hedberg [Hed98])**.** *Every type with decidable equality is a set.*

Proof. Let X be a type with decidable equality. We need to show that the identity type $x =_X y$ is a proposition, for every $x, y : X$. It suffices, by Lemma 2.6.9, to construct a family

$$\Pi_{(x,y:X)} x =_X y \to x =_X y$$

of weakly constant endofunctions. Let $x, y : X$. We show that there is a weakly constant endofunction on $x =_X y$ by case analysis on its proof of decidability. We have two cases: either we have a proof of $\neg(x =_X y)$ or we have some inhabitant $p : x =_X y$. In the former case, it is easy to see that the obvious map $x =_X y \to \mathbf{0} \to x =_X y$ is weakly constant. In the latter case, the mapping $x \mapsto p$, which sends everything to the identity proof p, constitutes a weakly constant endomap. □

Example 2.6.11. The type $\mathbb{N}$ of natural numbers is a set since it has decidable equality as shown in Example 2.6.6.

Example 2.6.12. Every proposition is a set, since every proposition has decidable equality by Example 2.6.3.

We also note the following facts as useful corollaries of the above examples.

Lemma 2.6.13. *For every type X, the type* is-prop(X) *is a proposition.*

Proof. Follows from Lemma 2.9.8 and Example 2.6.12. □

Lemma 2.6.14. *For every type X, the type* is-contractible(X)*, expressing that X is contractible, is a proposition*

Proof. Let X be a type and consider two proofs (c_1, φ_1), (c_2, φ_2) that it is contractible. By Lemma 2.5.5, it suffices to construct a proof $p : c_1 = c_2$ and a proof

$$q : \mathsf{transport}(c_1, c_2, p, \varphi_1) = \varphi_2.$$

The function φ_2 gives a proof $\varphi_1(c_2) : c_1 = c_2$. Since propositions are sets by Example 2.6.12, we must have that $\mathsf{transport}(c_1, c_2, \varphi_1(c_2), \varphi_1) = \varphi_2$ by function extensionality. □

2.7 Function extensionality and invertible functions

In this section, we discuss the function extensionality axiom as well as the notions of invertible and left-cancellable maps.

2.7.1 The function extensionality axiom

It is reasonable to expect that, in type theory, functions satisfy the principle of *function extensionality* with respect to the identity type. That is to say, given a pair of functions $f, g : X \to Y$, the identity type $f =_{(X \to Y)} g$ should be logically equivalent to the type

$$\Pi_{(x,x' : X)} f(x) =_Y f(x'),$$

expressing that functions f and g are extensionally equal. Unfortunately, however, this is not provable [Str93, Theorem 3.17] in the base type theory that we presented in Section 2.1. The usual solution for this is to postulate function extensionality in the form of an *axiom* in type theory (as we discussed in Section 2.1.2).

To discuss this axiom, we start by defining the pointwise equality of two functions.

Definition 2.7.1 (Pointwise equality)**.** Two dependent functions $f, g : \Pi_{(x:X)} Y(x)$ are said to be *extensionally equal* or *pointwise equal* if the type

$$f \sim g \quad :\equiv \quad \Pi_{(x:X)} f(x) =_{Y(x)} f(x')$$

is inhabited.

Under the homotopical interpretation of the identity type, this is precisely a homotopy between f and g.

Definition 2.7.2 (AXIOM: Naïve function extensionality)**.** The *naïve function extensionality* axiom states that $f \sim g$ implies $f = g$, for every pair of functions $f, g : X \to Y$.

Similar to the examples discussed in Section 2.5.5, the above formulation is the naïve form of the function extensionality axiom. As it is possible for the codomains of the functions in consideration to have nontrivial identity types, they can have different homotopies witnessing their extensional equality, meaning this is *structure* and not *property* in general. We will give examples of such higher groupoids in Section 2.10.2 and in Section 2.11.

2.7.2 Quasi-inverses

The interpretation of pointwise equality as homotopy suggests the following notion of homotopy equivalence between types.

Definition 2.7.3 (Quasi-inverse [Uni13, §4.1])**.** A function $g : Y \to X$ is said to be the *quasi-inverse* of a function $f : X \to Y$ if there are homotopies

$$g \circ f \sim \mathsf{id}_X \quad \text{and} \quad f \circ g \sim \mathsf{id}_Y.$$

Given a function $f : X \to Y$, its type of quasi-inverses is defined to be

$$\mathsf{QInv}(f) \quad :\equiv \quad \Sigma_{(g : Y \to X)} (g \circ f \sim \mathsf{id}_X) \times (f \circ g \sim \mathsf{id}_Y).$$

Definition 2.7.4. A function $f : X \to Y$ is said to be *invertible* if we have an inhabitant of $\mathsf{QInv}(f)$.

Once again, this is the naïve form of the concept in consideration, and will not be our official definition. The reason is that, since extensional equality is structure and not property, there could be distinct witnesses of the fact that a function $g : Y \to X$ constitutes inverse to some function $f : X \to Y$. Nevertheless, it will be convenient to have pinned down with these naïve forms for two reasons:

1. They are *logically equivalent* to their refined forms.
2. For the homotopy level of sets (in the sense of Definition 2.5.13), they are *equivalent* to their refined versions.

In Section 2.8, we will refine the notion of invertible map to that of *equivalence*, which is a property for arbitrary types and not just sets.

2.7.3 Left-cancellable maps

Yet another useful albeit naïve notion is that of a *left-cancellable map*.

Definition 2.7.5 (Left-cancellability [Uni13, p. 139])**.** A function $f : X \to Y$ is called *left-cancellable* if we have a map

$$f(x) =_Y f(x') \to x =_X x',$$

for every pair of inhabitants $x, x' : X$.

Similar the notion of invertible map, left-cancellability is not a proposition in general, but is always a proposition when the domain is a set. In Section 2.8.2, we will refine this to the notion of *embedding*, analogously to our refinement of invertible maps to equivalences.

2.8 Equivalences and embeddings

We presented the notions of invertible and left-cancellable maps in Section 2.7.2, and discussed the fact that these are not properties for arbitrary n-types. The problem here is that the notion of invertible function is the appropriate notion of equivalence only for *sets*, and it does not generalize to arbitrary n-types. In this section, we will look at the refined versions of these notions that are well-behaved for all types.

2.8.1 Equivalences

The notion of equivalence given in Definition 2.8.2 was first formulated by Voevodsky.

Definition 2.8.1 ([Uni13, Definition 4.2.4])**.** The *fibre* of a function $f : X \to Y$ over a point $y : Y$ is the type

$$\mathsf{fib}_f(y) \quad :\equiv \quad \Sigma_{(x:X)} f(x) =_Y y.$$

Definition 2.8.2 ([Uni13, Definition 4.4.1])**.** A function $f : X \to Y$ is called an *equivalence* if

$$\mathsf{is\text{-}equiv}(f) \quad :\equiv \quad \Pi_{(y:Y)}\, \mathsf{is\text{-}contractible}(\mathsf{fib}_f(y)).$$

We denote by $X \simeq Y$ the type of equivalences between types X and Y. Furthermore, given an equivalence $e : X \simeq Y$, we denote by e^{-1} its inverse, and by $\ulcorner e \urcorner$, its underlying map $X \to Y$.

Example 2.8.3. For every type X, the identity function $\mathsf{id}_X : X \to X$ on it obviously forms an equivalence. We denote this by $\mathsf{idequiv}_X : X \simeq X$.

Lemma 2.8.4. *Being an equivalence is a property. That is to say, the type* $\mathsf{is\text{-}equiv}(f)$ *is a proposition, for every function* $f : X \to Y$.

Proof. Follows directly from Lemma 2.9.8 and Lemma 2.6.14. □

In Section 2.7.2, we explained that being invertible is not a proposition for arbitrary types. The two notions are thus not equivalent. They are, however, *logically* equivalent.

Lemma 2.8.5 (🤌). *A function $f : X \to Y$ is invertible if and only if it is an equivalence.*

Having a well-behaved notion of equivalence for n-types is crucial in univalent type theory, since it enables the formulation of various extensionality axioms. In Section 2.8.3, we will present the official formulation of the function extensionality axiom using the above notion of equivalence, thereby circumventing the naïvety involved in the formulation we gave in Definition 2.7.2. Furthermore, this notion of equivalence is also crucial to the statement of the univalence axiom, since the formulation using invertible maps is provably false as shown in [Uni13, Exercise 4.6.(iii)].

2.8.2 Embeddings

As we refined the notion of invertible map to that of equivalence above, we now refine the notion of left-cancellable map (Definition 2.7.5) to that of an embedding.

Definition 2.8.6. A function $f : X \to Y$ is called an *embedding* if

$$\mathsf{is\text{-}embedding}(f) \quad :\equiv \quad \Pi_{(y:Y)}\, \mathsf{is\text{-}prop}(\mathsf{fib}_f(y)).$$

Lemma 2.8.7. *For every function $f : X \to Y$, the type* $\mathsf{is\text{-}embedding}(f)$ *is a proposition.*

Proof. Follows immediately from Lemma 2.9.8 and the fact that being proposition is a proposition (Lemma 2.6.13). □

Lemma 2.8.8 (🤌). *Every equivalence is an embedding.*

Proof. See [Esc19, §3.26]. □

Lemma 2.8.9 (🤌). *Every embedding is a left-cancellable map.*

Proof. See [Esc19, §3.26]. □

2.8.3 Function extensionality revisited

In Section 2.7, we formulated what we called the *naïve* function extensionality axiom, and promised to give the official definition later. Having pinned down the well-behaved notion of equivalence in Definition 2.8.2, we are now ready to revisit function extensionality as promised.

Definition 2.8.10 (happly)**.** For every pair of dependent functions $f, g : \Pi_{(x:X)} Y(x)$, we define the map

$$\mathsf{happly}_{f,g} : f = g \to f \sim g$$

as follows. Let $f, g : \Pi_{(x:X)} Y(x)$, and define the type family $E(h) :\equiv f \sim h$; we define

$$\mathsf{happly}_{f,g}(p) \quad :\equiv \quad \mathsf{transport}^E(f, g, p, \lambda x.\, \mathsf{refl}_x).$$

Definition 2.8.11 (AXIOM: Function extensionality [Uni13, Axiom 2.9.3])**.** The *function extensionality axiom* states that, for every pair of functions $f, g : \Pi_{(x:X)} Y(x)$, the map $\mathsf{happly}_{f,g}$ is an equivalence.

This is clearly a proposition since being an equivalence is a proposition as shown in Lemma 2.8.4. The above *official* formulation of function extensionality is logically equivalent to the naïve formulation from Definition 2.7.2, even though the above is formulated for dependent functions and might thus seem stronger at first glance. This fact was first observed by Voevodsky as explained in [Esc19, §3.18].

Lemma 2.8.12. *Function extensionality is logically equivalent to both of the following statements:*

- *naïve function extensionality from Definition 2.7.2,*
- *the dependent version of naïve function extensionality i.e. the statement that, for every pair of dependent functions $f, g : \Pi_{(x:X)} Y(x)$, we have that*

$$f \sim g \text{ implies } f = g.$$

Proof. See [Esc19, §3.18]. □

2.9 Homotopy levels of types

In Section 2.5.4, we explained that the homotopical interpretation of the identity type gives rise to a classification of types with respect to the complexity of their identity types. We also discussed the examples of contractible and propositional types as well as sets. In this section, we present the notion of an n-type, which generalizes this classification to an arbitrary homotopy level $n : \mathbb{N}$. This stratification of types lies at the heart of HoTT/UF.

2.9.1 Definition of the homotopy hierarchy

We define a predicate $\mathsf{is\text{-}of\text{-}hlevel} : \mathcal{U} \to \mathbb{N} \to \mathcal{U}$, expressing what it means for a type $X : \mathcal{U}$ to be of some homotopy level $n : \mathbb{N}$.

Definition 2.9.1 (*cf.* [Uni13, Definition 7.1.1])**.** The *homotopy level* of a type $X : \mathcal{U}$ is

defined by induction as follows:

$$\begin{aligned} \text{is-of-hlevel}(X, \text{zero}) &:\equiv \text{is-contractible}(X) \\ \text{is-of-hlevel}(X, \text{succ}(n)) &:\equiv \Pi_{(x,y:X)} \text{is-of-hlevel}((x =_X y), n). \end{aligned}$$

A type X is called an *n-type* if we have that X is of h-level n.

Remark 2.9.2. Note that the HoTT Book [Uni13, Definition 7.1.1] uses the indexing convention of starting from -2, which differs from our indexing convention above.

We can now reconstruct the homotopy levels that we have looked at as special cases of this.

Lemma 2.9.3 ([Uni13, Lemma 3.11.10])**.** *For every type X, the type is-of-hlevel$(X, 1)$ is inhabited if and only if type X is a proposition.*

Proof. Let X be a type.

($\Rightarrow$) This direction is easy to see. Suppose X satisfies is-of-hlevel$(X, 1)$, which is to say that, for every pair of inhabitants $x, y : X$, the type $x =_X y$ is contractible. It immediately follows that $x =_X y$ is inhabited, since contractible types are obviously inhabited.

($\Leftarrow$) Suppose that X is a proposition, meaning the type $x =_X y$ is inhabited, for every pair of inhabitants $x, y : X$. Since propositions are sets by Example 2.6.12, we also know that the type $x =_X y$ is a proposition. It follows from Lemma 2.5.12 that the identity type $x =_X y$ is contractible. □

Corollary 2.9.4. *For every type X, we have that X is a set if and only is-of-hlevel$(X, 2)$.*

2.9.2 Basic properties of homotopy levels

We now collect some key properties of the hierarchy of homotopy types. Our goal here is to summarize the theory of n-types to the extent that we need it in our development. We thus omit many interesting and crucial facts about the hierarchy of homotopy levels. The reader interested in a detailed treatment is referred to [Uni13, §7.1].

As contractible types are propositions by Example 2.5.11, it follows from Lemma 2.9.3 that, for every type X, we have

$$\text{is-of-hlevel}(X, 0) \Rightarrow \text{is-of-hlevel}(X, 1).$$

Furthermore, Example 2.6.12 means that, for every type X, we have

$$\text{is-of-hlevel}(X, 1) \Rightarrow \text{is-of-hlevel}(X, 2).$$

In fact, this inclusion into the next homotopy level can be generalized to an arbitrary homotopy level n:

Lemma 2.9.5 ([Uni13, Theorem 7.17]). *Homotopy levels are cumulative. That is to say, for every type X and every $n : \mathbb{N}$, we have that*

$$\textit{is-of-hlevel}(X, n) \Rightarrow \textit{is-of-hlevel}(X, \mathrm{succ}(n)).$$

Proof. Let X be a type. We proceed by induction.

Base case: zero. We have already shown that contractible types are propositions in Example 2.5.11, and that being proposition is equivalent to having a homotopy level of 1 in Lemma 2.9.3.

Inductive step: $\mathrm{succ}(n')$. Assume inductively that every n'-type is also an $(n'+1)$-type, and suppose X is an $(n'+1)$-type. This is to say that the identity type $x =_X y$ is an n'-type, for every pair of inhabitants $x, y : X$. By the inductive hypothesis, this implies that the identity type $x =_X y$ is also an $(n'+1)$-type, which means the type X is an $(n'+2)$-type as needed. □

Lemma 2.9.6 ([Uni13, Corollary 7.1.5]). *For every n-type X and every type Y, if $X \simeq Y$ then Y is also an n-type.*

Closure properties

Homotopy levels are closed under Σ-types.

Lemma 2.9.7 ([Uni13, Theorem 7.1.8]). *For every type $X : \mathcal{U}$, family $Y : X \to \mathcal{U}$, and $n : \mathbb{N}$, if X is an n-type and each Y_x is an n-type, then the dependent sum $\Sigma_{(x:X)} Y_x$ is also an n-type.*

Similarly, the dependent function type $\Pi_{(x:X)} Y_x$ is an n-type, whenever the family Y consists of n-types, irrespective of the homotopy level of the domain X.

Lemma 2.9.8 ([Uni13, Theorem 7.1.9]). *For every type $X : \mathcal{U}$, family $Y : X \to \mathcal{U}$, and every $n : \mathbb{N}$, if each Y_x is an n-type, then the dependent product type $\Pi_{(x:X)} Y_x$ is also an n-type.*

2.10 The univalence axiom

We have discussed two extensionality axioms so far: propositional extensionality and function extensionality. Since logical equivalence is the special case of *type equivalence* at the homotopy level of $n = 1$, propositional extensionality can be interpreted as stating that the identity type of propositions coincide with the type of equivalences between them. Similarly, function extensionality gives a characterization of the identity type of the function type, as it implies that there is an equivalence $(f \sim g) \simeq f = g$.

The *univalence axiom* can be construed as a generalized extensionality axiom, relating the identity types of arbitrary n-types to the types of equivalences between them. Somewhat surprisingly, the univalence axiom gives *all* standard extensionality

axioms as corollaries: propositional extensionality and function extensionality both follow from it.

2.10.1 Formal definition of the univalence axiom

To state the univalence axiom, we first define the map idtoeqv from the identity type of two types into their type of equivalences.

Definition 2.10.1 (). For every pair of types $X, Y : \mathcal{U}$, we define a map

$$\mathsf{idtoeqv} : X =_{\mathcal{U}} Y \to X \simeq Y.$$

Let X and Y be a pair of types. We define the type family $E(Z) :\equiv X \simeq Z$, and use Lemma 2.5.2 to define

$$\mathsf{idtoeqv}(p) \quad :\equiv \quad \mathsf{transport}^{E}(X, Y, p, \mathsf{idequiv}_X),$$

where idequiv refers to the identity equivalence from Example 2.8.3. The judgemental equality for transport gives $\mathsf{idtoeqv}(\mathsf{refl}_x) \equiv \mathsf{id}$.

Definition 2.10.2 (Univalent universe). A universe $\mathcal{U}$ is called *univalent* if, for every pair of types $X, Y : \mathcal{U}$, the map $\mathsf{idtoeqv} : (X =_{\mathcal{U}} Y) \to (X \simeq Y)$ defined above is an equivalence.

Definition 2.10.3 (The univalence axiom). The *univalence axiom* states that all universes are univalent.

We will usually speak of *the* univalence axiom, although in practice we will be assuming a specific universe to be univalent. We will explicitly specify the universe for which we are assuming univalence only when extra clarity is required.

The univalence axiom is known to be consistent with the base Martin-Löf Type Theory that we presented in Section 2.1.1. This was first shown by Voevodsky's *simplical sets model* [KL21] which validates univalence.

Theorem 2.10.4. *Every univalent universe $\mathcal{U}$ satisfies propositional extensionality and function extensionality.*

Proof. It is easy to see that the univalence axiom implies propositional extensionality, as it is clear that equivalence specializes to logical equivalence at the homotopy level of $n = 1$. The proof that it implies function extensionality is due to Voevodsky. A proof can be found in [Uni13, §4.9]. □

2.10.2 Universes are not sets, assuming univalence

When we discussed the **K** axiom in Section 2.5.4, we mentioned that it is inconsistent with univalence since the univalence axiom gives us the means to construct nontrivial

inhabitants of identity types. We now look at our first example of a type that is not a set under the assumption of univalence.

Example 2.10.5. Universes are *not* sets. To see this, consider an arbitrary universe $\mathcal{U}$. We take the copy $2_{\mathcal{U}}$ of the type of Booleans in universe $\mathcal{U}$. We have a proof

$$\mathsf{refl}_2 : 2 =_{\mathcal{U}} 2,$$

that this type is equal to itself. We then have an equivalence

$$\mathsf{idtoeqv}(\mathsf{refl}_2) : 2 \simeq 2.$$

Both sides of this equivalence can be seen to be the identity map by the computation rule for transport (Lemma 2.5.2). Using the univalence axiom, we can construct a distinct proof that it is equal to itself. Consider the function

$$\begin{aligned} \mathsf{swap}(0) &:\equiv 1 \\ \mathsf{swap}(1) &:\equiv 0 \end{aligned}$$

It is easy to see that this type also gives an equivalence of $2_{\mathcal{U}}$ with itself, and thus a path $p : 2 =_{\mathcal{U}} 2$ by univalence. If we have that $p = \mathsf{refl}_x$, however, it follows that $\mathsf{swap} = \mathsf{id}$, which is a contradiction since it implies $0 = 1$.

2.10.3 The structure identity principle

It is common practice in mathematics to regard two algebraic structures to be *the same* when they are isomorphic. Thanks to the univalence axiom, we can obtain a precise mathematical account of this informal practice in univalent foundations, actually proving that isomorphic structures are equal.

This is achieved through the so-called *structure identity principle*, which can be thought of as a generalized extensionality axiom for mathematical structures. The key idea is to define a generalized notion of isomorphism of structures, subject to some conditions ensuring that it is well behaved. The structure identity principle then describes the identity type of mathematical structures in terms of this generalized notion of isomorphism.

There are various forms of the structure identity principle in the literature. Three well-known formulations are due to Coquand and Danielsson [CD13], Aczel [Uni13, §9.8], and Escardó [Esc19, §3.33]. In this section, we present the structure identity principle of Escardó from *loc. cit.* We provide a high-level overview and do not delve into the full technical details.

When defining a mathematical structure in univalent foundations, we usually start by expressing the structure in consideration as a type family. Take the notion of monoid as an example. To define the type of monoids, we define a type family

$$\mathsf{Monoid\text{-}Structure} : \mathcal{U} \to \mathcal{U},$$

that takes a type $X : \mathcal{U}$ and expresses the type of monoid structures on it. This may be defined as a Σ-type, equipping the carrier type X in consideration with a unit

element $\epsilon : X$ and a binary operation $\mathrm{op} : X \to X \to X$, subject to the monoid axioms:

$$\begin{aligned}&\mathsf{Monoid\text{-}Structure}(X) :\equiv \\ &\quad \Sigma_{(\epsilon : X)} \Sigma_{(\mathrm{op} : X \to X \to X)} \mathsf{satisfies\text{-}monoid\text{-}axioms}(\epsilon, \mathrm{op}).\end{aligned}$$

The type of monoids would then be defined as

$$\mathsf{Monoid}_{\mathcal{U}} \quad :\equiv \quad \Sigma_{(X : \mathcal{U})} \mathsf{Monoid\text{-}Structure}(X).$$

In Escardó's formulation of the structure identity principle from [Esc19, §3.33], the isomorphism of two mathematical structures is expressed by imposing certain conditions on an *equivalence* between the carrier types of the structures in consideration. Consider two monoids M and N. An equivalence $e : |M| \simeq |N|$ between their carrier types is said to be homomorphic if the underlying map $\ulcorner e \urcorner : M \to N$ is a homomorphism of monoids.

We generalize this state of affairs by considering a mathematical structure as given by a type family $S : \mathcal{U} \to \mathcal{U}$. A notion of isomorphism for S-structures is given by a family

$$\iota_{s_A, s_B} : A \simeq B \to \mathcal{U},$$

for each pair of S-structures (A, s_A) and $(B, s_B) : \Sigma_{(X : \mathcal{U})} S(X)$. The idea is that ι expresses what it means for an equivalence to preserve the S-structure. Escardó's structure identity principle then stipulates the following conditions to ensure that such a notion of isomorphism is well-behaved:

1. For every type A and structure $s : S(A)$, the identity equivalence $\mathsf{idequiv}_A$ (from Example 2.8.3) satisfies $\iota_{s_A, s_A}(\mathsf{idequiv}_A)$.
2. For every type A, and every pair of S-structures s_A and t_A on it, the canonical map $s_A = t_A \to \iota_{s_A, A}(\mathsf{idequiv}_X)$ is an equivalence.

A structure $S : \mathcal{U} \to \mathcal{U}$ is called a *standard notion of structure* when it is equipped with a type family ι satisfying the above conditions. The structure identity principle can then be precisely stated as follows:

Theorem 2.10.6 (🖉 [Esc19, §3.33]). *Let $S : \mathcal{U} \to \mathcal{V}$ and let $A, B : \mathcal{U}$. Assuming that universe $\mathcal{U}$ is univalent, if S is a standard notion of structure as given by some ι, we then have*

$$((A, s_A) = (B, s_B)) \simeq ((A, s_A) \cong_\iota (B, s_B)),$$

where $(A, s_A) \cong_\iota (B, s_B)$ denotes the type of ι-isomorphisms between (A, s_A) and (B, s_B).

We will make use of the structure identity principle in Section 4.7, where we will construct an equivalence between the type of large, locally small, and small-complete spectral locales and the type of small distributive lattices.

2.11 Higher inductive types

We previously mentioned the notion of an *inductively generated type*, and explained in Section 2.3 that the formal account of inductive types is given by the mechanism of

W-types. The avoidance of the **K** axiom, which amounts to a generalization from sets to n-types, gives us the freedom to generalize the idea of an inductively generated type to the setting of n-types as well. This generalization gives rise to the notion of *higher inductive type*.

An inductive type is generated by a set of constructors, which inductively generate the type in consideration. A higher inductive type is generated by a set of n-constructors for each homotopy level $n : \mathbb{N}$. For the definition of a type X as a higher inductive type,

- the 0-constructors are the usual constructors generating the inhabitants of X,
- the 1-constructors augment the path type $x =_X y$ of X through the free addition of nontrivial path constructors,
- the 2-constructors augment the path type $p =_{(x =_X y)} q$ of the path type itself by adding path constructors between the path constructors.

By generalizing this to an arbitrary n, we obtain the notion of n-constructor.

In this thesis, we will use only two higher inductive types. We thus refrain from presenting the complete mathematical account of the general notion of a higher inductive type, which is a rather technical and intricate question. We will instead look at some easy examples and then present the specific higher inductive types that we will be using in our work.

2.11.1 The circle type

Definition 2.11.1 ([Uni13, §6.4])**.** The *circle* (denoted $\mathbb{S}^1$) is the higher inductive type generated by the following n-constructors:

- a single 0-constructor $\mathsf{base} : \mathbb{S}^1$,
- a single 1-constructor $\mathsf{loop} : \mathsf{base} =_{\mathbb{S}^1} \mathsf{base}$.

This type has the following induction principle: for every type family $P : \mathbb{S}^1 \to \mathcal{U}$, if there is

- a point $b : P(\mathsf{base})$, and
- a path $l : \mathsf{base} = \mathsf{base}$,

there is a function $f : \Pi_{(x:\mathbb{S}^1)} P(x)$ satisfying the equalities

- $f(\mathsf{base}) = b$, and
- $\mathsf{ap}_f(\mathsf{loop}) = l$.

Lemma 2.11.2. *The paths* $\mathsf{refl}_{\mathsf{base}}, \mathsf{loop} : \mathsf{base} =_{\mathbb{S}^1} \mathsf{base}$ *are not equal to each other.*

Proof sketch. If these two paths are equal, then it can be shown by the induction principle of the circle that all types are sets, which is not the case as we showed in Example 2.10.5. See [Uni13, Lemma 6.4.1] for the full proof. □

Corollary 2.11.3. *The circle type is not a set, which means it is neither contractible nor a proposition.*

From a homotopical perspective, the fact that loop is distinct from $\mathsf{refl}_{\mathsf{base}}$ *detects* that the circle type contains a hole.

2.11.2 Propositional truncation

We now look at another example of a higher inductive type that is essential for the use of univalent type theory as a mathematical foundation: the *propositional truncation* of a type.

Definition 2.11.4. The *propositional truncation* is given by the type former

$$\|\text{-}\| : \mathcal{U} \to \mathcal{U}.$$

For every type $X : \mathcal{U}$, the type $\|X\|$ is defined by the following n-constructors:

- A single 0-constructor $|\text{-}| : X \to \|X\|$.
- A single 1-constructor $\mathsf{trunc} : \prod_{(x,y:\|X\|)} x =_X y$.

This gives the induction principle

$$\prod_{(P:\mathcal{V})} \mathsf{is\text{-}prop}(P) \to (X \to P) \to (\|X\| \to P).$$

Observe that the trunc constructor here says exactly that the type $\|X\|$ is a proposition.

Lemma 2.11.5. *For every pair of types $X : \mathcal{U}$, and $Y : \mathcal{V}$, we have a function $\|X\| \to \|Y\|$, whenever we have a function $X \to Y$.*

Proof. Immediate by the recursion principle of propositional truncation since $\|Y\|$ is a proposition. □

It is easy to see that this defines a functor. This functor is in fact the *reflector* of types in a universe $\mathcal{U}$ into its full subcategory $\Omega_{\mathcal{U}}$ of propositions.

Lemma 2.11.6. *The propositional truncation $\|\text{-}\| : \mathcal{U} \to \Omega_{\mathcal{U}}$ is the left adjoint to the inclusion $\Omega_{\mathcal{U}} \overset{\mathsf{pr}_1}{\hookrightarrow} \mathcal{U}$.*

In Section 2.5.5, we discussed the distinction between structure and property. We will frequently encounter situations where a type expressing some statement of interest will be structure and *not* a property. We already gave an example of this in Example 2.5.15. The propositional truncation is particularly useful in such situations where we can obtain the property of interest simply by truncating the structure in consideration.

2.12 Propositions-as-types revisited

Now that we have introduced higher inductive types and presented the mechanism of propositional truncation, we are ready to present the *refined* form of the propositions-

as-types interpretation as we promised in Section 2.4.

In Example 2.5.15, we gave an example illustrating that it may be misleading to think of Σ-types as asserting the *existence* of objects in type theory. This is because a proof of a Σ-type *specifies* the object in consideration, and is thus structure and not property (in general). There is a similar problem with interpreting the sum type $P + Q$ as the logical disjunction of propositions P and Q. A proof of a sum type contains not only the information that one of the disjuncts holds, but also the information of *which disjunct* holds. It is thanks to propositional truncation that we can form the propositional versions of these, and thereby view all logical statements as the denotations of propositional types.

2.12.1 The refined propositions-as-types translation

We showed in Lemma 2.9.7 that propositions are closed under product types, meaning we have a function

$$(\text{-}) \wedge (\text{-}) : \Omega_{\mathcal{U}} \to \Omega_{\mathcal{V}} \to \Omega_{\mathcal{U}\sqcup\mathcal{V}}.$$

Furthermore, we established in Lemma 2.9.8 that the dependent product type gives a function

$$\Pi(\text{-}, \text{-}) : \Pi_{(X:\mathcal{U})} ((X \to \Omega_{\mathcal{V}}) \to \Omega_{\mathcal{U}\sqcup\mathcal{V}}),$$

taking a domain of quantification $X : \mathcal{U}$ and a $\mathcal{V}$-valued predicate on it, and returning a proposition in universe $\mathcal{U} \sqcup \mathcal{V}$.

Logical statement	Proposition
Falsum $\bot$	Empty type $\mathbf{0}$
Truth $\top$	Unit type $\mathbf{1}$
Conjunction $P \wedge Q$	Product $P \times Q$
Disjunction $P \vee Q$	Truncated sum $\|P + Q\|$
Implication $P \Rightarrow Q$	Function type $P \to Q$
Predicate on type X	Family $X \to \Omega$ of propositions
Universal quantification $\forall_{(x:X)} P(x)$	Dependent product $\Pi_{(x:X)} P(x)$
Existential quantification $\exists_{(x:X)} P(x)$	Truncated dependent sum $\left\|\Sigma_{(x:X)} P(x)\right\|$

Table 2.12.1: summary of the refined propositions-as-types principle.

Thanks to the existence of propositional truncations, we can now consider the truncated sum type, which gives a function $\|(\text{-}) + (\text{-})\| : \Omega_{\mathcal{U}} \to \Omega_{\mathcal{V}} \to \Omega_{\mathcal{U}\sqcup\mathcal{V}}$, and the truncated Σ-type former, which is of type

$$\|\Sigma(\text{-}, \text{-})\| : \Pi_{(X:\mathcal{U})} ((X \to \Omega_{\mathcal{V}}) \to \Omega_{\mathcal{U}\sqcup\mathcal{V}}).$$

We think of truncated Σ-types as asserting the *unspecified existence* of an object. Using these, we can give a complete translation of logical statements into the type Ω of propositions. We give a summary of this refined form of the propositions-as-types principle in Table 2.12.1.

As an example, we refine the naïve law of excluded middle (Definition 2.6.5) through the refined propositions-as-types translation.

Definition 2.12.2 (Law of excluded middle (LEM))**.** The *law of excluded middle* for universe $\mathcal{U}$ asserts that, for every proposition $P : \Omega_{\mathcal{U}}$, we have an inhabitant of the disjunction $\|P + \neg P\|$.

We will hereafter use the shorthand notation $\exists_{(x:X)} P(x)$ and $P \vee Q$ for truncated Σ-types and sum types respectively.

2.12.2 Specified vs. unspecified existence

As explained before, the distinction between structure and property is a hallmark feature of univalent type theory. Now that we have presented both of the notions of *specified* and *unspecified* existence that we will be working with, we adopt some vernacular conventions to distinguish between structure and property.

- We refrain from saying "x is Y" if the type family Y in consideration does not consist of propositions.
- For the anonymous inhabitation $\|A\|$ of a type A, we say that A *is inhabited*.
- We say *specified inhabitant* of type A to contrast it with the anonymous inhabitation $\|A\|$. Similarly, we say there is a *specified* or *chosen* element to emphasize that we are using Σ instead of $\exists$.
- For truncated Σ-types, we use the terminologies *there is some* and *there exists some*. We also use the phrase *there is a specified/unspecified*, when extra clarity is required.
- Unlike the HoTT Book [Uni13], we do not use the adjective *mere* for unspecified existence.

2.12.3 Subsets

As Ω is the (large) type of types that we view as properties, we think of Ω-valued functions as *predicates* or *subsets*.

Definition 2.12.3 ($\mathcal{V}$-subset)**.** A *$\mathcal{V}$-subset* (or *$\mathcal{V}$-valued subset*) of a type $X : \mathcal{U}$ is a predicate $S : X \to \Omega_{\mathcal{V}}$. The $\mathcal{V}$-powerset of X, denoted $\mathcal{P}_{\mathcal{V}}(X)$, is the type of all $\mathcal{V}$-subsets of X.

Remark 2.12.4. It is useful to observe here that, for every type $X : \mathcal{U}$, the type $\mathcal{P}_{\mathcal{V}}(X) \equiv X \to \Omega_{\mathcal{V}}$ lives in universe $\mathcal{U} \sqcup \mathcal{V}^+$. In particular, the type $\mathcal{P}_{\mathcal{U}}(X)$ of all $\mathcal{U}$-subsets lives in universe $\mathcal{U}^+$.

The situation here is very similar to topos logic. In topos theory, one works with a notion of subobject classifier Ω, which is axiomatized through a universal property stating that the morphisms $\mathsf{Hom}(X, \Omega)$ are in bijection with the subobjects of X.

In univalent type theory, the type $\Omega_{\mathcal{U}}$ plays the rôle of the subobject classifier as it classifies *subtypes*, which we will discuss in detail later. Unlike topos logic, however, its construction is completely internal to the type theory; the fact that it classifies subobjects is a theorem and not an axiom.

Another difference with topos logic here is that $\Omega_{\mathcal{U}}$ is not admitted as a type living in universe $\mathcal{U}$, since it quantifies over the types in universe $\mathcal{U}$. This brings us to the subject of predicativity, which we will discuss soon in Section 2.13.

Kuratowski finite subsets

In constructive mathematics, there are several distinct notions of finiteness. The two salient ones among them are *Bishop finiteness* and *Kuratowski finiteness*. We now briefly explain the latter, as it will be of interest to us in our development of locale theory.

We denote by $\mathsf{Fin}(n)$ the type with exactly n inhabitants, which can be constructed in our setting as the sum type consisting of n-many copies of the unit type $\mathbf{1}$. Furthermore, given a subset $P : X \to \Omega_{\mathcal{U}}$, we denote by $\mathbb{T}(P)$ the type $\Sigma_{(x:X)} P(x)$ i.e. the type of all inhabitants of x falling in P.

Definition 2.12.5 (Kuratowski finiteness)**.**

- (🤌) A type X is called *Kuratowski finite* if there exists a surjection $\mathsf{Fin}(n) \twoheadrightarrow X$, for some $n : \mathbb{N}$. This is called an *enumeration* of X.
- (🤌) Given a type $X : \mathcal{U}$, a subset $P : X \to \Omega_{\mathcal{U}}$ is called *Kuratowski finite* if the type $\mathbb{T}(P)$ is Kuratowski finite.

We denote by $\mathsf{KFin}(X)$ the type of Kuratowski finite subsets of a type X.

Lemma 2.12.6 (🤌 Kuratowski finite induction [dJon23, Lemma 4.2.10])**.** *For every type $X : \mathcal{U}$, and every property $P : \mathsf{KFin}(X) \to \Omega_{\mathcal{V}}$ of the collection of Kuratowski finite subsets of X, to show that P holds for all Kuratowski finite subsets of X, it suffices to show the following:*

- *The empty set satisfies P.*
- *All singleton sets satisfy P.*
- *The property P is closed under binary unions.*

2.13 Predicativity and propositional resizing

We now start studying the notion of *size* of a type, which captures what we mean by the predicativity of our foundational setting. We start by defining the notions of $\mathcal{U}$-small and locally $\mathcal{U}$-small types.

Definition 2.13.1 ([Rij17])**.** A type $X : \mathcal{U}$ is called *$\mathcal{V}$-small* if

$$\mathsf{is\text{-}}\mathcal{V}\text{-}\mathsf{small}(X) \quad :\equiv \quad \Sigma_{(Y:\mathcal{V})} X \simeq Y.$$

Definition 2.13.2 ([Rij17])**.** A type $X : \mathcal{U}$ is called *locally $\mathcal{V}$-small* if the identity type $x =_X y$ is $\mathcal{V}$-small, for every pair of inhabitants $x, y : X$. In formal notation, we denote this by $\mathsf{is\text{-}locally\text{-}}\mathcal{V}\text{-}\mathsf{small}(X)$.

Example 2.13.3. Assuming propositional extensionality, the type $\Omega_{\mathcal{U}}$ is locally $\mathcal{U}$-small. This type lives in $\mathcal{U}^+$ implying that the identity type $P = Q$ lives in $\mathcal{U}^+$. However, an equivalent $\mathcal{U}$-small copy of the identity type $P = Q$ is always given by the type $(P \Rightarrow Q) \times (Q \Rightarrow P)$, which is small since the implication between two $\mathcal{U}$-small propositions is always $\mathcal{U}$-small.

It is natural to ask whether the type is-$\mathcal{V}$-small(X) is a proposition. This turns out to be the case, but somewhat surprisingly, its propositionality *crucially relies* on the univalence axiom as captured by the lemma below.

Lemma 2.13.4 (). *The following are equivalent:*

1. *For every pair of universes $\mathcal{U}$ and $\mathcal{V}$, the type* is-$\mathcal{V}$-small(X) *is a proposition, for every type $X : \mathcal{U}$.*
2. *The univalence axiom holds.*

Proof. A proof can be found in [Esc20]. □

The fact that being small is a property under the assumption of univalence will find an important application in our development of the predicative theory of spectral locales. To be more specific, this will ensure in Lemma 4.2.3 that one of our definitions of being a spectral locale (Definition 4.2.1) is a property and not structure.

2.13.1 Subsets and families

In Section 2.12.3, we defined a subset of a type as a predicate on the type in consideration. We now look at an alternative description of subsets through *embeddings*.

Definition 2.13.5 ($\mathcal{V}$-family)**.** A *$\mathcal{V}$-family* (or *$\mathcal{V}$-small family*) on a type X is a function $I \to X$ whose domain I lives in $\mathcal{V}$.

We denote by $\mathsf{Fam}_{\mathcal{V}}(X)$ the type of $\mathcal{V}$-families on some type X. Also, we often use the notation $(x_i)_{i:I}$ for a family $x : I \to X$. Moreover, we sometimes use the comprehension notation $\{x_i \mid i : I, P(i)\}$ for families of the form $x : (\Sigma_{(i:I)} P(i)) \to X$, where the index type is a Σ-type.

Definition 2.13.6 (Subfamily)**.** Given a $\mathcal{V}$-family $(x_i)_{i:I}$ and a map $\alpha : J \to I$, where $J : \mathcal{V}$, the composition $x \circ \alpha : J \to X$ is called the *subfamily* obtained by reindexing along α.

It is convenient to use the family notation for reindexing maps themselves, thus writing $(x_{i_j})_{j:J}$ for reindexed families. Accordingly, we will simply speak of a subfamily $(x_{i_j})_{j:J}$ of a family $(x_i)_{i:I}$, and will refrain from working explicitly with a reindexing map $\alpha : I \to J$.

Definition 2.13.7 ($\mathcal{V}$-subtype)**.** A *$\mathcal{V}$-subtype* of a type X is a $\mathcal{V}$-family on X that is an embedding.

We denote the type of $\mathcal{V}$-subtypes by $\mathrm{Sub}_{\mathcal{V}}(X)$.

Lemma 2.13.8. *Assuming the univalence axiom, the type $\Omega_{\mathcal{U}}$ classifies the $\mathcal{U}$-subtypes of $\mathcal{U}$-types. In other words, for every type $X : \mathcal{U}$, we have an equivalence*

$$\mathrm{Sub}_{\mathcal{U}}(X) \simeq \mathcal{P}_{\mathcal{U}}(X).$$

Proof. A proof can be found in [Esc19, §3.31]. □

A family that is not necessarily an embedding can be thought of as containing *repetitions*. In other words, if a family is not an embedding, this means that the index set does not provide a unique representation for the elements of the subset in consideration. Whilst the embedding families correspond exactly to subsets, it turns out to be convenient in practice to drop the embedding requirement and represent subtypes by families that may have repetitions. In Section 2.14, we will discuss the set replacement principle, which will allow us to remove such repetitions when the need arises. This procedure works even in a constructive and predicative setting, provided that the set replacement principle is available. Moreover, in Section 3.5, we will revisit this question and discuss the applications of this to our development of locale theory.

2.13.2 Propositional resizing axioms

We will frequently encounter situations where we will be working with a subset

$$X \to \Omega_{\mathcal{U}^+}$$

given by large propositions (with respect to some base universe $\mathcal{U}$), and we would like it to be small i.e. a $\mathcal{U}$-valued one. We previously explained that a hallmark feature of univalent type theory is the *internal* definition of the notion of property, using which we define the type $\Omega_{\mathcal{U}}$ of $\mathcal{U}$-propositions. This is in contrast with the Calculus of Constructions [CH88] where the universe Prop of propositions can be made impredicative by construction thanks to the fact that being proposition is a judgement in CoC.

At this point of our foundational setting, there arises the question of how one can do impredicative mathematics in a setting where we work with $\Omega_{\mathcal{U}}$ as our subtype classifier. Voevodsky's proposal to address this problem was to consider the so-called *propositional resizing* rules [Voe11, p. 10]. There are slightly different forms of these rules, but in summary, they state that every proposition $P : \Omega_{\mathcal{U}}$ can be *resized down* to $\Omega_{\mathcal{U}_0}$.

Although the resizing rules are convenient, it is not known whether type theory extended with them remains strongly normalizing. This is a long-standing open problem in univalent type theory. Because of this, the resizing rules are studied in the form of an axiom.

Definition 2.13.9 (*cf.* [dJE21, Definition 2] and [Uni13, Axiom 3.5.5])**.**

- The *propositional $(\mathcal{U}, \mathcal{V})$-resizing axiom* says that every proposition $P : \Omega_{\mathcal{U}}$ is $\mathcal{V}$-small.
- The *global propositional resizing* axiom says that, for every pair of universes $\mathcal{U}$

and $\mathcal{V}$, the propositional $(\mathcal{U},\mathcal{V})$-resizing axiom holds.
- The Ω-$(\mathcal{U},\mathcal{V})$-*resizing axiom* states that, for every pair of universes $\mathcal{U}$ and $\mathcal{V}$, the type $\Omega_{\mathcal{U}}$ is $\mathcal{V}$-small.
- The type $\Omega_{\mathcal{U}}^{\neg\neg}$ denotes the type of propositions $P : \Omega_{\mathcal{U}}$ satisfying $\neg\neg P \Rightarrow P$. The $\Omega_{\neg\neg}$-$(\mathcal{U},\mathcal{V})$-*resizing axiom* states that the type $\Omega_{\mathcal{U}}^{\neg\neg}$ is $\mathcal{V}$-small.

Remark 2.13.10. Notice that the law of excluded middle immediately gives Ω-resizing. The law of excluded middle, considered for a universe $\mathcal{U}$, amounts to the assertion that $\Omega_{\mathcal{U}}$ is equivalent to the type $2_{\mathcal{U}}$. Since the type 2 has a copy in every universe, this automatically allows us to resize $\Omega_{\mathcal{U}}$ as we wish. This is to say that, one cannot do predicative mathematics in a classical setting. To put it another way, *predicative mathematics is a stricter branch of constructive mathematics.*

Lemma 2.13.11. *For every pair of universes $\mathcal{U}$ and $\mathcal{V}$, we have that Ω-$(\mathcal{U},\mathcal{V})$-resizing implies the propositional $(\mathcal{U},\mathcal{V})$-resizing axiom. Furthermore, if it is the case that propositional $(\mathcal{V},\mathcal{U})$-resizing holds, we have that propositional $(\mathcal{U},\mathcal{V})$-resizing implies Ω-$(\mathcal{U},\mathcal{V}^+)$-resizing*

Proof. See [dJE21, Proposition 3] and [Esc19, §3.36.4]. □

In the form of axioms, all of the above are independent of, but consistent with univalent type theory. When we speak of "the propositional resizing axiom" without further specification, we will be referring to the global propositional resizing axiom from the above definition.

By *impredicative mathematics* in the context of univalent foundations, we mean mathematics developed using any of the above forms of resizing. At this point, we recall our discussion of axioms in type theory from Section 2.1.2: an axiom in type theory is an inhabitant of a type that is declared to exist independently of the type's introduction rules, which are ensured to have well-defined computational behaviour.

It is exactly at this point that the motivation for the *predicativity* of our foundational setting emerges. At the time of writing, finding computational rules for the propositional resizing axioms above is an open problem. We previously explained that our foundational setting is an axiomatization of a class of univalent type theories, as we are assuming that certain axioms can be fulfilled instead of taking them as postulates in a fixed, concrete implementation of a type theory. To obtain computational applications, we could always take a concrete implementation of a type theory fulfilling these axioms, such as Cubical Type Theory [CCHM18] in which all of the axioms we use can be given computational rules. This is owing to the fact that Cubical Type Theory [CCHM18] provides a computational interpretation for the univalence axiom as well as higher inductive types (at least for the ones we use here).

There is no known type theory (to the author's knowledge), at the time of writing, that gives a computational interpretation to the propositional resizing axioms. In accordance with our motivation to use only axioms whose computational behaviours are well-understood, we *refrain from using all forms* of the propositional resizing axioms above. We thus commit to predicative mathematics, as a further form of our commitment

in this thesis to constructive mathematics.

Propositional resizing and propositional truncations

Another motivation for considering the propositional resizing axioms is that they give an alternative way to construct propositional truncations. The idea here is that we can take the double negation of a type $(X \to \bot) \to \bot$, which will always be a proposition. This proposition can be thought of as an approximation to the propositional truncation of X. Unfortunately, however, it does not satisfy the universal property of propositional truncation unless the law of excluded middle holds [KECA17, §7].

By quantifying over all propositions, however, one can obtain a large type

$$\Pi_{(P:\Omega_{\mathcal{U}})} (X \to P) \to P, \tag{2.13.12}$$

which does give the propositional truncation of X, although it cannot be admitted as a proposition in the same universe as it involves a large quantification. Escardó [Esc19, §36.5] shows that, if propositional $(\mathcal{U}, \mathcal{U}_0)$-resizing holds for every universe $\mathcal{U}$, then the above construction satisfies the universal property of truncation in every universe.

For the purposes of our work, we will not need the details of the construction here. We mention this use of propositional resizing in passing as it is an illustrative example of how higher inductive types can circumvent the need for impredicativity by playing the rôle of resizing principles. We refer the reader [KECA17, §7] and [dJon23, §2.11.1] for a more detailed discussion.

2.14 The set replacement principle

We now consider a principle that will find important applications in our development of the theory of spectral and Stone locales in Chapter 4. Similar to how the higher inductive type of propositional truncation circumvents the need to work with the impredicative encoding from (2.13.12), the set replacement principle serves as a resizing principle that will allow us to avoid impredicativity at certain points of our development of locale theory.

Definition 2.14.1 (AXIOM: Set replacement [Rij22, Axiom 18.1.8])**.** The *set replacement principle* states that, for every function $f : X \to Y$, if we have that

- Y is a set,
- X is $\mathcal{U}$-small, and
- Y is locally $\mathcal{V}$-small,

then the type $\mathsf{image}(f)$ is $(\mathcal{U} \sqcup \mathcal{V})$-small.

It is useful to explicitly record the following special case.

Corollary 2.14.2. *Let $X : \mathcal{U}$ be a type and $Y : \mathcal{U}^+$, a set. For every function $f : X \to Y$, to show that $\mathsf{image}(f)$ is $\mathcal{U}$-small, it suffices to show that Y is locally $\mathcal{U}$-small.*

An inhabitant of the set replacement principle can be constructed as a higher inductive type (as shown by Rijke [Rij17]), and it is thus a theorem in a foundational setting where all higher inductive types are available (such as Cubical Type Theory [CCHM18]). As explained in Section 2.1.2, we will simply assume that our foundational setting satisfies this principle as we do not commit to a concrete implementation of univalent type theory.

We also note a relevant result due to de Jong [dJon23, Theorem 2.11.24], showing that the set replacement principle is logically equivalent to the existence of small set quotients, which is another construction requiring the use of higher inductive types. For us, de Jong's result means that the above axiom is satisfied in any foundational setting where small set quotients are available.

Chapter 3

Basics of point-free topology

As we explained in the introduction, point-free topology is an approach to topology where the fundamental object of study is taken to be a *locale* rather than a topological space. The latter is a set of *points* equipped with a *topology*, which is a class of subsets of the points that is required to be closed under finite intersection and arbitrary union. A locale, in contrast, is a notion of space that is characterized solely by an abstract lattice of opens. The lattice-theoretic structure abstracting the behaviour of the class of open sets is called a *frame*, and the remarkable insight that nontrivial topological questions can be entertained by studying frames as spaces can be traced back to the work of Stone [Sto34; Sto36].

The motivation for taking a point-free approach to topology is twofold:

1. Locales are a constructively better-behaved notion of space than topological spaces.
2. The category of locales satisfies some desirable properties that the category of topological spaces does not, even in a classical setting.

For our purposes, the primary motivation is (1) as we would like to develop topology in a constructive foundational setting. The prime example of locales being constructively better behaved is the localic form of the Tychonoff Theorem: its point-set form is equivalent to the axiom of choice [Kel50] whereas its localic form is constructively provable [Coq03; Vic05]. Furthermore, there are numerous theorems in classical mathematics characterizing when a locale can be represented as the frame of opens of a topological space (i.e. is *spatial*). These show that the class of spatial locales encompasses a variety of interesting spaces. However, a common feature of such theorems is that they turn out to be equivalent to various classical principles, meaning the point-free approach to such spaces is essential to their constructive treatment.

In this chapter, we develop the basics of locale theory in the setting of constructive and predicative univalent foundations, whose technical details we presented in Chapter 2. The organization of this chapter is as follows:

Section 3.1: We present the notions of frame and locale, and address the technical

details of their predicative development.

Section 3.2: We discuss sublocales.

Section 3.3: We discuss compactness and the way-below relation.

Section 3.4: We discuss the notion of clopen in point-free topology as well as the well-inside relation.

Section 3.5: We present various notions of bases for locales, which play a pivotal role in our development of the theory of spectral locales.

Section 3.6: We present predicatively well-behaved definitions of some interesting classes of locales, including spectral, Stone, zero-dimensional, and regular locales.

Section 3.7: We present a predicative form of the adjoint functor theorem for frames.

3.1 Definition of the notion of frame

Our type-theoretic definition of a frame, in its most general form, is parameterized by three universes:

1. the universe where the carrier set lives,
2. the universe where the truth values of the order live,
3. the universe for join-completeness i.e. for the index types of families on which the join operation is defined.

We adopt the convention of using the universe variables $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ for these respectively.

Definition 3.1.1 (Frame). A $(\mathcal{U},\mathcal{V},\mathcal{W})$-*frame* L consists of:

- a type $|L| : \mathcal{U}$,
- a partial order $(\text{-}) \leq (\text{-}) : |L| \to |L| \to \Omega_{\mathcal{V}}$,
- a top element $\mathbf{1} : |L|$,
- an operation $(\text{-}) \wedge (\text{-}) : |L| \to |L| \to |L|$ giving the greatest lower bound $x \wedge y$ of every pair of elements $x, y : |L|$,
- an operation $\bigvee(\text{-}) : (I \to |L|) \to |L|$ giving the least upper bound $\bigvee \alpha$ of every $\mathcal{W}$-family $\alpha : I \to |L|$,

such that binary meets distribute over arbitrary joins, i.e. $x \wedge \bigvee_{i:I} y_i = \bigvee_{i:I} x \wedge y_i$ for every $x : |L|$ and $\mathcal{W}$-family $(y_i)_{i:I}$ in $|L|$.

As we have done in the definition above, we adopt the shorthand notation $\bigvee_{i:I} x_i$ for the join of a family $(x_i)_{i:I}$. We also adopt the usual abuse of notation and write L instead of $|L|$.

We also note here that the universe generality involved in Definition 3.1.1 will not be needed, a point we will discuss in detail in Section 3.1.2. Nevertheless, it will be useful to have formulated the notion of frame at this level of generality as this will allow us to discuss *why* it will not be needed.

The reader might have observed that we have not imposed a sethood condition on the carrier type in this definition. The reason for this is that sethood follows

automatically from the antisymmetry of the partial order on the type.

Lemma 3.1.2. *Every type X equipped with a partial order is a set.*

Proof. Let X be a type equipped with a partial order $(\leq)$. By Lemma 2.6.9, it suffices to construct a family of constant endomaps on the identity types of X. Let $x, y : X$. We define a map $x =_X y \to x =_X y$ as the composition of the maps:

$$x =_X y \underset{g}{\overset{f}{\rightleftarrows}} (x \leq y) \times (y \leq x),$$

where f is the obvious map obtained by reflexivity and path induction on the equality proof, whereas the map g is exactly the antisymmetry condition. Since the order is proposition-valued, the type $(x \leq y) \times (y \leq x)$ is a proposition and hence the composite $g \circ f$ is a constant map. □

Our presentation of this proof follows the AGDA proof of Escardó [Esc12].

3.1.1 Local smallness of frames

In Definition 2.13.2, we presented the notion of a locally small type. In the presence of a semilattice structure, the local smallness of a type can be characterized in terms of the order as follows:

Lemma 3.1.3 ( ). *For every $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame L, every pair of elements $x, y : L$, and every universe $\mathcal{T}$, we have that $x \leq y$ is $\mathcal{T}$-small if and only if the carrier of L is a locally $\mathcal{T}$-small type.*

Proof. Let $x, y : L$ and let $\mathcal{T}$ be a universe. It is a standard fact of lattice theory that $x \leq y \leftrightarrow x \wedge y = x$.

($\Rightarrow$) If the frame is locally $\mathcal{T}$-small in the sense that $x \leq y$ is $\mathcal{T}$-small for every two $x, y : L$, then the identity type $x = y$ must also be $\mathcal{T}$-small since it is equivalent to the conjunction $(x \leq y) \times (y \leq x)$.

($\Leftarrow$) Conversely, if the carrier of L is a locally $\mathcal{T}$-small type, then $x \leq y$ must be $\mathcal{T}$-small since it is equivalent to the identity type $x \wedge y = x$. □

Remark 3.1.4. As we will discuss in detail in Section 3.1.2, we will always be interested in *large and small-complete frames*, that is, frames where the join-completeness universe is below the carrier universe. The lemma above is best understood in this special case, where it says that the carrier is locally small if and only if the order has small truth values.

3.1.2 Primer on predicative lattice theory

The preliminary definition of the notion of frame from Definition 3.1.1 is highly general; all three universes involved in the definition are permitted to be distinct. It is

worth considering whether this level of generality is sensible. In order to address this question in full detail, we now provide a brief summary of de Jong and Escardó's investigation [dJE21; dJE23; dJon23] of predicative aspects of lattice theory in univalent foundations.

Frames may be viewed as certain cocomplete (-1)-*categories* (also called *posetal* or *thin* categories). Under this view, a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame is a category whose objects live in universe $\mathcal{U}$, whose (trivial) Hom-sets live in universe $\mathcal{V}$, and whose colimits are over $\mathcal{W}$-diagrams. In category theory, there is a well-known (classical) result due to Freyd [Fre64, Chapter 3, Exercise D] stating that every complete small category is a posetal. This is a so-called no-go theorem showing that the study of non-posetal complete categories *must be* limited to those that are large and small-complete.

Remark 3.1.5. We note here that we will be using the term *complete lattice* instead of *cocomplete lattice*. Although the latter is technically more accurate, it is nonstandard and somewhat confusing in the context of lattice of theory.

Although complete, small lattices can be constructed impredicatively, there are analogous no-go theorems for them in a predicative foundational setting. It can be shown that such lattices cannot be constructed *without* using some form of impredicativity. The first result of this form is due to Curi [Cur10a; Cur10b], who showed that the existence of a complete, small lattice is not provable in Aczel and Rathjen's Constructive[1] ZF set theory [AR10], which is a predicative system.

de Jong and Escardó [dJE21; dJE23; dJon23] proved an analogue of Curi's result in the setting of univalent foundations. Their result is more direct and is in the style of reverse constructive mathematics: they give a *resizing taboo*, showing directly that the existence of a nontrivial complete and small lattice implies a form of propositional resizing. For the sake of self-containment, we now provide a high-level overview of this result, and for the technical details we refer the reader to [dJon23, §6] and [dJE23, §4].

The aforementioned resizing taboo applies to a class of preorders much more general than join-complete lattices.

Definition 3.1.6 (Directed family)**.** Let X be a type equipped with a preorder. A family $(x_i)_{i:I}$ is called *directed* if

- the index type I is inhabited, and
- for every pair of indices $i, j : I$, there is some $k : I$ such that x_k is an upper bound of x_i and x_j.

Definition 3.1.7 ([dJE23, Examples 4.3])**.** A $(\mathcal{U}, \mathcal{V})$-preorder X is called

- $\mathcal{W}$*-sup-complete* if it has joins of $\mathcal{W}$-families,
- $\mathcal{W}$*-bounded-complete* if it has joins of bounded $\mathcal{W}$-families,
- $\mathcal{W}$*-directed-complete* if it has joins of directed $\mathcal{W}$-families (i.e. all "directed

[1] Note that in the context of set theory, the term *constructive* corresponds to what we call *constructive and predicative*. The term *intuitionistic* is used for what we call constructive i.e. mathematics without the use of classical principles.

joins").

When formulated in terms of $\mathcal{W}$-sup-complete lattices above, a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame is a $(\mathcal{U}, \mathcal{V})$-preorder that is (1) $\mathcal{W}$-sup-complete, (2) has finite meets, and (3) in which the finite meets distribute over the $\mathcal{W}$-joins.

Remark 3.1.8. Note that de Jong [dJon23] uses the notation $\mathcal{W}$-DCPO$_{\mathcal{U},\mathcal{V}}$ (see [dJon23, Definition 3.2.11]) for what we called a $\mathcal{W}$-directed-complete $(\mathcal{U}, \mathcal{V})$-preorder above. Furthermore, the order and the carrier universes are often left implicit so that one simply says $\mathcal{W}$-DCPO. We will also leave these two universes implicit, and will hence speak of $\mathcal{W}$-sup-complete preorders.

Remark 3.1.9. The term *complete small lattice* is not to be confused with *small-complete lattice*. To avoid possible sources of confusion, we always use the hyphenated form for the latter.

Each of the completeness conditions from Definition 3.1.7 implies a useful technical notion of completeness called $\delta_{\mathcal{W}}$-completeness:

Definition 3.1.10 ($\delta_{\mathcal{W}}$-complete poset [dJon23, Definition 6.2.1])**.** A poset X is called *$\delta_{\mathcal{W}}$-complete*, for some universe $\mathcal{W}$, if the family $\delta_{x,y,P} : \mathbf{1} + P \to X$, defined as

$$\delta_{x,y,P}(\mathsf{inl}(\star)) :\equiv x$$
$$\delta_{x,y,P}(\mathsf{inr}(p)) :\equiv y,$$

admits a join in X, for every $x \leq y$ and every $\mathcal{W}$-proposition P.

The AGDA formalization of the above notion is due to TYPETOPOLOGY contributor Ian Ray.

de Jong and Escardó also use a positive reformulation of a poset being nontrivial, which they define through a positive formulation of the strictly less than relation. They call this *strictly below* (see [dJE23, Definition 4.8] for the definition).

- A poset X is *nontrivial* [dJE23, Definition 4.4] if there are specified $x, y : X$ such that $x \neq y$ and $x \leq y$.
- A $\delta_{\mathcal{W}}$-complete poset X is *positive* [dJE23, Definition 4.13] if there are specified $x, y : X$ with x *strictly below* y.

Note that the above notion of positivity is formulated for $\delta_{\mathcal{W}}$-complete posets. The resizing taboo of interest is then formulated as below.

Theorem 3.1.11 ([dJE23, Theorem 4.23])**.**

1. *There is a nontrivial small $\delta_{\mathcal{W}}$-complete poset if and only if $\Omega_{\neg\neg}$-resizing holds in universe $\mathcal{W}$.*
2. *There is a positive small $\delta_{\mathcal{W}}$-complete poset if and only if Ω-resizing holds in universe $\mathcal{W}$.*

Because $\delta_{\mathcal{W}}$-completeness is implied by the completeness notions introduced above, we have the following corollary:

Corollary 3.1.12.

1. *There is a nontrivial, small $\mathcal{U}$-sup-lattice, $\mathcal{U}$-dcpo, or $\mathcal{U}$-bounded-complete lattice if and only if $\Omega_{\neg\neg}$-resizing holds in universe $\mathcal{U}$.*
2. *There is a positive, small $\mathcal{U}$-sup-lattice, $\mathcal{U}$-dcpo, or $\mathcal{U}$-bounded-complete lattice if and only if Ω-resizing holds in universe $\mathcal{U}$*

This implies that *there are no nontrivial small frames* unless a form of resizing holds:

Corollary 3.1.13. *There is a positive (resp. nontrivial) small frame if and only if Ω-resizing (resp. $\Omega_{\neg\neg}$-resizing) holds.*

3.1.3 Categories of frames and locales

Corollary 3.1.13 implies that attention *must be* restricted to large and small-complete frames in a predicative foundational setting. Furthermore, we know that a universe larger than $\mathcal{U}^+$ is never needed for the order, since Lemma 3.1.3 establishes that the order can always be resized to $\mathcal{U}^+$. For the purposes of our work, it is in fact sufficient to consider only those frames that are locally small (i.e. which have $\mathcal{U}$-small values) as this class encompasses most frames that are encountered in practice. In light of these, we hereby restrict attention to $(\mathcal{U}^+,\mathcal{U},\mathcal{U})$-frames, which we refer to as *large, locally small, and small-complete* frames.

Convention. We fix a *base* universe $\mathcal{U}$, and refer to types that have equivalent copies in $\mathcal{U}$ as *small types*. In contrast, we refer to types in $\mathcal{U}^+$ as *large types*. Accordingly,

$$\textit{we hereafter take frame/locale to mean } (\mathcal{U}^+,\mathcal{U},\mathcal{U})\textit{-frame/locale},$$

unless otherwise specified.

Remark 3.1.14. Even though large, locally small, and small-complete frames are sufficient for the work presented in this thesis, there do exist large and small-complete frames that cannot be locally small unless a form of resizing holds. We give a new resizing taboo showing this in Section 3.5.4.

Having established the above convention, we are now ready to define the category of frames, and thereby the category of locales.

Definition 3.1.15 (Frame homomorphism)**.** Let K and L be two frames. A function $h : |K| \to |L|$ is called a *frame homomorphism* if it preserves the top element, binary meets, and joins of small families. We denote by $\mathbf{Frm}_{\mathcal{U}}$ the category of frames and their homomorphisms (over our base universe $\mathcal{U}$). We often drop the subscript $\mathcal{U}$ when the base universe is clear from context.

A *locale* is a frame considered in the opposite category, denoted $\mathbf{Loc}_{\mathcal{U}} :\equiv \mathbf{Frm}_{\mathcal{U}}{}^{\mathrm{op}}$. We adopt the notational conventions of Mac Lane and Moerdijk [MM94] and Johnstone [Joh02, Definition 1.2.1]; to highlight that we view the category of locales as a category of spaces, we

- use the letters $X, Y, Z, \ldots$ (or sometimes $A, B, C, \ldots$) for locales,
- denote the frame corresponding to a locale X by $\mathcal{O}(X)$.

For variables that range over the frame of opens of a locale X, we use the letters $U, V, W, \ldots$ We use the letters f and g for continuous maps $X \to Y$ of locales. A *continuous map* $f : X \to Y$ is a frame homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. A *homeomorphism* of locales is a frame isomorphism between their defining frames.

3.1.4 Impredicative vs. predicative locale theory

The traditional development of locale theory takes place in an impredicative foundational setting, where it is possible to construct nontrivial complete, small lattices. In fact, the standard practice in an impredicative setting is to leverage this capability as to develop the theory by working *only* with complete small lattices. Using the propositional resizing axiom, one achieves this as follows:

1. Consider a complete small frame L in universe $\mathcal{U}$.
2. For every subset $P : L \to \Omega_{\mathcal{U}}$, the equivalence from Lemma 2.13.8 is used, leveraging the smallness of $|L|$, to take the corresponding $\mathcal{U}$-family whose join can then be computed in L.
3. For more general subsets $P : L \to \Omega_{\mathcal{V}}$, a copy $P' : L \to \Omega_{\mathcal{U}}$ can always be obtained by further appeal to propositional resizing.

Due to the availability of the above method in the presence of propositional resizing, the traditional development of locale theory focuses on complete small frames.

As a result of this focus on complete small frames, many fundamental constructions in the traditional development of the subject are applicable only within the context of complete small frames. An example of this is the construction of *Heyting implications* in frames, which is crucial to many vital concepts of locale theory, including fixsets [Joh02, Proposition 1.1.13.(iv)], open sublocales [Joh02, Examples 1.1.16.(a)], the double-negation nucleus [Joh02, Examples 1.1.16.(c)], among others.

Heyting implications are constructed for complete, small lattices as

$$x \Rightarrow y \quad :\equiv \quad \bigvee \{z \in L \mid z \wedge x \leq y\},$$

which involves a quantification over the entirety of the carrier set. When the frame in consideration has a large carrier set, this family is large, and its join is hence not available in our predicative setting.

Similarly, arbitrary meets can also be constructed from arbitrary joins when working with complete, small lattices, by defining the meet of a family to the be join of the set of lower bounds of the family in consideration. Unfortunately, however, the collection of lower bounds of a family is a large one, meaning this join does not exist in general in a predicative setting. Because of this, the meaning of the term *complete lattice* in a predicative setting is not quite the same as its meaning in impredicative foundations:

meet-complete and join-complete lattice mean different things. Many constructions of locale theory rely on the existence of arbitrary meets, none of which will be a priori available in our predicative foundational setting.

Our solution for such problems will be to restrict attention to locales with *small* bases, drawing inspiration from the literature on locally presentable categories [AR94]. We will also show that certain rich classes of locales, such as the category of spectral locales, admit small bases by default. We return to this issue in Sections 3.5 and 3.7.

3.1.5 Basic examples of locales

We now look at three basic examples of locales:

1. the terminal locale $\mathbf{1}_{\mathcal{U}}$ (of the category $\mathbf{Loc}_{\mathcal{U}}$),
2. the discrete locale over a (small) set X,
3. the Sierpiński locale, which is the free frame on one generator.

The terminal locale

The *terminal locale* (denoted $\mathbf{1}_{\mathcal{U}}$) is the locale defined by the initial frame $\Omega_{\mathcal{U}}$:

Example 3.1.16 (Initial frame)**.** For every universe $\mathcal{U}$, the (large) type $\Omega_{\mathcal{U}}$ forms a poset under the order defined by implication. This poset is locally small since each proposition $P \Rightarrow Q$ is small. Note that the antisymmetry of this order is exactly propositional extensionality. Furthermore, this poset forms a frame: the top element is the true proposition $\top_{\mathcal{U}}$, the meet operation is given by the conjunction of two propositions, and the join of a family of propositions $(P_i)_{i:I}$ is defined as

$$\bigvee_{i:I} P_i :\equiv \exists_{(i:I)} P_i,$$

which is a small proposition for every small family of propositions.

The fact that these operations define meet and join operations is easy to see. For the distributivity law, it is easily checked that $(P \wedge \exists_{(i:I)} Q_i) \leftrightarrow \exists_{(i:I)}(P \wedge Q_i)$, for every $P : \Omega_{\mathcal{U}}$ and every family $(Q_i)_{i:I}$, using the recursion principle of propositional truncation.

Before proving the universal property of the initial frame, we first establish the following lemma:

Lemma 3.1.17 ( )**.** *Let $\beta : 2_{\mathcal{U}} \to \Omega_{\mathcal{U}}$ be the function defined as $\beta(0) :\equiv \bot$, $\beta(1) :\equiv \top$. Every $P : \Omega_{\mathcal{U}}$ is equal to*

1. *the join $\bigvee_{p:P} \top_{\mathcal{U}}$, and*
2. *the join of the directed family $\{\beta(b) \mid b : 2, \beta(b) \leq P\}$, which is formally given by the map $(\Sigma_{(b:2_{\mathcal{U}})} \beta(b) \leq P) \xrightarrow{\beta \circ \mathrm{pr}_1} \Omega_{\mathcal{U}}$.*

Proof. (1) is easy to see since $P = \top$ if and only if P holds. For (2), we show that

$$P = \bigvee \{\beta(b) \mid b : \mathbf{2}, \beta(b) \leq P\}$$

by antisymmetry. If P holds then $P = \top$ and hence $\beta(1) \equiv \top \leq P$. For the other direction, observe that P is an upper bound of the family since $\bot \equiv \beta(0)$ is trivially below P and $\top \leq P$ implies that P holds. The family $\{\beta(b) \mid b : \mathbf{2}, \beta(b) \leq P\}$ is directed since it is always inhabited by $\beta(0)$, and for every pair of Booleans $b_1, b_2 : \mathbf{2}$ with $\beta(b_1) \leq P$ and $\beta(b_2) \leq P$, the disjunction $b_1 \vee b_2$ is either b_1 or b_2. □

Lemma 3.1.18 (). *The frame $\Omega_{\mathcal{U}}$ is the initial object in the category $\mathbf{Frm}_{\mathcal{U}}$.*

Proof. Let L be a frame. By Lemma 3.1.17, $P : \Omega_{\mathcal{U}}$ can be expressed as the small join $\bigvee_{p:P} \top$. This implies that every frame homomorphism $h : \Omega_{\mathcal{U}} \to L$ is necessarily unique as it must satisfy the equality:

$$h(P) = h\left(\bigvee_{p:P} \top\right) = \bigvee_{p:P} h(\top) = \bigvee_{p:P} \mathbf{1}_L.$$

We accordingly define the unique map $\Omega_{\mathcal{U}} \to L$ as above. It is easy to see that this defines a frame homomorphism. □

Remark 3.1.19. The universes in the above proof can be generalized, but as explained in Section 3.1.2, we focus only on the category $\mathbf{Frm}_{\mathcal{U}}$ in this thesis. The reader interested in the generalization of universes is referred to the AGDA formalization.

The discrete locale

The *discrete locale* over a small type $X : \mathcal{U}$ is the locale defined by the powerset frame $\mathcal{P}_{\mathcal{U}}(X)$, which we define below.

Example 3.1.20 (Powerset frame)**.** Let $X : \mathcal{U}$ be a small type. The *powerset* $X \to \Omega_{\mathcal{U}}$ is a poset under the order defined by subset inclusion, assuming propositional and function extensionality. This poset is locally small thanks to the smallness of X, since it implies that the quantification over X is small. Moreover, this poset forms a frame:

- the top open is given by the full subset $x \mapsto \top_{\mathcal{U}}$,
- the meet of two subsets $S, T : X \to \Omega_{\mathcal{U}}$ is given by set intersection, defined as

$$S \cap T :\equiv x \mapsto x \in S \wedge x \in T,$$

- the join of a family of subsets $(S_i)_{i:I}$ is given by set union, defined as

$$\left(\bigvee_{i:I} S_i\right) :\equiv x \mapsto \exists_{(i:I)} x \in S_i,$$

  which is a small subset for every small family of subsets.

We will often write X for the full subset $x \mapsto \top$.

As we mentioned in the above example, the smallness of type X is crucial for the local smallness of the frame $\mathcal{P}_{\mathcal{U}}(X)$, since the inclusion order $S \subseteq T :\equiv \Pi_{(x:X)}\, x \in S \Rightarrow x \in T$ is, in general, not small without the use of a form of resizing. The resizing taboo below proves this.

Lemma 3.1.21. *If the powerset lattice over a large proposition P is locally small, then the negation $\neg P$ is a small proposition.*

Proof. Let $P : \Omega_{\mathcal{U}^+}$. Recall that in a powerset frame $\mathcal{P}(X)$, we denote the top subset $x \mapsto \top$ by X, which implies that we denote the top subset of $\mathcal{P}(P)$ by P itself. We have that $P \subseteq \emptyset$ if and only if $\neg P$ holds. Assuming the local smallness of the lattice $\mathcal{P}(P)$, the type

$$P \subseteq \emptyset$$

is small, and hence so is $\neg P$. □

If $\neg P$ is small, then obviously so is $\neg\neg P$ meaning we have:

Corollary 3.1.22. *If the powerset frame over every large proposition is locally small, then we have $\Omega_{\neg\neg}$-resizing from $\mathcal{U}^+$ to $\mathcal{U}$, for every universe $\mathcal{U}$.*

Given a large type $X : \mathcal{U}^+$, note that the local smallness of the powerset lattice $\mathcal{P}_{\mathcal{U}}(X)$ is a consequence of propositional resizing, since the inclusion $S \subseteq T$ is always a proposition. Whether the local smallness of this lattice is *equivalent* to a well-known form of resizing is unclear to the author at the time of writing.

We will look at the Sierpiński locale next, which is an example of a frame that is locally small and small-complete *despite* being defined on a large type.

The Sierpiński locale

Example 3.1.23 (Sierpiński frame)**.** A subset $S : \Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}}$ of $\mathcal{U}$-propositions is called *Scott open* if it preserves joins of directed families of propositions. The set of Scott open subsets forms a poset under the inclusion order, which can be shown to be locally small. This poset forms a small-complete frame and the locale it defines is called the *Sierpiński locale*.

Proof. It is easy to see that the full subset is Scott open. Given Scott continuous subsets $S, T : \Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}}$, to see that the intersection $S \cap T$ is Scott continuous,

consider a directed family of propositions $(P_i)_{i:I}$. We have

$$\begin{aligned}
(S \cap T)\left(\bigvee_{i:I} P_i\right) &= \left(\bigvee_{i:I} S(P_i)\right) \wedge \left(\bigvee_{i:I} T(P_i)\right) && \text{(Scott-continuity)} \\
&= \bigvee_{i,j:I} S(P_i) \wedge T(P_j) \\
&= \bigvee_{i:I} S(P_i) \wedge T(P_i) && (\dagger)
\end{aligned}$$

For the step (†), the $\geq$ direction is obvious. The $\leq$ direction follows from the directedness of the families $(S(P_i))_{i:I}$ and $(T(P_i))_{i:I}$.

To see that (small) unions of Scott open subsets are Scott open, let $(S_i)_{i:I}$ be a family of Scott open subsets and let $(P_j)_{j:J}$ be a directed family of propositions.

$$\begin{aligned}
\left(\bigvee_{i:I} S_i\right)\left(\bigvee_{j:J} P_j\right) &= \exists_{(i:I)} S_i\left(\bigvee_{j:J} P_j\right) \\
&= \exists_{(i:I)}\left(\bigvee_{j:J} S_i(P_j)\right) && \text{(Scott-continuity of } S_i\text{)} \\
&= \exists_{(i:I)}\exists_{(j:J)} S_i(P_j) \\
&= \exists_{(j:J)}\exists_{(i:I)} S_i(P_j) \\
&= \bigvee_{j:J}\left(\left(\bigvee_{i:I} S_i\right)(P_j)\right)
\end{aligned}$$

We also need to show that the pointwise order $\Pi_{(P:\Omega_\mathcal{U})} S(P) \Rightarrow T(P)$ is $\mathcal{U}$-small, for every pair of Scott open subsets S and T. We define a small version of the order above as $S \subseteq_s T :\equiv \Pi_{(P:\Omega_\mathcal{U})} S(P) \Rightarrow T(P)$, and show that $S \subseteq_s T$ is logically equivalent to $S \subseteq T$. The nontrivial direction is $S \subseteq_s T \Rightarrow S \subseteq T$. Assume $S \subseteq_s T$ and let $P : \Omega_\mathcal{U}$. By Lemma 3.1.17, we know that P can be expressed as the directed join $\bigvee \{\beta(b) \mid b : \mathbf{2}_\mathcal{U}, \beta(b) \leq P\}$. The Scott openness of S and T implies that

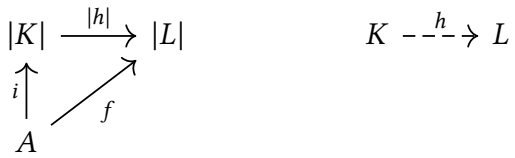
$$\begin{aligned}
S\left(\bigvee \{\beta(b) \mid b : \mathbf{2}_\mathcal{U}, \beta(b) \leq P\}\right) &= \bigvee \{S(\beta(b)) \mid b : \mathbf{2}_\mathcal{U}, \beta(b) \leq P\} \\
&\leq \bigvee \{T(\beta(b)) \mid b : \mathbf{2}_\mathcal{U}, \beta(b) \leq P\} \\
&= T\left(\bigvee \{\beta(b) \mid b : \mathbf{2}_\mathcal{U}, \beta(b) \leq P\}\right).
\end{aligned}$$

□

We denote the Sierpiński locale at universe $\mathcal{U}$ by $\mathbb{S}_\mathcal{U}$. If the universe is clear from context, drop the subscript and write $\mathbb{S}$.

Given a small set A, a frame K is said to be *freely generated* by A if there is a function $i : A \to |K|$ such that, for every frame L and every function $f : A \to |L|$, there is a unique frame homomorphism $h : K \to L$ satisfying $f = |h| \circ i$, as captured by the

diagram:



The universal property of the Sierpiński locale $\mathbb{S}$ is that its frame $\mathcal{O}(\mathbb{S})$ is freely generated by the unit type $\mathbf{1}$. Since functions $\mathbf{1} \to |L|$ are exactly the opens of frame L, this universal property can be simplified as given below.

Definition 3.1.24 (Universal property of Sierpiński)**.** A locale X has the *universal property of Sierpiński* if it has a specified open U satisfying the property that, for every locale Y and every open $V : \mathcal{O}(Y)$, there is a map $f : Y \to X$ satisfying $V = f^*(U)$.

Proposition 3.1.25 (🤌)**.** *The Sierpiński frame, which we constructed in Example 3.1.23, satisfies the above universal property. The specified open is given by the identity map.*

Conceptually, the universal property above states that the Sierpiński locale *classifies the opens* of every locale. We postpone the proof that our construction satisfies this property until Section 6.2.6, as the proof requires some technical preparation.

The frame of ¬¬-stable propositions

Example 3.1.26. We denote by $\Omega_{\mathcal{U}}^{\neg\neg}$ the type of propositions $P : \Omega_{\mathcal{U}}$ satisfying $\neg\neg P \Rightarrow P$, which are called the $\neg\neg$-*stable* propositions. These form a frame when ordered under implication, with meets given by conjunction as in Example 3.1.16. The join, however, is different from the one in Example 3.1.16. Given a family $(P_i)_{i:I}$ of $\neg\neg$-stable propositions, its join in $\Omega_{\mathcal{U}}^{\neg\neg}$ is defined to be

$$\neg\neg\left(\bigvee_{i:I} P_i\right) \equiv \neg\neg\exists_{(i:I)} P_i.$$

We denote by $\mathbf{1}_{\mathcal{U}}^{\neg\neg}$ the locale defined by the above frame.

The above frame can be described more naturally as a *sublocale* of the terminal locale, namely the one given by the double-negation nucleus [Joh82, §II.2.4]:

$$\neg\neg : \Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}}.$$

Unfortunately, we have to postpone this description at this point, as we have not presented the notions of sublocale and nuclei yet. We will address this in the next section.

3.2 Sublocales

There are several equivalent ways to represent the notion of *sublocale* in the traditional development of locale theory [Joh02, Proposition 1.1.13]:

1. Regular frame quotients of $\mathcal{O}(X)$ i.e. isomorphism classes of surjective frame homomorphisms into $\mathcal{O}(X)$.
2. Frame congruences on $\mathcal{O}(X)$.
3. Nuclei on $\mathcal{O}(X)$.
4. Fixsets of $\mathcal{O}(X)$, which are subsets that are (1) closed under arbitrary meets and (2) form exponential ideals with respect to Heyting implication.

Some of the notions of sublocale above are not predicatively available. For example, it is not clear how fixsets would be defined in our setting since they presuppose the existence of arbitrary meets and Heyting implications in the frame in consideration, neither of which is predicatively available as we explained in Section 3.1.4.

For the purposes of this thesis, we will take nuclei as the primary notion of sublocale.

Definition 3.2.1 (Nucleus)**.** A *nucleus on a locale* X is an endofunction $j : \mathcal{O}(X) \to \mathcal{O}(X)$ that is inflationary, preserves binary meets, and is idempotent.

We define $\mathsf{Nucleus}(X) :\equiv \Sigma_{(j:\mathcal{O}(X)\to\mathcal{O}(X))}\mathsf{is\text{-}nucleus}(j)$. Furthermore, we adopt the standard convention of using the letters $j, k, l \ldots$ for variables ranging over nuclei.

Example 3.2.2 (Closed nucleus)**.** For every locale X and every open $U : \mathcal{O}(X)$, the map

$$\mathbf{c}(U) :\equiv V \mapsto U \vee V$$

is a nucleus. It is inflationary, since $V \leq \mathbf{c}(U)(V) \equiv U \vee V$, for every pair of opens $U, V : \mathcal{O}(X)$. It is also idempotent since

$$\mathbf{c}(U)(\mathbf{c}(U)(V)) \equiv U \vee (U \vee V) \equiv U \vee V.$$

The preservation of binary meets follows from dual distributivity.

Lemma 3.2.3 ([Tos20, Proposition 3.27])**.** *For every locale X and nucleus $j : \mathcal{O}(X) \to \mathcal{O}(X)$, the type*

$$\Sigma_{(U:\mathcal{O}(X))}(U = j(U))$$

of fixed points of j forms a frame, the locale defined by which is denoted X_j. This satisfies the property that the inclusion $\mathcal{O}(X_j) \hookrightarrow \mathcal{O}(X)$ is the right adjoint of the map

$$\begin{aligned} \mathcal{O}(X) &\to \mathcal{O}(X_j) \\ U &\mapsto j(U) \end{aligned}$$

Lemma 3.2.4. *For every nucleus $j : \mathcal{O}(X) \to \mathcal{O}(X)$, the type $\Sigma_{(U:\mathcal{O}(X))}(U = j(U))$ is equivalent to the image of j.*

Proof. Standard fact of locale theory. See, for example, [Joh82, p. 49]. □

Definition 3.2.5. A locale Y is called a *sublocale* of locale X if there is some nucleus $j : \mathcal{O}(X) \to \mathcal{O}(X)$ such that Y is homeomorphic to X_j.

Conceptually, the notion of nucleus axiomatizes the monad arising from the adjunction of a frame quotient $q : \mathcal{O}(X) \twoheadrightarrow \mathcal{O}(Y)$ with its right adjoint $q_* : \mathcal{O}(Y) \to \mathcal{O}(X)$. Defining $j_q :\equiv q_* \circ q$, the j_q-fixed opens are the representatives of the equivalence class given by the quotient $q : \mathcal{O}(X) \twoheadrightarrow \mathcal{O}(Y)$.

Remark 3.2.6. We will also work with inflationary and binary-meet-preserving functions that are not necessarily idempotent. Such functions are called *prenuclei*. We also note that, to show a prenucleus j to be idempotent, it suffices to show $j(j(U)) \leq j(U)$ as the other direction follows from inflationarity. In fact, the notion of a nucleus could be defined as a prenucleus satisfying the inequality $j(j(U)) \leq j(U)$, but we define it as in Definition 3.2.1 for the sake of simplicity and make implicit use of this fact in our proofs of idempotency.

Lemma 3.2.7. *Every prenucleus is a monotone map.*

Proof. It is a well-known fact of lattice theory that meet preservation implies monotonicity [DP02, Lemma 2.8]. □

3.3 Compactness and the way-below relation

3.3.1 Compact opens

Compactness is a fundamental topological property. In point-set topology, a subspace $Y \subseteq X$ of a space is *compact* if every open cover of Y has a finite subcover. As this is a property involving only the open subsets of the space X, it can be readily translated into the point-free setting to define the notion of compact open in a locale. Our type-theoretic definition below closely follows Johnstone [Joh82, Paragraph II.3.1], except we refine it by working with Kuratowski finite families.

We first do some preparation.

Lemma 3.3.1 (🖉). *For every directed family of opens $(U_i)_{i:I}$, every Kuratowski finite set of elements of the family has an upper bound in the family.*

Proof. Let $(U_i)_{i:I}$ be a directed family of opens. We show by Kuratowski finite subset induction (Lemma 2.12.6) that every Kuratowski finite set of elements of the family $(U_i)_{i:I}$ has an upper bound in the family.

Base case (1): the empty subset. The family is inhabited by the assumption of directedness and any member of the family is vacuously an upper bound of the empty set.

Base case (2): the singleton subset. We need to show that the singleton subset $\{U_i\}$ has an upper bound in the family $(U_i)_{i:I}$, for every $i : I$. This is immediate since

every open U_i is in the family by definition and is an upper bound of the singleton containing itself.

Induction step: binary unions. Let F_1 and F_2 be two subsets of elements of the family and assume by induction that they have upper bounds U_1, U_2 in the family. We need to show that the union $F_1 \cup F_2$ has an upper bound in the family $(U_i)_{i:I}$. By the directedness of the family, there must be some U_3 in the family that is above both U_1 and U_2. This clearly implies that U_3 is an upper bound of the subset $F_1 \cup F_2$. □

Definition 3.3.2. Let X be a locale and consider a family $(U_i)_{i:I}$ of opens. The family $U^{\uparrow} : \mathsf{List}(I) \to \mathcal{O}(X)$, defined as

$$U^{\uparrow}(i_0, \ldots, i_{n-1}) :\equiv U_{i_0} \vee \cdots \vee U_{i_{n-1}},$$

is called the *closure of* $(U_i)_{i:I}$ *under finite joins.*

Lemma 3.3.3 (*cf.* [Joh82, §II.3.1])**.** *For every open U of a locale X, the following are equivalent:*

1. *For every family of opens $(V_i)_{i:I}$ with $U \leq \bigvee_{i:I} V_i$, there is a Kuratowski finite subfamily $(V_{i_j})_{j:J}$ such that $U \leq \bigvee_{j:J} V_{i_j}$.*
2. *For every family of opens $(V_i)_{i:I}$ with $U = \bigvee_{i:I} V_i$, there is a Kuratowski finite subfamily $(V_{i_j})_{j:J}$ such that $U = \bigvee_{j:J} V_{i_j}$.*
3. *For every directed family $(V_i)_{i:I}$ with $U \leq \bigvee_{i:I} V_i$, there is an index $i : I$ such that $U \leq V_i$.*

Proof. We start with the equivalence (1) ⇔ (2). For the forward implication, suppose some open U satisfies Condition (1) and let $(V_i)_{i:I}$ be a family of opens with $U = \bigvee_{i:I} V_i$. It must be the case that there is a Kuratowski finite subfamily $(V_{i_j})_{j:J}$ such that $U \leq \bigvee_{j:J} V_{i_j}$. It remains to show that $\bigvee_{j:J} V_{i_j} \leq U$, but we already have $\bigvee_{j:J} V_{i_j} \leq \bigvee_{i:I} V_i = U$. For the backward implication, let U be an open satisfying Condition (2) and consider a family of opens $(V_i)_{i:I}$ with $U \leq \bigvee_{i:I} V_i$. This means

$$U = U \wedge \bigvee_{i:I} V_i,$$

so by frame distributivity we have

$$U = \bigvee_{i:I} U \wedge V_i.$$

By the assumption of Condition (2), this means that there is a finite subcover

$$U = \bigvee_{j:J} U \wedge V_{i_j}.$$

This implies, again by distributivity, that $U = U \wedge \bigvee_{j:J} V_{i_j}$, which holds if and only if $U \leq \bigvee_{j:J} V_{i_j}$ so we are done.

For the equivalence (3) ⇔ (1), we first address the implication (3) ⇒ (1). Let U be an open satisfying (3) and let $(V_i)_{i:I}$ be a family with $U \leq \bigvee_{i:I} V_i$. We consider the closure of $(V_i)_{i:I}$ under finite joins, as defined above. Because directification preserves the join, we have that $U \leq \bigvee_{s:\mathsf{List}(I)} V_s^{\uparrow}$, and we therefore know, by the assumption of Condition (3), that there exists a list of indices $i_0, \ldots, i_{n-1}$ satisfying $U \leq V_{i_0} \vee \cdots \vee V_{i_{n-1}}$. Membership in the list $V_0, \ldots, V_{i_{n-1}}$ defines a Kuratowski finite subfamily of $(V_i)_{i:I}$, an upper bound of which is in the family and is above U. For the backward implication, let U be an open satisfying Condition (1) and consider a directed family $(V_i)_{i:I}$ such that $U \leq \bigvee_{i:I} V_i$. There must be a Kuratowski finite subfamily $(V_{i_j})_{j:J}$ with $U \leq \bigvee_{j:J} V_{i_j}$. By Lemma 3.3.1, this subfamily must have an upper bound V_k in $(V_i)_{i:I}$, meaning we have

$$U \leq \bigvee_{j:J} V_{i_j} \leq V_k.$$

□

Definition 3.3.4 (Compactness)**.**

- An open $U : \mathcal{O}(X)$ is called *compact* if it satisfies the equivalent conditions from Lemma 3.3.3.
- A locale X is called *compact* if its top open $\mathbf{1}_X$ is compact.

In the formalization, we take Condition (3) of Lemma 3.3.3 to be our primary definition of compactness.

Example 3.3.5. The terminal locale $\mathbf{1}_{\mathcal{U}}$ is compact. It follows directly from the definition of the join that, for every family of $\mathcal{U}$-propositions $(P_i)_{i:I}$, we have that $\top \leq \bigvee_{i:I} P_i$ if and only if some P_i holds i.e. $\top \leq P_i$ for some $i : I$.

Example 3.3.6. The discrete locale on $\mathbb{N}$ is **not** compact, since this would mean by Lemma 3.3.3 that $\mathbb{N}$ is a Kuratowski finite type.

Example 3.3.7. The Sierpiński locale is compact. Let $(S_i)_{i:I}$ be a directed family of Scott open subsets of $\Omega_{\mathcal{U}}$ with $\top \leq \bigcup_{i:I} S_i$, which is to say that the union $\bigcup_{i:I} S_i$ contains every proposition in $\Omega_{\mathcal{U}}$. There must then be a Scott open subset S_i with $\bot \in S_i$. By the monotonicity of Scott open subsets, S_i must then contain every proposition since $\bot$ implies every $\mathcal{U}$-proposition. Hence we have $\top \subseteq S_i$.

Lemma 3.3.8 (). *Compact opens are closed under finite joins, that is, the bottom open* $\mathbf{0}$ *is always compact and the join of two compact opens is compact.*

Proof. Well-known fact of locale theory [Joh82, Lemma II.3.2]. □

We denote by $\mathsf{K}(X)$ the type of compact opens of a locale X. In other words, we

define

$$\mathsf{K}(X) :\equiv \Sigma_{(U:\mathcal{O}(X))} U \text{ is compact.}$$

3.3.2 The way-below relation

In point-free topology, there is a generalization of compactness called the *way-below relation*. Whilst an open $U : \mathcal{O}(X)$ is compact if every open cover of U contains a finite subcover for U, it is said to be way below some other open V if every open cover of V has a finite subcover for U. In other words, the statement that U is way below V (written $U \ll V$) expresses that U is *compact relative to* the open V.

Definition 3.3.9 (Way-below relation)**.** We say that an open U of a locale X is *way below* an open V, written $U \ll V$, if for every directed family $(W_i)_{i:I}$ with $V \leq \bigvee_{i:I} W_i$, there is some $i : I$ with $U \leq W_i$.

Lemma 3.3.10. *The statement $U \ll V$ is a proposition, for every pair of opens U and V.*

Note that for every open $U : \mathcal{O}(X)$, the proposition $U \ll U$ is judgementally equal to Condition (3) from Lemma 3.3.3.

Lemma 3.3.11.

- *For every pair of opens $U, V : \mathcal{O}(X)$, if $U \ll V$ then $U \leq V$.*
- *If $U \ll V$ and $V \leq W$, then $U \ll W$, for every $U, V, W : \mathcal{O}(X)$.*
- *If $U \leq V$ and $V \ll W$, then $U \ll W$, for every $U, V, W : \mathcal{O}(X)$.*

Proof. See [Joh82, Lemma VII.2.2]. Proofs are unchanged in our foundational setting. □

Finally, we also note that Lemma 3.3.8 generalizes to the way-below relation.

Lemma 3.3.12. *For every open $U : \mathcal{O}(X)$,*

1. *the bottom open $\mathbf{0}_X$ is way below U, and*
2. *for every pair of opens V and W, if $V \ll U$ and $W \ll U$ then $V \vee W \ll U$.*

Proof.
(1) Let $(U_i)_{i:I}$ be a directed family such that $\mathbf{0}_X \leq \bigvee_{i:I} U_i$. The family must be inhabited by some U_k since it is directed, and so we have $\mathbf{0}_X \leq U_k$.
(2) Let U_1, U_2 be a pair of opens of X with $U_1 \ll U$ and $U_2 \ll U$. Let $(V_i)_{i:I}$ be a family of opens such that $U \leq \bigvee_{i:I} V_i$. There must be $i, j : I$ such that $U_1 \leq V_i$ and $U_2 \leq V_j$. Since the family $(V_i)_{i:I}$ is directed, there must be some $k : I$ with $U_1 \leq V_i \leq V_k$ and $U_2 \leq V_j \leq V_k$. It follows that $U_1 \vee U_2 \leq V_k$. □

3.4 Clopens and the well inside relation

3.4.1 Clopens

We now look at the point-free analogue of the notion of clopen subset from point-set topology.

Definition 3.4.1 (Clopen)**.** An open $U : \mathcal{O}(X)$ of a locale X is called a *clopen* if it is a complemented element of the lattice $\mathcal{O}(X)$. That is to say, there is a specified U' satisfying

$$U \wedge U' = \mathbf{0}_X \quad \text{and} \quad U \vee U' = \mathbf{1}_X.$$

Note that it is not necessary to use propositional truncation in the above definition:

Lemma 3.4.2. *For every open $U : \mathcal{O}(X)$, the type expressing that U is clopen is a proposition.*

Proof. Follows directly from the fact that Boolean complements are unique. □

Example 3.4.3. The clopens of the terminal locale $\mathbf{1}_{\mathcal{U}}$ are exactly the decidable propositions (i.e. $\top_{\mathcal{U}}$ and $\bot_{\mathcal{U}}$, as shown by Lemma 2.6.4).

We record some elementary facts about clopens for the sake of self-containment.

Lemma 3.4.4. *In every locale X,*

- *the top open $\mathbf{1}_X$ and the bottom open $\mathbf{0}_X$ are clopen,*
- *clopens are closed under binary joins and binary meets.*

Proof. It is easy to see that $\mathbf{1}_X$ and $\mathbf{0}_X$ are clopen, since they clearly complement each other. Given two clopens $F_1, F_2 : \mathcal{O}(X)$, with complements F_1' and F_2', the meet $F_1 \wedge F_2$ (resp. the join $F_1 \vee F_2$) is complemented by the join $F_1' \vee F_2'$ (resp. the meet $F_1' \wedge F_2'$). □

3.4.2 The well-inside relation

Similar to how the way-below relation generalizes the notion of a compact open, the *well-inside relation* generalizes the notion of a clopen. The statement that U is well-inside V (written $U \Subset V$) expresses that the open U is clopen relative to the open V.

Definition 3.4.5 (The well-inside relation)**.** We say that an open U of a locale X is *well inside* an open V, written $U \Subset V$, if there is some open W with $U \wedge W = \mathbf{0}_X$ and $V \vee W = \mathbf{1}_X$. Such a W is called a meet-complement of U and a join-complement of V.

We note here that the use of unspecified existence is essential in the above definition, since the untruncated version is structure and not property. The example below illustrates this:

Example 3.4.6 (🤌). Consider the terminal locale $\mathbf{1}_{\mathcal{U}}$, which has the unit proposition $\top_{\mathcal{U}}$ as its top open, and the empty proposition $\bot_{\mathcal{U}}$ as its bottom open. We have that $\bot \leqslant \top$, but this can be witnessed by two distinct propositions. The proposition $\bot$ is a meet-complement of itself since $\bot \wedge \bot = \bot$ and also a join-complement of $\top$ since $\top \vee \bot = \top$. It is also the case that $\top$ is a meet-complement of $\bot$ and a join-complement of itself.

Lemma 3.4.7 (🤌). *Given opens U, V, W of a locale,*

1. *if $U \leqslant V$ and $V \leq W$, then $U \leqslant W$;*
2. *if $U \leq V$ and $V \leqslant W$, then $U \leqslant W$.*

Proof. See [Joh82, Lemma III.3.1.1.(iii)]. The proof is unchanged in our foundational setting. □

In compact locales, the well-inside relation implies the way-below relation.

Lemma 3.4.8. *In every compact locale X, if $U \leqslant V$ then $U \ll V$, for every pair of opens $U, V : \mathcal{O}(X)$.*

Proof. The proof is well-known. See [Joh82, p. 303]. □

Since every open $U : \mathcal{O}(X)$ is compact if and only if $U \ll U$ and is clopen if and only if $U \leqslant U$, we immediately obtain the following corollary:

Corollary 3.4.9. *In every compact locale, clopens are compact.*

What about the converse of this implication? In Section 4.5.1, we will prove that the way-below relation implies the well-inside relation in *regular locales*, as we define them in Definition 3.6.5.

3.5 Bases

In the context of point-set topology, a *base* (or *basis*) for a topology X [Eng89, p. 12] is a collection $\mathcal{B}$ of open subsets of X such that every open $U \in \mathcal{O}(X)$ can be expressed as the union of a subcollection of $\mathcal{B}$. These open sets given by $\mathcal{B}$ are called the *basic opens*.

In the context of point-free topology, this notion of base is abstracted by the order-theoretic notion of a *generating lattice*. In [Joh02, p. 548], for example, a base for a locale X is defined as a subset $\mathcal{B} \subseteq \mathcal{O}(X)$ such that every element of the frame $\mathcal{O}(X)$ is expressible as a join of members of $\mathcal{B}$. This definition can be readily translated into the context of type theory:

Definition 3.5.1. Let X be a locale. A family of opens $(B_i)_{i:I}$ is said to form a base for X if, for every $U : \mathcal{O}(X)$, there is a specified directed subfamily $(B_{i_j})_{j:J}$ such that $U = \bigvee_{j:J} B_{i_j}$. This is called the *basic covering family* for the open U.

Note that we did not require the collection of basic opens to be given by an embedding family in the above definition. Technically, the precise translation of the notion of generating lattice would include this requirement since subsets correspond to embedding families. However, it turns out to be quite useful for our purposes to consider a variant of this notion where we permit repetitions in the family of basic opens. We call this an *intensional base* and distinguish it from the notion of base with the embedding requirement, which we call an *extensional base*. We discuss the difference between the two further in Section 3.5.1.

Another variant of the notion of base that we consider is that of a *weak base*, where we weaken the specified existence requirement for basic covering families to *unspecified* existence. We discuss this further in Section 3.5.2. The motivation for this may not be clear prima facie, but it turns out to be quite important for the predicative development of the theory of spectral and Stone locales. We discuss this in detail in Chapter 4.

3.5.1 Intensional vs. extensional bases

We consider a variant of the notion of base that we call *extensional base*:

Definition 3.5.2 (Extensional base)**.** A base $(B_i)_{i:I}$ is said to be *extensional* if the function $B : I \to \mathcal{O}(X)$ is an *embedding* i.e. it is a subtype in the sense of Definition 2.13.7.

The requirement that the family of basic opens be an embedding states that every basic open has a unique representation in the base. Although this captures exactly the standard notion of locale base, it is often more convenient to work with bases that *do have* repetitions. An example of this is the discrete locale:

Example 3.5.3. Let X be a small type. The function $\beta : \mathsf{List}(X) \to \mathcal{P}(X)$, defined as

$$\beta(x_0, \ldots, x_{n-1}) \quad :\equiv \quad y \mapsto y =_X x_0 \vee \cdots \vee y =_X x_{n-1},$$

gives a (small) intensional base for the frame $\mathcal{P}(X)$. However, this is not an extensional base since every permutation of a list will give the same subset. One can instead work with the *image* of β to remove the repetitions. Notice, however, that the image of β is exactly the type of Kuratowski finite subsets of X, the smallness of which requires the set replacement principle. So taking the notion of extensional base as the standard definition would force us to adopt an extraneous assumption.

The process of taking the image to obtain an extensional base, which we mentioned in the above example, can be generalized to arbitrary bases. We call this the *extensionalization of an intensional base*. It crucially relies on the set replacement principle.

Lemma 3.5.4. *For every small base $(B_i)_{i:I}$ on a locale X, the image of $B : I \to \mathcal{O}(X)$ is a small type, assuming the set replacement principle.*

Proof. Let $(B_i)_{i:I}$ be a small base. By the set replacement principle (Corollary 2.14.2), showing (1) that I is a small type and (2) that $\mathcal{O}(X)$ is a locally small set is sufficient to show that $\mathsf{image}(B)$ is a small type. The former is immediate by assumption and the latter is given by Lemma 3.1.3. □

Since replacing an index of a base with the image does not affect the joins of the covering families given by the base, we have the following corollary:

Corollary 3.5.5 (Base extensionalization)**.** *For every small intensional base $(B_i)_{i:I}$ on a locale X, the inclusion $\mathsf{image}(B) \hookrightarrow \mathcal{O}(X)$, formally given by the first projection, is a small extensional base that contains exactly the same opens.*

3.5.2 Weak vs. strong bases

Consider a family $(B_i)_{i:I}$. Neither the statement that it forms an intensional base nor the statement that it forms an extensional base is a proposition. This is due to the fact that there are multiple ways for a base to give covering families with the same join. For example, one could choose to include the bottom element or not in a basic covering family. Although this will not change the join of the family, the family will formally be a different one.

In Chapter 4, we will encounter cases of locales where an extensional base can be shown to be unique. In the context of such locales, the fact that forming a base is not a property will be the limiting factor preventing us from obtaining a propositional notion of base. To address this problem, we define the notion of *weak base* where the requirement of specified existence is relaxed to unspecified existence.

Definition 3.5.6 (Weak base)**.** A *weak base* for a locale X is a family $(B_i)_{i:I}$ such that for every $U : \mathcal{O}(X)$, there is an unspecified directed subfamily $(B_{i_j})_{j:J}$ with $U = \bigvee_{j:J} B_{i_j}$.

Remark 3.5.7. It is important to note here that this notion of weak base is *not* well behaved in general: many results that one would like to prove using a base requires the specified existence of covering families. This should therefore be viewed as an auxiliary technical notion that has certain uses, and is well behaved in the context of certain classes of locales where the structure of the covering of the families can be obtained from their unspecified existence.

In contexts where we need to be careful about the distinction, we use the explicit term *strong base* for Definition 3.5.1. Unless otherwise specified, all bases are strong. The intensional-extensional distinction, which we specified in the context of strong bases, also applies to weak bases.

3.5.3 Directification of bases

We required the basic covering families given by a base to be directed in all of the definitions above. In the context of a frame, where we have all finite joins, this requirement is not essential since one can always choose to work with the closure of the base under finite joins (as in Definition 3.3.2), in which the basic covering families are directed. More specifically, here is how one directifies a base:

1. Given a base $(B_i)_{i:I}$, we take its closure under finite joins.
2. For each basic covering family given by the base $(B_i)_{i:I}$, we directify it also by closing it under finite joins. This is made possible by Step (1).

Step (1) above can be skipped if the base in consideration happens to be closed under finite joins already.

Lemma 3.5.8 (🤌). *For every base $(B_i)_{i:I}$ of a locale X, the family*

$$B^{\uparrow} : \mathsf{List}(I) \to \mathcal{O}(X),$$

obtained by closing it under finite joins (as in Definition 3.3.2) forms a base with directed covering families.

Proof. Let U be an open of X. Denote by $(B_{i_j})_{j:J}$ the basic covering family that B gives for U. We define the directed covering family given by $B^{\uparrow}$ to be the subfamily $\alpha : \mathsf{List}(J) \to \mathcal{O}(X)$, defined as

$$\alpha(j_0, \ldots, j_{n-1}) :\equiv B_{i_{j_0}} \vee \cdots \vee B_{i_{j_{n-1}}},$$

which is obviously directed and is a subfamily of $B^{\uparrow}$. □

We note that the closure of a small base under finite joins is small, since $\mathsf{List}(I)$ inhabits the same universe as I does. Furthermore, we note that we could have defined the notion of base, in light of Lemma 3.5.8, without the requirement that the basic covering families be directed. We nevertheless prefer to include this requirement in the definition for technical convenience.

3.5.4 Examples

We now look at some more examples of bases, starting with the terminal locale $\mathbf{1}_{\mathcal{U}}$.

Bases for the terminal locale

In Lemma 3.1.17, we showed how a proposition can be decomposed into a join of decidable propositions. Having pinned down the notion of base, we now proceed to extend Lemma 3.1.17 by showing that decidable propositions form a base for the terminal locale:

Example 3.5.9 (🤌). Recall the family $\beta : \mathbf{2} \to \Omega_{\mathcal{U}}$ from Lemma 3.1.17, defined as

$$\beta(0) :\equiv \bot_{\mathcal{U}}$$
$$\beta(1) :\equiv \top_{\mathcal{U}}$$

This forms a base for the terminal locale $\mathbf{1}_{\mathcal{U}}$. For every proposition $P : \Omega_{\mathcal{U}}$, the family $\{\top \mid p : P\}$ is a covering family for P by Lemma 3.1.17. We apply the directification procedure from Lemma 3.5.8 to force the basic covering families to be directed.

Note that the covering families given above are not directed without the use of directification. As it is somewhat technically inconvenient to work with a directified base, we also show how the above family $\beta : \mathbf{2} \to \Omega_{\mathcal{U}}$ can be shown to form a directed base directly.

Example 3.5.10. The family $\beta : \mathbf{2} \to \Omega_{\mathcal{U}}$ from Example 3.5.9 forms a directed base for $\mathbf{1}_{\mathcal{U}}$. For every proposition $P : \Omega_{\mathcal{U}}$, the basic covering family is given by

$$\left(\Sigma_{(b:\mathbf{2})}\beta(b) \leq P\right) \xrightarrow{\beta \circ \mathrm{pr}_1} \Omega_{\mathcal{U}}.$$

This family is always inhabited by empty proposition $\bot_{\mathcal{U}}$. Given two $b_1, b_2 : \mathbf{2}$ with $\beta(b_1) \Rightarrow P$ and $\beta(b_2) \Rightarrow P$, we have that $\beta(b_1 \vee b_2) \Rightarrow P$, and hence the least upper bound $b_1 \vee b_2$ is in the family. The fact that it covers P was given in Lemma 3.1.17.

Base construction for the Sierpiński locale

To construct a base for the Sierpiński locale, we first do some preparation.

Definition 3.5.11. For every proposition $P : \Omega_{\mathcal{U}}$, its *upset* (or *principal filter*), denoted $\uparrow P$, is defined as

$$\uparrow P :\equiv Q \mapsto (P \Rightarrow Q).$$

This defines a function $\uparrow P : \Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}}$.

Lemma 3.5.12. *For every decidable proposition $P : \Omega_{\mathcal{U}}$, the principal filter $\uparrow P$ is a Scott open subset $\Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}}$.*

Proof. Let $P : \Omega_{\mathcal{U}}$ be a decidable proposition and let $(Q_i)_{i:I}$ be a directed family of propositions. We need to show that $P \Rightarrow \exists_{(i:I)} Q_i$ if and only if $\exists_{(i:I)} P \Rightarrow Q_i$. If P is inhabited, then it is easy to see that $P \Rightarrow \exists_{(i:I)} Q_i$ holds, since $\top \Rightarrow \exists_{(i:I)} Q_i$ iff $\exists_{(i:I)} Q_i$ iff $\exists_{(i:I)} \top \Rightarrow Q_i$. If P is *not* inhabited, $\exists_{(i:I)} P \Rightarrow Q_i$ must be trivially true by the elimination principle for bottom, since the family $(Q_i)_{i:I}$ is inhabited. We therefore have

$$\exists_{(i:I)} P \Rightarrow Q_i \leftrightarrow \top \leftrightarrow P \Rightarrow \exists_{(i:I)} Q_i.$$

□

Lemma 3.5.13. *The principal filter* $\uparrow \bot_{\mathcal{U}}$ *is the full subset of* $\Omega_{\mathcal{U}}$.

Proof. This is obvious since every proposition is above $\bot$. □

Example 3.5.14. The family $(\uparrow \beta(b))_{b:2}$ forms a base for the Sierpiński locale. Every Scott open subset $S : \Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}}$ can be expressed as a join

$$S = \bigvee_{b:2} \{\uparrow \beta(b) \mid \beta(b) \in S\}$$

For every proposition $P \in S$, since we know (from Example 3.5.10) that

$$P = \bigvee_{\substack{b:2 \\ \beta(b) \leq P}} \beta(b),$$

the Scott continuity of S implies

$$\bigvee_{\substack{b:2 \\ \beta(b) \leq P}} S(\beta(b)),$$

since we have $S\left(\bigvee_{\substack{b:2 \\ \beta(b) \leq P}} \beta(b)\right)$. Conversely, given a proposition P such that there is some $b : 2$ with $\beta(b) \in S$ and $\beta(b) \leq P$, it is immediate by the monotonicity of S that $P \in S$.

3.6 Important classes of locales

Having pinned down the notion of base in Section 3.5, we now look at various classes of locales that can be defined simply by requiring the existence of a base satisfying certain conditions. This approach is inspired by the notions of locally presentable and locally accessible categories [AR94], and de Jong and Escardó's application of it to the predicative development of domain theory.

Definition 3.6.1 (Spectral locale)**.** A locale X is called *spectral* if it has some small base $(B_i)_{i:I}$ satisfying the following three conditions.

- ($\mathrm{SP_b}1$) Every B_i in the base is compact.
- ($\mathrm{SP_b}2$) It contains the top open $\mathbf{1}_X$.
- ($\mathrm{SP_b}3$) It is closed under binary meets i.e. for every two $i, j : I$, there exists some $k : I$ such that $B_k = B_i \wedge B_j$.

A base satisfying these conditions is called a *spectral base*.

In Chapter 4, we will give various alternative characterizations (using the univalence axiom) of this notion of spectral locale. However, we assign conceptual priority to this definition as it is an instance of a general technique that usually results in the correct

notion in the context of predicative univalent foundations.

Definition 3.6.2 (Zero-dimensional locale)**.** A locale X is called *zero-dimensional* if it has some base $(B_i)_{i:I}$ consisting of *clopens*. We use the term *zero-dimensional base* to refer to such bases.

Definition 3.6.3. A locale X is called *Stone* if it has some base $(B_i)_{i:I}$ satisfying the following three conditions.

($\mathsf{ST}_\mathsf{b}1$) Every B_i in the base is compact
($\mathsf{ST}_\mathsf{b}2$) It contains the top open $\mathbf{1}_X$.
($\mathsf{ST}_\mathsf{b}3$) Every B_i in the base is clopen.

A base satisfying the above conditions is called a *Stone base*.

Definition 3.6.4 (Locally compact locale)**.** A locale X is called *locally compact* if its frame is a continuous lattice, which is to say it has some base $(B_i)_{i:I}$ satisfying the condition that, for every open $U : \mathcal{O}(X)$, the basic opens in the covering family for U are all way below U.

Definition 3.6.5 (Regular locale)**.** A locale X is called *regular* if it has some base $(B_i)_{i:I}$ such that for every open $U : \mathcal{O}(X)$, the covering subfamily $U = \bigvee_{j:J} B_{i_j}$ satisfies $B_{i_j} \ll U$, for every $j : J$

Lemma 3.6.6. *Every zero-dimensional locale is regular.*

Proof. Let X be a locale with a zero-dimensional base $(B_i)_{i:I}$ and consider some $U : \mathcal{O}(X)$. The basic covering family $U = \bigvee_{j:J} B_{i_j}$ consists of basic opens that satisfy $B_{i_j} \ll B_{i_j}$. Since each B_{i_j} is below U, it follows from Lemma 3.4.7 that each B_{i_j} satisfies $B_{i_j} \ll U$. □

3.7 Adjoint functor theorem for frames

In Section 3.1.4, we explained that many essential constructions of traditional locale theory, such as Heyting implications and arbitrary meets, are not available in our predicative context. In this section, we show how this situation can be remedied by giving a predicative proof of the adjoint functor theorem for frames with small bases. The key idea here is that our notion of base serves the rôle of the solution set condition.

Definition 3.7.1. Let $(A, \leq_A)$ and $(B, \leq_B)$ be two preordered sets. Given monotone maps $f : A \to B$ and $g : B \to A$, the map g is said to be the *right adjoint* of f (and f the *left adjoint* of g) if it satisfies

$$f \dashv g \quad :\equiv \quad f(a) \leq_B b \Leftrightarrow a \leq_A g(b) \text{ for all } a : A, b : B.$$

Lemma 3.7.2 (). *The statement $f \dashv g$ expressing that f and g form an adjunction is a proposition. Furthermore, for every monotone map $f : A \to B$, its type of right adjoints, i.e. $\Sigma_{(g:B\to A)} f \dashv g$, is a proposition as right adjoints are unique.*

In the case of non-posetal categories, the adjoint functor theorem involves the so-called solution set condition. In our predicative foundational setting, a small base plays the rôle of the solution set, allowing us to construct the right adjoint of a frame homomorphism. A somewhat interesting observation here is that the unspecified existence of a *weak base* suffices for proving the adjoint functor theorem.

Theorem 3.7.3 (Posetal adjoint functor theorem)**.** *Let X and Y be two locales and let $h : \mathcal{O}(Y) \to \mathcal{O}(X)$ be a monotone map of frames. Assume that Y has an unspecified, small, and weak base. The map h has a right adjoint if and only if it preserves the joins of small families.*

Proof. Let $h : \mathcal{O}(Y) \to \mathcal{O}(X)$ be a monotone map of frames and assume that Y has an unspecified, small, and weak base.

The forward direction is easy: suppose h has a right adjoint $g : \mathcal{O}(X) \to \mathcal{O}(Y)$ and let $(U_i)_{i:I}$ be a family in $\mathcal{O}(Y)$. By the uniqueness of joins, it is sufficient to show that $h(\bigvee_i U_i)$ is the join of the family $(h(U_i))_{i:I}$. It is clearly an upper bound by the fact that h is monotone. Given some other upper bound V of $(h(U_i))_{i:I}$, we have $h(\bigvee_i U_i) \leq V$ since $h(\bigvee_i U_i) \leq V \leftrightarrow (\bigvee_i U_i) \leq g(V)$ meaning it is sufficient to show $U_i \leq g(V)$ for each U_i. Since $U_i \leq g(V)$ if and only if $h(U_i) \leq V$, we are done as the latter can be seen to hold directly from the fact that V is an upper bound of the family in consideration.

For the converse, first note that we may appeal to the induction principle of propositional truncation, since the type of right adjoints of h is a proposition by Lemma 3.7.2. We therefore work with a specified small base $(B_i)_{i:I}$. Suppose $h(\bigvee_i U_i) = \bigvee_{i:I} h(U_i)$ for every small family of opens $(U_i)_{i:I}$. We define the right adjoint of h as

$$g(U) \quad :\equiv \quad \bigvee \{B_i \mid i : I, h(B_i) \leq U\}\,.$$

We need to show that g is the right adjoint of h i.e. that $h(V) \leq U \leftrightarrow V \leq g(U)$ for every pair of opens $V : \mathcal{O}(Y), U : \mathcal{O}(X)$.

For the $(\Rightarrow)$ direction, assume $h(V) \leq U$. It must be the case that $V = \bigvee_{j:J} B_{i_j}$ for some specified covering family $(i_j)_{j:J}$. This means that we just have to show $B_{i_j} \leq g(U)$ for every $j : J$, which is the case since $h(B_{i_j}) \leq h(V) \leq U$.

For the $(\Leftarrow)$ direction, assume $V \leq g(U)$. This means that we have:

$$\begin{aligned} h(V) &\leq h(g(U)) \\ &= h\left(\bigvee \{B_i \mid i : I, h(B_i) \leq U\}\right) \\ &= \bigvee \{h(B_i) \mid i : I, h(B_i) \leq U\} \\ &\leq U \end{aligned}$$

so that $h(V) \leq U$, as required. □

3.7.1 Heyting implications and the open nucleus

Using the adjoint functor theorem, we construct the right adjoint of a frame homomorphism on a frame with a small (weak) base. This will find several applications in our development. Most importantly, it will allow us to define Heyting implications thereby enabling us to work with the notion of *open nucleus* in a predicative foundational setting. This will be crucial to our construction of the patch locale in Chapter 5.

Definition 3.7.4 (). Let $f : X \to Y$ be a continuous map of locales, where Y has an unspecified small base. By Theorem 3.7.3, the frame homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ has a right adjoint, denoted by

$$f_* : \mathcal{O}(X) \to \mathcal{O}(Y).$$

Definition 3.7.5 (Heyting implication). Let X be a locale with some small base. For every open $U : \mathcal{O}(X)$, the map $(\text{-}) \wedge U : \mathcal{O}(X) \to \mathcal{O}(X)$ is a frame homomorphism by the frame distributivity law. It thus has a right adjoint as constructed in Definition 3.7.4. We denote this right adjoint by $U \Rightarrow (\text{-}) : \mathcal{O}(X) \to \mathcal{O}(X)$. The operation $(\text{-}) \Rightarrow (\text{-})$ is known as *Heyting implication.*

Lemma 3.7.6 (). *Let X be a locale with a small weak base, and consider an open $U : \mathcal{O}(X)$. The map $\mathbf{o}(U) : \mathcal{O}(X) \to \mathcal{O}(X)$, defined as*

$$\mathbf{o}(U) \quad :\equiv \quad V \mapsto U \Rightarrow V,$$

forms a nucleus.

Proof. See [Joh82, §II.2.4]. □

Definition 3.7.7 (Open nucleus). The nucleus $\mathbf{o}(U)$ constructed in the above lemma is called the *open nucleus* induced by U.

Chapter 4

Spectral and Stone locales

In the previous chapter, we presented the basics of a predicative approach to constructive locale theory in univalent foundations. We now start developing the theory of spectral and Stone locales in this setting. Our predicative approach is inspired by locally presentable categories [AR94], and their use by de Jong and Escardó [dJon23; dJon21b] in their predicative development of domain theory in univalent foundations. The organization of this chapter is as follows:

- Section 4.1: We provide a high-level overview of the notion of spectral locale along with some history.
- Section 4.2: We discuss how the standard, impredicative definition of the notion of spectral locale can be translated into our predicative setting, and prove that this translation is equivalent to Definition 3.6.1 under the assumption of the set replacement principle and the univalence axiom.
- Section 4.3: We look at some examples, as well as counterexamples, of spectral locales.
- Section 4.4: We construct the frame of ideals over a small distributive lattice in preparation for our predicative development of Stone duality in Section 4.7.
- Section 4.5: We define zero-dimensional and regular locales, and record some of their salient properties.
- Section 4.6: We define Stone locales and start developing their theory.
- Section 4.7: We prove that there is an equivalence between the type **Spec** of large, locally small, and small-complete spectral locales and the type **DLat** of small distributive lattices.
- Section 4.8: We extend the type equivalence from Section 4.7 to show that there is an equivalence of categories between **DLat** and **Spec**.

4.1 Introduction

In the introduction to Chapter 3, we mentioned that the idea of studying frames as spaces can be traced back to the work of Stone [Sto34; Sto36]. Stone's work originated in the context of what are now called *Stone spaces*. In the context of classical point-set topology, these can be defined as the spaces of *ultrafilters* of Boolean algebras [Smy93, §2.5]. The remarkable insight that arose in Stone's work was that such spaces can be studied in a *completely algebraic* way, since they are fully characterized by their Boolean algebras of clopen subsets. This is captured by Stone's representation theorem [Joh82, §II.4.4] stating that there is a dual categorical equivalence between the category of Stone spaces and continuous maps and the category of Boolean algebras and their homomorphisms. This is often referred to as *Stone duality* for Boolean algebras, and it is historically notable as the first duality of this kind to be discovered.

Stone later generalized [Sto37] this duality from Boolean algebras to distributive lattices (as explained in [DST19]). The fundamental idea here is that, whilst Stone spaces are fully characterized by their Boolean algebras of clopen subsets, spectral spaces are fully characterized by their distributive lattices of compact opens. Compact open subsets coincide with the clopen subsets in Stone spaces, so this generalization from Stone spaces to spectral ones consists in the observation that it is really the *algebra of compact opens* that characterizes the space in consideration. For the detailed history of the subject, we refer the reader to the survey in [DST19].

In classical point-set topology, a spectral space is defined[1] [DST19, Definition 1.1.5] as a space X satisfying the following conditions:

1. Space X is compact and T_0.
2. The collection of compact open subsets forms a base for the topology on X.
3. Compact open subsets are closed under binary intersections.
4. Space X is sober.

In this thesis, we will work with the localic manifestation of the notion of spectral space, which we call a *spectral locale* (called *coherent locale* by some, including Johnstone [Joh82, §II.3.2]). Classically, it can be shown that every spectral locale is spatial, but this is equivalent to the Prime Ideal Theorem for distributive lattices in a constructive setting [Joh83, p. 46]. When one adopts a point-free approach towards spectral spaces, Stone duality is provable in a completely constructive way.

4.2 Spectrality in a predicative setting

In Section 3.6, we gave several examples of how various classes of locales can be defined solely by stipulating the unspecified existence of a base subject to certain conditions. We also explained (1) that this approach can be considered an adaptation of locally accessible and presentable categories to the posetal setting, and (2) that de Jong and Escardó drew inspiration from the literature on these to develop a satisfactory account of domain theory [dJon23; dJon21b] in constructive and predicative univalent foundations.

[1] Note that in [DST19, Definition 1.1.5] the term *quasi-compact* is used in this definition. We prefer to use the term *compact* here.

We recall that the definition of spectral locale (Definition 3.6.1) we gave in Section 3.6 was the following:

- a locale X is said to be *spectral* if it has some small spectral base, where
- *spectral base* means a base consisting of compact opens that is closed under finite meets.

We also explained that we gave conceptual priority to this definition, as it is the instance of a general technique that often leads to predicatively well-behaved definitions. It is not a priori clear, however, that this definition corresponds to the traditional definition of spectral locale [Joh82, Paragraph II.3.2]. In Section 4.2.1 below, we motivate an alternative predicatively well-behaved definition through the traditional definition, and then show the equivalence of the two in Section 4.2.3. The proof of equivalence is not straightforward and seems to rely crucially on both the univalence axiom and the set replacement principle (which is equivalent to the existence of small set quotients as we explained in Section 2.14).

4.2.1 The traditional definition from a predicative point of view

The traditional definition of a *spectral locale* is a locale whose compact opens form a base closed under finite meets [Joh82, §II.3.2]. A careful analysis of this definition in a predicative setting shows that it is well-behaved only within the context of complete and small locales, among which only the trivial ones can be constructed in a predicative setting (as we discussed in detail in Section 3.1.2). Let us examine this definition in detail to elucidate why it is not satisfactory in a predicative context.

The direct translation of the traditional definition of spectral locale would amount to the requirement that the inclusion

$$\mathsf{K}(X) \overset{\mathsf{pr}_1}{\hookrightarrow} \mathcal{O}(X)$$

form a base closed under finite meets. When working with $(\mathcal{U},\mathcal{U},\mathcal{U})$-frames, this family is always small since the carrier is a small set. Notice, however, that when working with large, locally small, and small-complete frames, there is no reason for the type $\mathsf{K}(X)$ to be small in general. This is problematic for at least two reasons:

1. Such a base would not give us the constructions that we discussed in Section 3.1.4; it is essential that the base is small for this purpose.
2. The natural form of Stone duality to consider for spectral locales in our predicative setting is that between spectral locales and *small* distributive lattices. As we will explain in detail later, taking the frame of (small) ideals over a small distributive lattice always gives a large, locally small, and small-complete frame. If satisfying Stone duality in this sense is taken as a desideratum for a well-behaved definition of the notion of spectral locale, the definition permitting a large type of compact opens is not adequate.

It turns out, as we will discuss in this chapter soon, that *both* of the above problems can be remedied simply by stipulating that the type $\mathsf{K}(X)$ be small. This leads us to the following predicative formulation of the notion of spectral locale:

Definition 4.2.1 (Spectral locale)**.** A locale X is called *spectral* if it satisfies the following conditions:

(SP1) It is compact (i.e. the empty meet is compact).
(SP2) Compact opens are closed under binary meets.
(SP3) The type $\mathsf{K}(X)$ forms a weak base.
(SP4) The type $\mathsf{K}(X)$ is small.

In addition to the motivations discussed above, we also note that we do not have any examples of locales satisfying Conditions (SP1)–(SP3), but not Condition (SP4).

Finally, we record the following characterization of the order in spectral frames:

Lemma 4.2.2. *Let X be a spectral locale. The order of the frame $\mathcal{O}(X)$ is equivalent to the order defined as*

$$U \leq_s V \quad :\equiv \quad \Pi_{(K:\mathsf{K}(X))}\, K \leq U \Rightarrow K \leq V.$$

Proof. ($\Rightarrow$) It is immediate that $U \leq V$ implies $U \leq_s V$, for every pair of opens $U, V : \mathcal{O}(X)$. The nontrivial direction is the converse: ($\Leftarrow$) let U and V be a pair of opens with $U \leq_s V$. By (SP3), we know that there is an unspecified basic covering family $(K_i)_{i:I}$ for U that consists of compact opens. Since we know $U = \bigvee_{i:I} K_i$, it suffices to show $(\bigvee_{i:I} K_i) \leq V$. For every $i : I$, we know that K_i is a compact open with $K_i \leq U$, and we thus have $K_i \leq V$ by our assumption that $U \leq_s V$. □

4.2.2 Propositionality of being spectral

We would like the type expressing the spectrality of a locale to be a proposition, for which we need to ensure that all four conditions in Definition 4.2.1 are propositions. Surprisingly, it seems to be the case that we *have to* use the univalence axiom to force Condition (SP4) to be a proposition, since the propositionality of being small is logically equivalent to univalence by Lemma 2.13.4. Moreover, to ensure that Condition (SP3) is a proposition, we make use of the notion of *weak base* from Definition 3.5.6.

Lemma 4.2.3. *Assuming the univalence axiom, the type expressing that X is spectral is a proposition for every locale X.*

Proof. It is easy to see that Conditions (SP1)–(SP3) are all propositions. Assuming the univalence axiom, Condition (SP4) is a proposition by Lemma 2.13.4. □

In the next section, we show that Definition 3.6.1 and Definition 4.2.1 are equivalent.

4.2.3 Relating the two approaches

In Section 3.6, we defined various classes of locales as locales with unspecified bases satisfying certain conditions. In Definition 3.6.1 from that section, we also gave a definition of the notion of spectral locale, using the aforementioned general technique

that applies to many other classes of locales. Having pinned down Definition 4.2.1, which is well motivated from the perspective of the standard, impredicative definition, we now proceed to prove that these two definitions are equivalent under the assumption of univalence and set replacement. Our motivation for proving this equivalence is twofold:

1. It captures that the choice of the base *does not matter* under the conditions of spectrality, even in a constructive and predicative setting, which is a fact that is important to observe in itself.
2. In Sections 4.7 and 4.8, we prove Stone duality for Definition 4.2.1, which justifies that we have identified the right category of spectral locales. The equivalence of these two definitions is necessary for this justification to apply to Definition 3.6.1, since Stone duality seems applicable only to Definition 4.2.1. In other words, proving the equivalence of the two definitions seems to be a natural step for proving that Definition 3.6.1 enjoys Stone duality.

We now start to do some preparation towards the proof of this equivalence.

Lemma 4.2.4. *For every locale X with a base $(B_i)_{i:I}$, every compact open of X falls in the base. In other words, for every compact open K, there exists an index $i : I$ such that $K = B_i$.*

Proof. Let $K : \mathcal{O}(X)$ be a compact open. As $(B_i)_{i:I}$ is a strong base, there is a specified directed family of indices $(i_j)_{j:J}$ such that $K = \bigvee_{j:J} B_{i_j}$. By the compactness of K, there is some $k : J$ such that $K \leq B_{i_k}$. Clearly, $B_{i_k} \leq K$ is also the case since K is an upper bound of the family $(B_{i_j})_{j:J}$, which means we have $K = B_{i_k}$. □

Corollary 4.2.5 (🖋). *For every locale X with a small base $(B_i)_{i:I}$ consisting of compact opens, we have an equivalence of types:*

$$\mathsf{image}(B) \simeq \mathsf{K}(X).$$

This equivalence is particularly useful for us since it serves the rôle of a resizing principle for the type $\mathsf{K}(X)$:

Lemma 4.2.6. *For every locale X, if X has a small base that consists of compact opens, then the type $\mathsf{K}(X)$ is small, under the assumption of the set replacement principle.*

Proof. Let X be a locale with a small base $(B_i)_{i:I}$. It was shown in Lemma 3.5.4, using the set replacement principle, that having a specified small base implies the smallness of $\mathsf{image}(B)$. We have just established in Corollary 4.2.5 above that

$$\mathsf{K}(X) \simeq \mathsf{image}(B),$$

which thus implies that $\mathsf{K}(X)$ has $\mathsf{image}(B)$ as a small copy. □

Lemma 4.2.7. *Assuming the univalence axiom, every locale with an unspecified small spectral base is spectral.*

Proof. First, notice that the conclusion is a proposition since being spectral is a proposition by Lemma 4.2.3 (which uses univalence). This means that we may appeal to the induction principle of propositional truncation and assume we have a specified spectral base $(B_i)_{i:I}$. We now verify each condition from Definition 4.2.1:

(SP1) The top element is basic and hence compact so Condition (SP1) holds.

(SP2) Given two compact opens K_1 and K_2, they must be basic by Lemma 4.2.4, meaning there exist k_1, k_2 such that $K_1 = B_{k_1}$ and $K_2 = B_{k_2}$. Because the base is closed under binary meets, we know that there are some k_3 with $B_{k_3} = K_1 \wedge K_2$, which implies that $K_1 \wedge K_2$ is compact as $(\mathrm{SP_b}1)$ says that the base consists of compact opens. This concludes that Condition (SP2) holds.

(SP3) Consider an open $U : \mathcal{O}(X)$. We know that there is a specified small family $(i_j)_{j:J}$ of indices such that $U = \bigvee_{j:J} B_{i_j}$. The subfamily $(B_{i_j})_{j:J}$ is then clearly a small directed family with each B_{i_j} compact, which was what we needed.

(SP4) Follows directly from Lemma 4.2.6.

□

Lemma 4.2.8 (🤌). *If a locale X is spectral then it has a specified, small, and extensional base.*

Proof. Let X be a spectral locale. We claim that the inclusion $\mathsf{K}(X) \hookrightarrow \mathcal{O}(X)$, which is formally given by the first projection, is an extensional and spectral small base. The fact that it is small is given by Condition (SP4). By (SP1) and (SP2), we know that this base contains the top open $\mathbf{1}_X$ and is closed under binary meets.

It remains to show that $\mathsf{K}(X) \hookrightarrow \mathcal{O}(X)$ forms a strong base. For the basic covering family of an open $U : \mathcal{O}(X)$, we pick the inclusion:

$$\left(\Sigma_{(K:\mathsf{K}(X))}(K \leq U)\right) \xhookrightarrow{\mathrm{pr}_1} \mathsf{K}(X),$$

which is again small by (SP4) and the local smallness of the frame $\mathcal{O}(X)$. It is clear that U is an upper bound of this family so it remains to show that it is the least upper bound. Consider some V that is an upper bound of this family. By Lemma 4.2.2, it suffices to show $K \leq U$ implies $K \leq V$, for every compact open K. Every compact $K \leq U$ falls in the above family by construction which implies $K \leq V$ as V is assumed to be an upper bound of the family. □

Lemma 4.2.9 (🤌). *From every unspecified intensional spectral base on a locale X, we can obtain a specified extensional spectral base.*

Proof. Let X be a locale with an unspecified intensional spectral base. By Lemma 4.2.7, we know that it is spectral, and therefore that it has a specified extensional base by Lemma 4.2.8. □

In other words, the type of intensional spectral bases has *split support* in the terminology of [KECA17]. A type X is said to have split support if $\|X\| \to X$. The mere existence of a base allows us to obtain the unique, canonical base $\mathsf{K}(X) \hookrightarrow \mathcal{O}(X)$, which can be thought of as the largest possible base satisfying the conditions of spectrality. Note here that the propositionality of being spectral, which we discussed in Section 4.2.2, is crucial since the proof uses Lemma 4.2.7.

We also record here the following fact about the closure of a spectral base under finite joins:

Lemma 4.2.10. *For every spectral base $(B_i)_{i:I}$, its closure under finite joins also forms a spectral base.*

Proof. We know by Lemma 3.3.8 that the closure of the family of basic opens also consists of compact opens. It is easy to see that this also forms a base. □

We are finally ready to prove the main characterization theorem for spectral locales, showing that various seemingly different approaches to the notion of spectral locale are all equivalent.

Theorem 4.2.11. *For every locale X, the following are logically equivalent:*

1. *X has an unspecified intensional small spectral base (Definition 3.6.1).*
2. *X is spectral in the sense of Definition 4.2.1.*
3. *X has an unspecified extensional small spectral base.*
4. *The inclusion $\mathsf{K}(X) \hookrightarrow X$ is an extensional spectral base, where to an open U we assign the directed family of compact opens below it as in the construction of Lemma 4.2.8.*
5. *X has a specified intensional small spectral base.*
6. *X has a specified extensional small spectral base.*

Proof. First, observe the obvious implications: we have that (6) ⇒ (3) ⇒ (1) and that (6) ⇒ (5) ⇒ (1). The converse of (5) ⇒ (1) was given in Lemma 4.2.9, which concludes the equivalence (5) ⇔ (1). For (6) ⇔ (4), observe that every extensional small spectral base must be equivalent to $\mathsf{K}(X)$, since $\mathsf{image}(B) \simeq \mathsf{K}(X)$ by Corollary 4.2.5. For (2) ⇔ (6), the forward direction was given by Lemma 4.2.8. For the converse, recall that we know (1) ⇒ (2) by Lemma 4.2.9, and that (6) ⇒ (5) ⇒ (1). □

Note that Conditions (1)–(4) above are all propositions, whereas Conditions (5) and (6) need not be propositions in general.

4.2.4 The category of spectral locales

Having defined the notion of spectral locale, we now present the category of spectral locales by defining the notion of *spectral map*:

Definition 4.2.12. A continuous map $f : X \to Y$ is called a *spectral* if $f^*(K)$ is compact for every compact open K of Y.

We denote by $\mathbf{Spec}_{\mathcal{U}}$ the category of spectral locales and spectral maps over the base universe $\mathcal{U}$. If the base universe is clear from context, we drop the subscript and simply write **Spec**.

This category satisfies some desirable properties thanks to the definition of spectral locale ensuring the existence of a small base. For example, we have the following:

Lemma 4.2.13. *For every map $f : X \to Y$ of spectral locales, the right adjoint of the frame homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ exists.*

Proof. We showed in Theorem 4.2.11 that spectrality implies having a small base. We can therefore construct the right adjoint of f^* as in Definition 3.7.4. □

Characterization in terms of perfect maps

Using the existence of right adjoints, the characterization of spectral maps in terms of their right adjoints can be adapted to our setting.

Definition 4.2.14 (). A continuous map $f : X \to Y$ of locales is called *perfect* if the right adjoint of its frame homomorphism is Scott continuous.

Lemma 4.2.15 (). *Let $f : X \to Y$ be a perfect map where Y is a locale with some small base. The frame homomorphism f^* respects the way-below relation, that is, $U \ll V$ implies $f^*(U) \ll f^*(V)$, for every pair of opens $U, V : \mathcal{O}(Y)$.*

A proof of this fact can be found in [Esc99] and it works in our predicative setting.

Corollary 4.2.16. *Perfect maps are spectral as they preserve the compact opens.*

Lemma 4.2.17. *For spectral locales X and Y, a continuous map $f : X \to Y$ is spectral if and only if it is perfect.*

Proof. The forward direction is given by Corollary 4.2.16. For the backward direction, assume $f : X \to Y$ to be a spectral map. We have to show that the right adjoint $f_* : \mathcal{O}(X) \to \mathcal{O}(Y)$ of its defining frame homomorphism is Scott continuous. Letting $(U_i)_{i:I}$ be a directed family in $\mathcal{O}(X)$, we show $f_*(\bigvee_{i:I} U_i) = \bigvee_{i:I} f_*(U_i)$. The $\bigvee_{i:I} f_*(U_i) \leq f_*(\bigvee_{i:I} U_i)$ direction is easy. For the $f_*(\bigvee_{i:I} U_i) \leq \bigvee_{i:I} f_*(U_i)$ direction, we appeal to Lemma 4.2.2. Let K be a compact open with $K \leq f_*(\bigvee_{i:I} U_i)$. By the

adjunction $f^* \dashv f_*$, we have that $f^*(K) \leq \bigvee_{i:I} U_i$ and since $f^*(K)$ is compact, by the spectrality assumption of f^*, there exists some $l : I$ such that $f^*(K) \leq U_l$. Again by adjointness, $K \leq f_*(U_l)$ which implies $K \leq \bigvee_{i:I} f_*(U_i)$. □

4.3 Examples of spectral locales

We now examine each of our running examples of locales to determine which are spectral.

4.3.1 The terminal locale

The first example of a spectral locale is the terminal locale:

Example 4.3.1. The terminal locale $\mathbf{1}_{\mathcal{U}}$ defined in Example 3.1.16 is spectral. We have already constructed a base for it in Example 3.5.9. This base satisfies the definition of spectral base from (Definition 3.6.1):

- ($\mathrm{SP_b}1$) It consists of compact opens since $\top_{\mathcal{U}}$ and $\bot_{\mathcal{U}}$ are both compact opens of the terminal locale (as shown in Example 3.3.5 and Lemma 3.3.8).
- ($\mathrm{SP_b}2$) It contains the top open $\top_{\mathcal{U}}$.
- ($\mathrm{SP_b}3$) It is closed under binary meets since the conjunction of every pair of Booleans is a Boolean.

4.3.2 The discrete locale

The discrete locale is a counterexample since it does not satisfy Condition (SP1).

Example 4.3.2. The powerset frame $\mathcal{P}_{\mathcal{U}}(\mathbb{N})$ is not spectral, because it is not compact as discussed in Example 3.3.6.

4.3.3 The Sierpiński locale

The Sierpiński locale is another example of a spectral locale. Proving this, however, is a bit more involved. We first do some preparation.

Lemma 4.3.3. *For every decidable proposition $B : \Omega_{\mathcal{U}}$, the principal filter map*

$$\uparrow B : \Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}}$$

from Definition 3.5.11 is a compact open of the Sierpiński locale.

Proof. Let $B : \Omega_{\mathcal{U}}$ be a decidable proposition and let $(S_i)_{i:I}$ be a directed family of Scott open subsets such that $\uparrow B \subseteq \bigcup_{i:I} S_i$. We proceed by case analysis on proposition B:

- *Case*: $B = \bot$. By Lemma 3.5.13, we know that the principal filter $\uparrow \bot$ is the top

open of the locale $\mathbb{S}$, which we know to be a compact open of the Sierpiński locale from Example 3.3.7.
- *Case*: $B = \top$. In this case, we have $P \in \uparrow\top$ if and only if $P = \top$, for every proposition $P : \Omega_{\mathcal{U}}$. This implies that there is some S_i such that $\top \in S_i$. It must then be the case that every $P \in \uparrow\top$ is in S_i, since $P = \top$ implies $P \in S_i$.

□

Example 4.3.4. The Sierpiński locale $\mathbb{S}_{\mathcal{U}}$ from Example 3.1.23 is spectral. Recall from Example 3.5.14 that we constructed a base for the Sierpiński locale as the family

$$(\uparrow\beta(b))_{b:2},$$

where $\beta : 2 \to \Omega_{\mathcal{U}}$ is the function defined as $\beta(0) :\equiv \top_{\mathcal{U}}$, and $\beta(1) :\equiv \bot_{\mathcal{U}}$. Observe that this base meets the the conditions from Definition 3.6.1:

($\mathrm{SP_b}1$) We showed in Lemma 4.3.3 that this base consists of compact opens.

($\mathrm{SP_b}2$) The top open is in the base since it is equal to $\uparrow\bot$, as mentioned in Lemma 3.5.13.

($\mathrm{SP_b}3$) Let $b_1, b_2 : 2$. The meet $\beta(b_1) \wedge \beta(b_2)$ falls in the base since β maps the Boolean conjunction $b_1 \wedge b_2$ to it.

4.4 Distributive lattices and their spectra

A distributive lattice is a poset with finite meets and finite joins, satisfying the distributivity law [DP02, Definition 4.4]. These are also called *bounded* distributive lattices in the literature [DP02, Definition 2.12]. In this section, we give a predicative construction of the frame of ideals over a small distributive lattice. This is in preparation for Sections 4.7 and 4.8 where we develop a predicative version of the Stone duality between spectral locales and distributive lattices.

4.4.1 The definition of distributive lattice

Unlike our definition of the notion of frame from Definition 3.1.1, we define distributive lattices through their meet and join operations instead of their orders, for reasons that we will explain in Remark 4.4.5. The definition that we give below follows [Joh82, §I.1.4–§I.1.5].

Definition 4.4.1 (Distributive $\mathcal{U}$-lattice). A *distributive $\mathcal{U}$-lattice* consists of

- a set $|L| : \mathcal{U}$,
- an element $\mathbf{1} : |L|$,
- an element $\mathbf{0} : |L|$,
- an operation $(\text{-}) \wedge (\text{-}) : |L| \to |L| \to |L|$ that
  - is associative,
  - is commutative,

  - has $\mathbf{1}$ as a unit element: $x \wedge \mathbf{1} = x$, for every $x : |L|$,
  - is idempotent: $x \wedge x = x$, for every $x : |L|$,
  - satisfies the *absorptive law*:

$$x \wedge (x \vee y) = x,$$

    for every pair of elements $x, y : |L|$,

- an operation $(\text{-}) \vee (\text{-}) : |L| \to |L| \to |L|$ that
  - is associative,
  - is commutative,
  - has $\mathbf{0}$ as a unit element: $x \vee \mathbf{0} = x$, for every $x : |L|$,
  - is idempotent: $x \vee x = x$, for every $x : |L|$,
  - satisfies the *dual absorptive law*:

$$x \vee (x \wedge y) = x,$$

    for every pair of elements $x, y : |L|$.

As usual, we engage in the abuse of notation of denoting the carrier set $|L|$ by L.

Remark 4.4.2. We will occasionally speak of "the distributive lattice structure on type X". By this, we refer to the algebraic structure that the carrier set $|L|$ of a distributive lattice L is equipped with. We do not give a separate formal definition of this, but refer the reader to the discussion in Section 2.10.3 to clarify how we formally implement this structure in type theory.

Lemma 4.4.3. *For every distributive $\mathcal{U}$-lattice L, the relations*

$$x \leq_1 y :\equiv (x \wedge y) = x,$$
$$x \leq_2 y :\equiv (x \vee y) = y$$

form equivalent partial orders with $\mathcal{U}$-small truth values.

Proof. It is direct from the smallness of the carrier set of L that the types $x \wedge y = x$ and $x \vee y = y$ are $\mathcal{U}$-small. The fact that these are equivalent is a well-known fact of lattice theory [DP02, Lemma 2.8], but we nevertheless provide the proof for the sake of self-containment.

Let $x, y : L$ be a pair of elements with $x \wedge y = x$.

$$\begin{aligned} x \vee y &= (x \wedge y) \vee y \\ &= y \vee (y \wedge x) && \text{(commutativity)} \\ &= y && \text{(dual absorptive law)} \end{aligned}$$

Conversely, let $x, y : L$ be a pair of elements with $x \vee y = y$.

$$\begin{aligned} x \wedge y &= x \wedge (x \vee y) \\ &= x && \text{(absorptive law)} \end{aligned}$$

In light of the equivalence, we show just $(\leq_1)$ to be a partial order. Reflexivity is exactly the idempotency of the operation (-) $\wedge$ (-). Antisymmetry is easy to see since for every $x, y : L$, if we have $x \leq_1 y$ and $y \leq_1 x$, then we have $x = (x \wedge y) = y$. Finally, for transitivity, let $x, y, z : L$ with $x \leq_1 y$ and $y \leq_1 z$. We have that

$$\begin{aligned} x \wedge z &= (x \wedge y) \wedge z \\ &= x \wedge (y \wedge z) \\ &= x \wedge y \\ &= x. \end{aligned}$$

□

Remark 4.4.4. Note that sethood is an explicit requirement in the definition of distributive lattice above. Unlike the case for frames, sethood does not follow automatically from the definition in this setting. This is because we rely on the sethood of the carrier type to conclude that the type families $(\leq_1)$, $(\leq_2)$ defined in the above lemma are proposition-valued.

In light of the above lemma, we use the two orders interchangeably. Given a distributive lattice L, we denote its order by (-) $\leq_L$ (-), or simply by (-) $\leq$ (-) if the lattice in consideration is clear from context.

Remark 4.4.5. Observe here that there is a crucial difference compared to the definition of the notion of frame from Definition 3.1.1. One could have defined a notion of distributive $(\mathcal{U}, \mathcal{V})$-lattice with $\mathcal{U}$ being the universe of the carrier set, and $\mathcal{V}$ being the universe of the order. However, the order can always be resized to universe $\mathcal{U}$ by using the equivalence $x \leq y \Leftrightarrow x \wedge y = x$ as explained in Lemma 3.1.3, which would give an equivalent distributive $\mathcal{U}$-lattice as defined above. This is to say that, when working with small lattices, such an ostensibly more general notion of $(\mathcal{U}, \mathcal{V})$-lattice is in fact no more general than the above notion of $\mathcal{U}$-lattice.

Lemma 4.4.6. *For every distributive lattice L, the operations $(\wedge)$ and $(\vee)$ are meet and join operations with respect to the order* (-) $\leq_L$ (-).

Proof. Well-known fact of lattice theory [DP02, Theorem 2.10]. □

Lemma 4.4.7. *In every distributive lattice L, the dual of the distributivity law holds:*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

for every $x, y, z : L$.

Proof. Well-known fact of lattice theory [DP02, Lemma 4.3]. □

Definition 4.4.8 (🤌). A function $h : K \to L$, from a distributive lattice K into a distributive lattice L, is called a *distributive lattice homomorphism* if it preserves the finite meets and the finite joins.

A homomorphism $s : K \to L$ is said to be a *distributive lattice isomorphism* if it has a quasi-inverse $r : L \to K$ that is also a homomorphism.

By $\mathbf{DLat}_{\mathcal{U}}$, we denote the category of distributive $\mathcal{U}$-lattices and their homomorphisms.

We can simplify the characterization of distributive lattice isomorphisms by using the style of Escardó's structure identity principle discussed in Section 2.10.3.

Lemma 4.4.9 (🤌). *A type equivalence $e : |K| \simeq |L|$, between the carrier sets of two distributive lattices, is said to be monotone if both directions are monotone maps.*

The type of monotone equivalences between distributive lattice is equivalent to the type of isomorphisms.

Proof. Every equivalence preserves meets and joins as it has both a right adjoint and a left adjoint. □

4.4.2 Ideals of distributive lattices

We explained in Section 2.12.3 that, in a predicative setting, the notion of subset is parameterized by universes. In other words, we work with subsets described by $\mathcal{U}$-propositions, which we call $\mathcal{U}$-subsets. As a result of this, one could consider a generalized notion of $\mathcal{V}$-ideal over a $\mathcal{U}$-lattice. There may indeed be cases where such generalized ideals are interesting to study, but for the purposes of our work, we will not need this. We accordingly define the following notion of ideal:

Definition 4.4.10 (🤌 Ideal). An ideal of a distributive $\mathcal{U}$-lattice L is a subset $I : \mathcal{P}_{\mathcal{U}}(L)$ satisfying the following conditions:

(I1) It is inhabited i.e. $\exists_{(x:L)} x \in I$.
(I2) It is downward closed i.e. $y \in I$ and $x \leq y$ imply $x \in I$, for every pair of elements $x, y : L$.
(I3) If $x \in I$ and $y \in I$, then $(x \vee y) \in I$, for every pair of elements $x, y : L$.

Lemma 4.4.11. *Being an ideal is a proposition.*

Proof. It is easily verified that each conjunct is a proposition. □

We define

$$\mathsf{Ideal}(L) \quad :\equiv \quad \Sigma_{(I:L\to\Omega_{\mathcal{U}})}\mathsf{is\text{-}ideal}(I).$$

We will usually talk about ideals in contexts with a fixed base universe $\mathcal{U}$. In such contexts, we will occasionally speak of *small ideals* to put extra emphasis on the fact that our ideals are $\mathcal{U}$-valued. The reader should keep in mind, however, that we never work with large ideals. Finally, we also note explicitly that the type $\mathsf{Ideal}(L)$ is large, since $\Omega_{\mathcal{U}}$ lives in universe $\mathcal{U}^+$.

Properties of ideals

Lemma 4.4.12. *Every ideal over a distributive lattice L contains the bottom element of L.*

Proof. Immediate from (I2). □

Lemma 4.4.13. *For every ideal I, the join of a list of elements in I is in I.*

Proof. Straightforward induction on lists. □

Principal ideals

Lemma 4.4.14. *For every distributive $\mathcal{U}$-lattice L and every $x : L$, the subset defined as*

$$\downarrow x : L \to \Omega_{\mathcal{U}}$$
$$\downarrow x :\equiv y \mapsto y \leq_L x$$

is an ideal.

Proof. Let L be a distributive lattice and let $x : L$.

(I1) The subset $\downarrow x$ is always inhabited by the bottom element $\mathbf{0}_L$.

(I2) Downward closure follows from the transitivity of the order.

(I3) For closure under binary joins, let $y, z \in \downarrow x$. It must be the case that $y \vee z \leq x$, since $y \vee z$ is the least upper bound of y and z.

□

Definition 4.4.15 (Principal ideal)**.** The ideal $\downarrow x$ from the above lemma is called the *principal ideal* generated by the element x.

Lemma 4.4.16. *The principal ideal map $\downarrow(\text{-}) : L \to \mathsf{Ideal}(L)$ is an embedding.*

Proof. Let $x, y : L$ such that $\downarrow x = \downarrow y$. Since $x \in \downarrow x$ and $y \in \downarrow y$, we have that $x \leq y$ and $y \leq x$, and we thus know $x = y$ by antisymmetry. □

Intersection of ideals

Subset intersection specializes to a meet operation on ideals since we have:

Lemma 4.4.17 (). *For every pair of small ideals $I, J : L \to \Omega_{\mathcal{U}}$, the intersection $I \cap J$ is a small ideal.*

Thus far, we have concluded that the lattice of $\mathcal{U}$-ideals is a meet-semilattice with respect to the inclusion order. In Section 4.4.3, we will proceed to construct joins in this meet-semilattice, thereby showing that it is a frame.

Lemma 4.4.18 (). *For every pair of elements $x, y : L$, we have $\downarrow x \cap \downarrow y = \downarrow(x \wedge y)$.*

Proof. This is obvious since we have that, $z \leq x \wedge y$ if and only if $z \leq \downarrow x$ and $z \leq \downarrow y$, for every $z : L$. □

Corollary 4.4.19. *The intersection of two principal ideals is a principal ideal.*

4.4.3 Joins in the lattice of ideals

We start by doing some preparation for the construction of joins in the frame of ideals.

Definition 4.4.20 (Finite covering)**.** Let L be a small distributive lattice. We define the *finite covering* type family $(\text{-}) \lhd (\text{-}) : \mathsf{List}(L) \to \mathsf{Fam}_{\mathcal{U}}(\mathsf{Ideal}(L)) \to \mathcal{U}$ by induction on lists as follows:

$$\begin{aligned} \epsilon \lhd I &:\equiv \mathbf{1}_{\mathcal{U}} \\ xt \lhd I &:\equiv \left(\Sigma_{(n:N)} x \in I_n\right) \times (t \lhd I) \end{aligned}$$

Conceptually, the type $(x_0 \dots x_{m-1}) \lhd I$ being inhabited expresses that I contains a finite subcover for each element x_i of the list. However, this involves nontrivial structure and is not a property so it should be thought of as the type of all possible ways in which the family I can give a finite subcover for the list in consideration.

Lemma 4.4.21 (). *For every distributive lattice L, every family of ideals $(I_k)_{k:K}$, and every list $x_0 \dots x_{m-1}$ of elements of L, if we have $(x_0 \dots x_{m-1}) \lhd I$, then, for every j with $0 \leq j < m$, there is some index $k : K$ such that $x_j \in I_k$.*

Proof. Straightforward induction on lists. □

Lemma 4.4.22 (). *For every family of ideals $(I_n)_{n:N}$ and every pair of lists s, t of elements of L, if $s \lhd I$ and $t \lhd I$ then $st \lhd I$.*

Proof. Straightforward induction. □

Lemma 4.4.23. *For every family of ideals $(I_n)_{n:N}$ and every pair of elements $x, y : L$, if $x \lhd I$ then $x \wedge y \lhd I$.*

Proof. Immediate by the downward closure of ideals (Condition (I2)). □

By extending this to lists, we obtain:

Corollary 4.4.24 (🤌). *For every family of ideals $(I_n)_{n:N}$, every element $x : L$, and every list $y_0, \dots, y_{m-1} : \mathsf{List}(L)$, if $(y_0, \dots y_{m-1}) \lhd I$ then $(x \wedge y_0), \dots, (x \wedge y_{m-1}) \lhd I$.*

Given a list $s :\equiv (x_0, \dots x_{m-1})$ of elements of a lattice, we informally abbreviate the join $(x_0 \vee \cdots \vee x_{m-1})$ to $\bigvee_{x \in s} x$.

Lemma 4.4.25 (🤌). *For every small family of ideals $(I_n)_{n:N}$ over a distributive lattice L, the following subset of L is a small ideal:*

$$x \mapsto \exists (y_0 \dots y_{m-1}) : \mathsf{List}(L).\ (y_0 \dots y_{m-1} \lhd I) \times (x = y_0 \vee \cdots \vee y_{m-1}).$$

Proof. Let L be a distributive $\mathcal{U}$-lattice, and let $(I_n)_{n:N}$ a family of small ideals over it. First observe that the type $\exists_{(s : \mathsf{List}(L))} (s \lhd I) \times (x = \bigvee_{y \in s} y)$ is a $\mathcal{U}$-small proposition, for every $x : L$, since $\mathsf{List}(L)$ is a small type and both conjuncts live in universe $\mathcal{U}$.

We now check the conditions of being an ideal.

(I1) The above subset is always inhabited by the bottom element $\mathbf{0}_L$, which can be expressed as the join of the empty list. The empty list is vacuously covered by every family of ideals.

(I2) Consider an element $x : L$ falling in this subset, and let $y : L$ with $y \leq x$. We know that $x = z_0 \vee \cdots \vee z_{m-1}$, for some list $(z_0 \dots z_{m-1}) \lhd I$. To show that y also falls in the above subset, we pick the list $(y \wedge z_0,\ \dots,\ y \wedge z_{m-1})$. By Corollary 4.4.24, we have that

$$(y \wedge z_0)\ \dots\ (y \wedge z_{m-1}) \lhd I.$$

It remains to show that $y = (y \wedge z_0) \vee \cdots \vee (y \wedge z_{m-1})$. This follows from the fact that $y \leq x$ i.e. $y \wedge x = y$, since it implies:

$$\begin{aligned} y &= y \wedge x \\ &= y \wedge (z_0 \vee \dots \vee z_{m-1}) \\ &= (y \wedge z_0) \vee \cdots \vee (y \wedge z_{m-1}). \end{aligned}$$

(I3) Let $x, y : L$ be a pair of elements falling in the above subset. We know that there are lists $a_0 \dots a_{k-1}$ and $b_0 \dots b_{l-1}$ such that

- $x = a_0 \vee \cdots \vee a_{k-1}$ and $(a_0 \dots a_{k-1}) \lhd I$; and also
- $y = b_0 \vee \cdots \vee b_{k-1}$ and $(b_0 \dots b_{k-1}) \lhd I$.

We obviously have

$$x \vee y = a_0 \vee \cdots \vee a_{k-1} \vee b_0 \vee \cdots \vee b_{l-1},$$

and we know by Lemma 4.4.22 that $(a_0 \dots a_{m-1} b_0 \dots b_{n-1}) \lhd I$.

□

Lemma 4.4.26 (). *For every small distributive lattice L, the meet-semilattice of small ideals over L is small complete.*

Proof. Let $(I_n)_{n:N}$ be a small family of ideals. We define the join $\bigvee_{n:N} I_n$ to be the ideal constructed in Lemma 4.4.25. We need to show that this is the least upper bound of the family $(I_n)_{n:N}$.

Let $n : N$. Too see that $I_n \subseteq \bigvee_{n:N} I_n$, let $x \in I_n$. Clearly, $x \lhd I$ since $x \in I_n$ and x is the join of the singleton list consisting of itself.

Let J be an ideal and assume that it is an upper bound of the family $(I_n)_{n:N}$. To see that $(\bigvee_{n:N} I_n) \subseteq J$, let $x \in \bigvee_{n:N} I_n$. This means that

$$x = y_0 \vee \cdots \vee y_{m-1},$$

for some list $(y_0 \dots y_{m-1}) \lhd I$. Therefore, it suffices to show by the closure of ideals under finite joins that each $y_k \in J$. Consider some k with $0 \leq k < m$. We know by Lemma 4.4.21 that there is some $n : N$ such that $y_k \in I_n$. It follows that $y_k \in J$ since we have $I_n \subseteq J$ by the assumption that J is an upper bound of the family of ideals in consideration. □

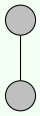
Lemma 4.4.27 (). *For every small distributive lattice L, the small ideals over L form a frame.*

Proof. We have already established in Lemma 4.4.26 that $\mathsf{Ideal}(L)$ is a small-complete meet-semilattice. It is obvious that it is locally small since (1) the inclusion order involves only a quantification over the small set L, and (2) we are working with small ideals.

It remains only to show that binary meets distributive over the joins. Let $I : \mathsf{Ideal}(L)$ and let $(J_n)_{n:N}$ be a small family of ideals over lattice L. We show that the meet $I \wedge (\bigvee_{n:N} J_n)$ is the least upper bound of the family $(I \wedge J_n)_{n:N}$. It is easy to see that it is an upper bound. Consider some index $n : N$. Every $x : L$ with $x \in I \wedge J_n$ obviously satisfies $x \in I$ and $x \in \bigvee_{n:N} J_n$. To see that it is the least upper bound, let K be an ideal and assume that it is an upper bound of the family $(I \wedge J_n)_{n:N}$. We need to show that $I \wedge (\bigvee_{n:N} J_n) \subseteq K$. Let $x \in I \wedge (\bigvee_{n:N} J_n)$. We know that $x = y_0 \vee \cdots \vee y_{m-1}$ for some list $(y_0 \vee \cdots \vee y_{m-1}) \lhd I$. By the closure of ideals under finite joins, it suffices to show that each $y_l \in K$, for $0 \leq l < m - 1$. Notice that each y_l must satisfy (1) $y_l \in I$ by the downward-closure of ideals since we have $y_l \leq x \in I$, and (2) $y_l \in J_n$ for some $n : N$ (by Lemma 4.4.21). This means that we have $y_l \in I \cap J_n$ for some $n : N$ and hence $y_l \in K$ since K is an upper bound of the $(I \wedge J_n)_{n:N}$. □

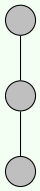
Definition 4.4.28 (). The locale defined by the frame of ideals over a small distributive lattice L, constructed in Lemma 4.4.27, is called the *spectrum* of L. We hereafter denote this by $\mathrm{Spec}(L)$.

Example 4.4.29. Consider the two-element chain 2, given by the Hasse diagram:



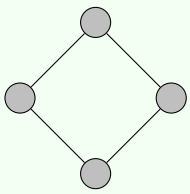
Formally, this can be constructed as the type **2** with meet and join given by Boolean conjunction and disjunction. The locale $\mathrm{Spec}(2)$ is exactly the terminal locale from Example 3.1.16.

Example 4.4.30. Consider the three-element chain 3, given by the Hasse diagram:



The locale $\mathrm{Spec}(3)$ is exactly the Sierpiński locale from Example 3.1.23.

Example 4.4.31. Consider the four-element distributive lattice $\mathbf{M}_2$ given by the Hasse diagram:



The spectrum of this lattice is the discrete locale on **2**. More precisely, when we take the copy of this lattice living in universe $\mathcal{U}$ and construct its frame of $\mathcal{U}$-ideals, we get the frame $\mathcal{P}_{\mathcal{U}}(\mathbf{2})$.

In Section 4.7, we will show that *every spectral locale arises this way*, as the spectrum of some small distributive lattice. This is indeed what justifies the term "spectral" locale. We need to first show, however, that the spectrum is itself a spectral locale. We address this in Section 4.4.4.

4.4.4 Spectrality of the frame of ideals

We now verify that the locale $\mathrm{Spec}(L)$ from the previous section satisfies the definition of spectral locale. We start with some preparation.

Base construction for the spectrum

Lemma 4.4.32. *The top open of the locale* $\mathrm{Spec}(L)$ *is the principal ideal* $\downarrow \mathbf{1}_L$*, for every small distributive lattice* L*.*

Proof. It is obvious that the ideal $\downarrow \mathbf{1}_L$ contains every element of L. It is thus equal to the full subset of L by propositional extensionality. □

Lemma 4.4.33 (🤌). *For every directed family of ideals* $(I_n)_{n:N}$ *over a small distributive lattice* L*, and for every list* $x_0 \dots x_{m-1}$ *of elements of* L*, if* $(x_0 \dots x_{m-1}) \lhd I$*, then there is some index* $n : N$ *such that* $(x_0 \vee \cdots \vee x_{m-1}) \in I_n$*.*

Proof sketch. Let $(I_n)_{n:N}$ be a directed family of ideals over some small distributive lattice L, and let $x_0 \dots x_{m-1}$ be a list of elements of L. We know that for each x_k in the list, there is some index $n_k : N$ with $x \in I_{n_k}$. By the directedness of the family $(I_n)_{n:N}$, there must be an upper bound of the collection of ideals we get for the x_k's in the list. Since this upper bound contains each x_k, it must contain the join $x_0 \vee \cdots \vee x_{m-1}$ by Lemma 4.4.13. □

Lemma 4.4.34 (🤌). *For every small distributive lattice* L *and every element* $x : L$*, the principal ideal* $\downarrow x$ *is a compact open of the locale* $\mathrm{Spec}(L)$*.*

Proof. Let L be a small distributive lattice and let $x : L$. To see that $\downarrow x$ is compact, let $(I_n)_{n:N}$ be a directed family of small ideals with $\downarrow x \subseteq \bigvee_{n:N} I_n$. We need to show that there is some index $n : N$ such that $\downarrow x \subseteq I_n$. Since $\downarrow x \subseteq \bigvee_{n:N} I_n$, it must be the case that $x \in \bigvee_{n:N} I_n$ as $x \in \downarrow x$. This is to say that $x = y_0 \vee \cdots \vee y_{m-1}$, for some list $(y_0 \dots y_{m-1}) \lhd I$. By Lemma 4.4.33, we know that there must be some ideal I_n such that $(y_0 \vee \cdots \vee y_{m-1}) \in I_n$. This means that $x \in I_n$, which was what we needed. □

Lemma 4.4.35 (🤌). *The locale* $\mathrm{Spec}(L)$ *is compact.*

Proof. Since $\downarrow \mathbf{1}_L$ is the top open of $\mathrm{Spec}(L)$ (by Lemma 4.4.32), it suffices to show that this is compact. This follows directly from Lemma 4.4.34. □

Lemma 4.4.36 (🤌). *For every small distributive lattice* L*, the principal ideal map*

$$\downarrow(\text{-}) : L \to \mathsf{Ideal}(L)$$

forms a small, extensional, and strong base for the locale $\mathrm{Spec}(L)$*.*

Proof. We have already shown in Lemma 4.4.16 that this map is an embedding and we know that the type L is small. It remains to show that this map forms a strong base.

Let $I : \mathsf{Ideal}(L)$. We let the basic covering family for I be

$$\left(\Sigma_{(x:L)} x \in I\right) \xhookrightarrow{\mathrm{pr}_1} L \xrightarrow{\downarrow(\text{-})} \mathsf{Ideal}(L).$$

It is easy to see that this is a small subfamily of $\downarrow(\text{-}) : L \to \mathsf{Ideal}(L)$. The fact that I is an upper bound of this family is easy to see from the downward closure of ideals (i.e. Condition (I2)). To see that it is the *least* upper bound, consider some ideal J and suppose that it is an upper bound of the above family. For every $x \in I$, we know $\downarrow x \subseteq J$ by the assumption that J is an upper bound. It follows that $x \in J$ since $x \in \downarrow x$. We thus have $I \subseteq J$, which was what we needed. □

Spectrality of the base

Having constructed a base for the spectrum in Lemma 4.4.36, we now proceed to show that this base is spectral.

Lemma 4.4.37 (🤌). *The base for* $\mathsf{Spec}(L)$*, constructed in Lemma 4.4.36, is a spectral base, for every small distributive lattice* L*.*

Proof.

- ($\mathrm{SP_b}1$) The fact that this base consists of compact opens was already established in Lemma 4.4.34.
- ($\mathrm{SP_b}2$) This base contains the top open since the top open is equal to $\downarrow \mathbf{1}_L$, as shown in Lemma 4.4.32.
- ($\mathrm{SP_b}3$) Let $x, y : L$. By Lemma 4.4.18, we know that $\downarrow x \cap \downarrow y = \downarrow(x \wedge y)$ meaning the meet $\downarrow x \cap \downarrow y$ falls in the base.

□

We have therefore established that $\mathsf{Spec}(L)$ satisfies the equivalent conditions from Theorem 4.2.11. Note also that we have the following:

Lemma 4.4.38 (🤌). *We have an equivalence*

$$\mathsf{K}(\mathsf{Spec}(L)) \quad \simeq \quad \Sigma_{(I:\mathsf{Ideal}(L))} \exists_{(x:L)} (I = \downarrow x).$$

In other words, the type of compact opens of $\mathsf{Spec}(L)$ *is equivalent to its type of principal ideals.*

Proof. By Corollary 4.2.5, the image of $\downarrow(\text{-}) : L \to \mathsf{Ideal}(L)$ is equivalent to the type of compact opens of $\mathsf{Spec}(L)$, since it is a base consisting of compact opens as shown in Lemma 4.4.36. □

4.5 Zero-dimensional and regular locales

We now consider the classes of *zero-dimensional* and *regular* locales, in preparation for the development of the theory of Stone locales in Section 4.6. Some of the results we will prove about Stone locales generalize to to regular locales, of which the zero-dimensional locales are a special case (Lemma 3.6.6). Although we are not interested in the theory of zero-dimensional and regular locales in itself, it will be useful in Section 4.6 to have collected some key properties of these classes of locales.

4.5.1 Regular locales

We briefly mentioned the notion of regular locale in Definition 3.6.5, where we defined it to be a locale X that has an unspecified base $(B_i)_{i:I}$ satisfying the condition that, for every $U : \mathcal{O}(X)$, the basic covering family for U consists of opens well inside U.

We start by recalling that the way-below relation implies the well-inside relation in regular locales:

Lemma 4.5.1 (🤌). *In every regular locale X, if $U \ll V$ then $U \Subset V$, for every pair of opens $U, V : \mathcal{O}(X)$.*

Proof. Let X be a regular locale. Since the conclusion is a proposition, we may appeal to the induction principle of propositional truncation and assume that we have a specified base $(B_i)_{i:I}$ satisfying the condition of regularity. Let U and V be a pair of opens with $U \ll V$, and let $(B_{i_j})_{j:J}$ be the basic covering family for V. Since $V = \bigvee_{j:J} B_{i_j}$, there exists some $j : J$ such that $U \leq B_{i_j}$, by the assumption that $U \ll V$. We thus have $U \leq B_j \Subset V$, which implies $U \Subset V$ by Lemma 3.4.7. □

Since an open U is compact if and only if $U \ll U$, and clopen if and only if $U \Subset U$, we obtain the immediate corollary below.

Corollary 4.5.2 (🤌). *In every regular locale, compact opens are clopen.*

When combined with Corollary 3.4.9, this gives:

Lemma 4.5.3. *In every compact regular locale, the compact opens coincide with the clopens.*

Proof. By Corollary 3.4.9, we know that clopens are compact in compact locales. Corollary 4.5.2 above tells us that the backward implication holds in regular locales. Since the types expressing being clopen and being compact are both propositions, this logical equivalence implies that we have a type equivalence $\mathsf{K}(X) \simeq \mathsf{C}(X)$. □

4.5.2 Zero-dimensional locales

A *zero-dimensional space*, in the context of point-set topology, is a topological space whose class of clopen subsets forms a base for the topology [DP02, Exercise 11.1] [HW41,

Chapter II]. The notion of dimension involved here is that of *inductive dimension*, which goes back to Urysohn [Ury25] and Menger [Men28]. We refrain from discussing the general notion of inductive dimension for topological spaces, and refer the reader to [HW41, Chapter III] for further reading on the subject.

We already mentioned the point-free analogue of this notion of zero-dimensionality in Section 3.6, where we defined a zero-dimensional locale as a locale that has some small base consisting of clopens. We now revisit the predicative manifestation of the notion of zero-dimensional locale and examine it in detail.

In Theorem 4.2.11, we proved that several alternative characterizations of the notion of spectral locale all coincide. The above definition merits the question of whether an analogous characterization can be obtained for zero-dimensional locales. Unlike the case for spectral locales, however, it turns out that *there is no analogue* of Definition 4.2.1 for zero-dimensionality (as we will soon show in Example 4.5.7). This illustrates how Theorem 4.2.11 is reliant on the compactness of the locale in consideration.

Motivated by the impredicative definition of zero-dimensional locale as a locale in which the clopens form a base, one may consider the definition below as a candidate for a predicative definition. We refrain from using the term zero-dimensional for this notion, the reason for which will be explained shortly.

Definition 4.5.4. A locale X is said to be *clopen-based* if

(CB1) the inclusion $\mathsf{C}(X) \overset{\mathsf{pr}_1}{\hookrightarrow} \mathcal{O}(X)$ forms a weak base, and
(CB2) the type $\mathsf{C}(X)$ is small.

Example 4.5.5. The terminal locale $\mathbf{1}_{\mathcal{U}}$ is clopen-based: observe that the base from Example 3.5.9 contains both of the decidable propositions $\top_{\mathcal{U}}$ and $\bot_{\mathcal{U}}$. Furthermore, the type $\mathsf{C}(\mathbf{1}_{\mathcal{U}})$ is small since the type of decidable propositions is equivalent to the type $\mathbf{2}$.

It is easy to see that the property of being clopen-based is stronger than zero-dimensionality:

Lemma 4.5.6. *Every clopen-based locale is zero-dimensional.*

Proof. Let X be a clopen-based locale. We show that the inclusion $\mathsf{C}(X) \overset{\mathsf{pr}_1}{\hookrightarrow} \mathcal{O}(X)$ forms a small, intensional, and strong base. This family consists of clopens by construction. For every open $U : \mathcal{O}(X)$, the covering family for it is given by the inclusion

$$\left(\Sigma_{(C:\mathsf{C}(X))} C \leq U\right) \overset{\mathsf{pr}_1}{\hookrightarrow} \mathcal{O}(X), \tag{$\dagger$}$$

which is easily seen to be a small family thanks to local smallness. It is clear U is an upper bound of this family. To see that it is the least upper bound, let V be an upper bound of the family. We need to show that $U \leq V$. By the weak base assumption, we know there is an unspecified family of clopens covering U. As the conclusion is a proposition, we may appeal to the induction principle of propositional truncation and work with a specified basic covering family $(C_i)_{i:I}$. Since we know $U = \bigvee_{i:I} C_i$,

we just have to show that V is above every C_i in the family. This follows from our assumption that V is an upper bound of the family (†). □

Seeing that the property of being clopen-based is stronger than the property of being zero-dimensional, it is natural to ask whether we want to pick the former or the latter as the predicative manifestation of zero-dimensionality. We answer this through the following example[2]:

Example 4.5.7. The locale $\mathbf{1}_{\mathcal{U}}^{\neg\neg}$ is zero-dimensional but not clopen-based unless $\Omega_{\neg\neg}$-resizing holds.

Proof. The base from Example 3.5.9 also forms a base for $\mathbf{1}_{\mathcal{U}}^{\neg\neg}$, but the type of clopens is a large type since every proposition in $\Omega_{\mathcal{U}}^{\neg\neg}$ is a clopen (of the locale $\mathbf{1}_{\mathcal{U}}^{\neg\neg}$). In other words, whilst the collection of clopens of $\mathbf{1}_{\mathcal{U}}^{\neg\neg}$ does form a base, it is not a small collection. Requiring it to be small amounts to requiring $\Omega_{\mathcal{U}}^{\neg\neg}$ to be $\mathcal{U}$-small, so it is easy to see that this locale being clopen-based gives $\Omega_{\neg\neg}$-resizing. □

It is a reasonable desideratum of the definition of zero-dimensionality that the locale $\mathbf{1}_{\mathcal{U}}^{\neg\neg}$ satisfy it. We thus conclude that the notion of clopen-based locale is overly restrictive in our predicative setting and adopt Definition 3.6.2 as the right definition of this notion. When the locale in consideration is compact, the two notions coincide:

Lemma 4.5.8. *A compact locale X is clopen-based if and only if it is zero-dimensional.*

Proof. We have already established the forward direction in Lemma 4.5.6, which holds irrespective of compactness. For the backward direction, consider a compact and zero-dimensional locale X. By Lemma 4.5.3, we know that there is a type equivalence $\mathsf{K}(X) \simeq \mathsf{C}(X)$ since the zero-dimensionality of X implies that it is regular. This means that the zero-dimensional base $(B_i)_{i:I}$ of X consists of compact opens, and we thus know that there is an equivalence

$$\mathsf{K}(X) \simeq \mathsf{image}(B) \simeq \mathsf{C}(X),$$

since $\mathsf{K}(X) \simeq \mathsf{image}(B)$ by Corollary 4.2.5. This implies that $\mathsf{C}(X)$ is a small type so we are done. □

4.6 Stone locales

Having pinned down the right notion of zero-dimensional locale in Section 4.5.1, we are now ready to define the class of *Stone locales*:

Definition 4.6.1 (). A locale X is called *Stone* if it is compact and zero-dimensional.

[2]This example was suggested to the author by Igor Arrieta (private communication).

Although this definition is perfectly sufficient for our purposes, it is illustrative to look at alternative characterizations as we did for spectral locales in Theorem 4.2.11. We must also justify that the above definition is equivalent to the definition of Stone locale from Definition 3.6.3.

Lemma 4.6.2 (). *In every Stone locale, the compact opens coincide with the clopens.*

Proof. Every Stone locale is clearly compact and regular, since zero-dimensionality implies regularity by Lemma 3.6.6. We thus know by Lemma 4.5.3 that the compact opens coincide with the clopens. □

4.6.1 Stone bases are spectral

We recall Definition 3.6.3 where we defined a Stone locale as a locale with an unspecified Stone base. A *Stone base* for a locale X is a base $(B_i)_{i:I}$ such that

- every basic open B_i is both compact and clopen, and
- the top open $\mathbf{1}_X$ is an element of the base.

It is easy to see that having an unspecified Stone base implies being Stone:

Lemma 4.6.3. *If a locale X has an unspecified Stone base then X is Stone as in the above definition i.e. is compact and zero-dimensional.*

Proof. Zero-dimensionality is obvious. For compactness, note that the top open $\mathbf{1}_X$ being in the base implies that it is compact, since the base is stipulated to consist of compact opens. □

Lemma 4.6.4. *In every locale X with an unspecified Stone base, every clopen is compact.*

Proof. Let X be a locale with an unspecified Stone base. By Lemma 4.6.3, we know that X is compact. Corollary 3.4.9 says that clopens are compact in every compact locale. □

Lemma 4.6.5. *Every Stone base is spectral.*

Proof. Let X be a locale and let $(B_i)_{i:I}$ a specified Stone base for it. Conditions ($\mathrm{SP_b}1$) and ($\mathrm{SP_b}2$) from Definition 3.6.1 clearly hold by construction. It remains to show Condition ($\mathrm{SP_b}3$) i.e. the fact that the base $(B_i)_{i:I}$ is closed under binary meets. Consider two indices $i, j : I$. The basic opens B_i and B_j are both clopen, and so their meet $B_i \wedge B_j$ must also be clopen by Lemma 3.4.4. Since we know by Lemma 4.6.4 that clopens are compact in the presence of a Stone base, the meet $B_i \wedge B_j$ must be compact and must hence fall in the base, since compact opens are basic in every locale with a base (Lemma 4.2.4). □

Lemma 4.6.6. *For every locale X and every base $(B_i)_{i:I}$ for X, the following are equivalent:*

1. *the base $(B_i)_{i:I}$ is Stone,*
2. *the base $(B_i)_{i:I}$ is spectral and also zero-dimensional,*
3. *the locale X is compact and the base $(B_i)_{i:I}$ is zero-dimensional.*

Proof. We have already established (1) ⇒ (3) in Lemmas 4.6.3 and 4.6.5. For the converse, first observe that the top open $\mathbf{1}_X$ must fall in the base since compact opens are basic in every locale equipped with a base (by Lemma 4.2.4). Furthermore, Corollary 3.4.9 says that clopens are compact in every compact locale, meaning each B_i must be compact in addition to being clopen.

The implication (2) ⇒ (3) is direct. Moreover, we also have (1) ⇒ (2), since it is immediate by construction that every Stone base is zero-dimensional and we have already shown in Lemma 4.6.5, that Stone bases are spectral. □

4.6.2 Alternative characterization

In our discussion of spectral locales in Section 4.2, we mentioned that the formulation of the notion of spectral locale in Definition 4.2.1 is the obvious translation of the standard, impredicative notion of spectral locale into our predicative setting. Furthermore, we also stressed that the equivalence of Definition 3.6.1 with Definition 4.2.1 is crucial to our development.

It is natural to ask if there is a definition analogous to Definition 4.2.1 for Stone locales. We have in fact already given such a definition in Lemma 4.5.8 where we proved that zero-dimensionality and being clopen-based coincide when the locale at hand is compact. The latter can be taken to be a definition similar to Definition 4.2.1 since it amounts to the following conditions:

(ST1) Compactness.

(ST2) The inclusion $\mathsf{C}(X) \overset{\mathsf{pr}_1}{\hookrightarrow} \mathcal{O}(X)$ forming a weak base.

(ST3) The type $\mathsf{C}(X)$ being small.

Lemma 4.6.7. *Every locale that satisfies conditions (ST1)–(ST3) is Stone as in Definition 4.6.1.*

Proof. This follows directly from Lemma 4.5.8. □

Lemma 4.6.8 (🤌). *Every Stone locale is spectral.*

Proof. By Lemma 4.6.7, we know that every Stone locale is compact and has a specified zero-dimensional base. By Lemma 4.6.6, this specified base must be one that is also spectral. □

Theorem 4.6.9. *For every locale X, the following are logically equivalent, assuming univalence and the set replacement principle.*

1. *Locale X has an unspecified Stone base.*
2. *Locale X is compact and zero-dimensional.*
3. *Locale X is compact and clopen-based.*
4. *Locale X is compact and has an unspecified extensional small base of clopens.*
5. *Locale X is compact and the inclusion $\mathsf{C}(X) \hookrightarrow X$ is an extensional base of clopens, where to an open U we assign the family of all clopens below it as in the construction of Lemma 4.5.6.*
6. *Locale X is compact and has a specified intensional small base of clopens.*
7. *Locale X is compact and has a specified extensional small base of clopens.*

Proof. We make use of Theorem 4.2.11. We have already established that each of the conditions in the statement implies one of the equivalent conditions from Theorem 4.2.11:

- Conditions (1) and (2) both imply spectrality, which was shown in Lemma 4.6.6.
- Condition (3) is equivalent to Condition (2) of Theorem 4.2.11, as shown in Lemma 4.5.8.
- Conditions (5), (6), and (7) clearly imply, respectively, Conditions (1), (2), and (4) of Theorem 4.2.11.

We know by Corollary 4.2.5 that, for every locale X with a base $(B_i)_{i:I}$ consisting of compact opens, $\mathsf{K}(X) \simeq \mathsf{image}(B)$. It is also easy to see that each of the conditions in the statement implies zero-dimensionality, meaning $\mathsf{K}(X) \simeq \mathsf{C}(X)$ by Lemma 4.5.3. From this, we can observe that each condition in the statement is the corresponding condition (as given in the above implications) with the added requirement of consisting of clopens. □

Notice also that the first five of the above conditions are propositions, but the last two are not in general.

4.6.3 The category of Stone locales

We can now define the category of Stone locales. Unlike the category **Spec** of spectral locales, there is no special notion of morphism for Stone locales thanks to the following:

Lemma 4.6.10. *Every continuous map reflects clopens.*

Proof. Let $f : X \to Y$ be a continuous map of locales and consider a clopen $F : \mathcal{O}(Y)$. We denote the complement of F by F'. The application $f^*(F)$ must also be clopen, since it is complemented by $f^*(F')$:

$$f^*(F) \wedge f^*(F') = f^*(F \wedge F') = f^*(\mathbf{0}_Y) = \mathbf{0}_X,$$
$$f^*(F) \vee f^*(F') = f^*(F \vee F') = f^*(\mathbf{1}_Y) = \mathbf{1}_X.$$

□

Since the clopens coincide with the compact opens in Stone locales, we have the following corollary:

Corollary 4.6.11. *Every continuous map $f : X \to Y$ of Stone locales is a spectral map.*

We denote by $\mathbf{Stone}_{\mathcal{U}}$ the category of Stone locales and continuous maps over the base universe $\mathcal{U}$. If the base universe is clear from context, we drop the subscript and simply write **Stone**.

Lemma 4.6.12. *The category* $\mathbf{Stone}_{\mathcal{U}}$ *is a full subcategory of* $\mathbf{Spec}_{\mathcal{U}}$.

Proof. We have already established in Lemma 4.6.8 that every Stone locale is spectral, meaning this is a subcategory. That it is *full* is given by Corollary 4.6.11. □

4.6.4 Examples and counterexamples

The terminal locale

Example 4.6.13. The terminal locale from Example 3.1.16 is Stone. It was shown to be compact in Example 3.3.5, It was shown to be clopen-based in Lemma 4.5.6, and it is hence zero-dimensional by Lemma 4.5.6.

The discrete locale

Example 4.6.14. The discrete locale is not Stone in general. As explained in Example 4.3.2, the powerset frame $\mathcal{P}_{\mathcal{U}}(\mathbb{N})$ is not spectral so it cannot be Stone.

The Sierpiński locale

The *Sierpiński locale* is an illustrative example of a locale that is not Stone *despite* being spectral.

Example 4.6.15. The Sierpiński locale $\mathbb{S}_{\mathcal{U}}$ is not Stone.

Proof. We explained in Example 4.3.4 that the Sierpiński locale has exactly one nontrivial compact open: the principal filter $\uparrow\top$. This is the singleton $\{\top\}$ since a proposition falls in this subset if and only if it is inhabited. Suppose towards a contradiction that the singleton $\{\top\}$ has a complement i.e. there existed some $F : \mathcal{O}(\mathbb{S}_{\mathcal{U}})$ satisfying

$$\{\top\} \cap F = \emptyset, \tag{†}$$
$$\{\top\} \cup F = \mathcal{O}(\mathbb{S}_{\mathcal{U}}). \tag{‡}$$

We know by (‡) that every proposition $P : \Omega_{\mathcal{U}}$ satisfies either $P = \top$ or $P \in F$. Combining this with (†), we obtain that $P \in F$ if and only if $\neg(P = \top)$ i.e. $P = \bot$. This

is to say that F is the singleton subset $\{\bot\}$. Notice, however, that the Scott openness of the subset F implies that we also have $\top \in F$, since Scott open subsets are upward closed. It follows that $\bot = \top$, a contradiction. □

Example 4.6.16. The Sierpiński locale is spectral but *not* Stone. Its spectrality was given in Example 4.3.4, and we established in the above lemma that that it is not Stone.

4.7 Stone duality for spectral locales

We showed in Lemma 4.4.37 that there is a spectral locale associated with every small distributive lattice L. This is the spectrum $\mathrm{Spec}(L)$ whose defining frame is the frame of small ideals over L. In this section, we will extend this to show:

1. that this is the case for every spectral locale i.e. *every* spectral locale is homeomorphic the spectrum of some small distributive lattice, namely its small distributive lattice of compact opens,
2. that every distributive lattice is isomorphic to the small lattice of compact opens of its spectrum.

Thanks to this, we will add yet another equivalent condition to the list of conditions from Theorem 4.2.11: being homeomorphic to $\mathrm{Spec}(L)$ for some small distributive lattice L. We will thus obtain maps

$$\mathbf{Spec}_{\mathcal{U}} \underset{\mathrm{Spec}(\text{-})}{\overset{\mathsf{K}}{\rightleftarrows}} \mathbf{DLat}_{\mathcal{U}},$$

where $\mathbf{Spec}_{\mathcal{U}}$ denotes the type of large, locally small, and small-complete spectral locales, and $\mathbf{DLat}_{\mathcal{U}}$ denotes the type of small distributive lattices (over base universe $\mathcal{U}$), and then show that these maps form an equivalence.

Although we extend these maps to functors and show that they form a categorical equivalence in Section 4.8, we note that just the type equivalence that we present in this section is quite useful in the context of univalent foundations, since it implies that these two types are the same in a strong sense. For example, one can show that, for every pair X and Y of spectral locales, we have an equivalence

$$(X \cong Y) \simeq (\mathsf{K}(X) \cong \mathsf{K}(Y)),$$

between the type $X \cong Y$ of locale homeomorphisms and the type $\mathsf{K}(X) \cong \mathsf{K}(Y)$ of distributive lattice isomorphisms, and use this to show that two locales are homeomorphic by showing that their distributive lattices of compact opens are isomorphic.

4.7.1 Transporting a lattice structure along an equivalence

In Definition 2.13.1, we defined the notion of $\mathcal{V}$-smallness: a type $X : \mathcal{U}$ is called $\mathcal{V}$-small if it has an equivalent copy in universe $\mathcal{V}$. Consider a type X equipped with some

algebraic structure. If we have an equivalence $e : X \simeq Y$, we can *transport* the algebraic structure through e to obtain an equivalent copy of the algebraic structure on Y. If the equivalence here is between a larger universe and a smaller one, this gives us a way to *resize the algebraic structure* in consideration by constructing an isomorphic copy in the smaller universe.

Lemma 4.7.1 (🤌). *For every distributive $\mathcal{U}$-lattice L and every $\mathcal{V}$-type X, if there is an equivalence $|L| \simeq X$, then there is a distributive lattice structure σ on X such that we have an isomorphism $L \cong (X, \sigma)$.*

Proof. Let L be a distributive $\mathcal{U}$-lattice, and let $A : \mathcal{V}$ be a type such that we have a specified equivalence $e : |L| \simeq A$. Denote by $s : |L| \to A$ and $r : A \to |L|$ the two directions of this equivalence. We define the following lattice structure on A:

$$\begin{aligned} \mathbf{0}_A &:\equiv s(\mathbf{0}_L) \\ \mathbf{1}_A &:\equiv s(\mathbf{1}_L) \\ a \wedge_A b &:\equiv s(r(a) \wedge_L r(b)) \\ a \vee_A b &:\equiv s(r(a) \vee_L r(b)) \end{aligned}$$

It is easy to see that these define a distributive lattice structure on A. We denote this distributive $\mathcal{V}$-lattice by L'. The isomorphism between L and L' is given by the equivalence e itself. It remains to show that both directions of the equivalence are homomorphisms of distributive lattices. The map s preserves $\mathbf{0}_L$ and $\mathbf{1}_L$ by construction. To see that it preserves meets and joins, let $x, y : L$. Since r is a retraction of s, we have:

$$\begin{aligned} s(x \wedge_L y) &= s(r(s(x)) \wedge_L y) \\ &= s(r(s(x)) \wedge_L r(s(y))) \\ &\equiv s(x) \wedge_A s(y) \end{aligned} \qquad \begin{aligned} s(x \vee_L y) &= s(r(s(x)) \vee_L y) \\ &= s(r(s(x)) \vee_L r(s(y))) \\ &\equiv s(x) \vee_A s(y) \end{aligned}$$

Conversely, the fact that r is a lattice homomorphism follows directly from the fact that it is a retraction of s. □

4.7.2 Distributive lattice of compact opens of a spectral locale

We now explicitly define the distributive lattice of compact opens of a spectral locale.

Lemma 4.7.2 (🤌). *For every spectral locale X, the type $\mathsf{K}(X)$ forms a distributive lattice (which is a priori large).*

Proof. The frame structure on $\mathcal{O}(X)$ can be restricted to operations:

$$\begin{aligned} (\text{-}) \wedge_K (\text{-}) &: \mathsf{K}(X) \to \mathsf{K}(X) \to \mathsf{K}(X) \\ (\text{-}) \vee_K (\text{-}) &: \mathsf{K}(X) \to \mathsf{K}(X) \to \mathsf{K}(X) \\ \mathbf{1}_K &: \mathsf{K}(X) \\ \mathbf{0}_K &: \mathsf{K}(X) \end{aligned}$$

This is thanks to the following:

- the bottom open $\mathbf{0}_X$ is always compact by Lemma 3.3.8,
- compact opens are always closed under binary joins (again by Lemma 3.3.8),
- the meet of two compact opens is compact in spectral locales by (SP2),
- the top open is compact in spectral locales by (SP1).

□

In Definition 4.2.1, we required the type $\mathsf{K}(X)$ of a spectral locale X to be small, which means that, for every spectral locale X, we have that $\mathsf{K}(X)$ is equivalent to a small type. This means that we can *resize* the distributive lattice of compact opens from the above lemma.

Lemma 4.7.3 (🦔). *For every spectral locale X, the large distributive lattice $\mathsf{K}(X)$ from the above lemma has an isomorphic small copy.*

Proof. We appeal to Lemma 4.7.1 with the equivalence stipulated in Condition (SP4). □

Remark 4.7.4. In the formalization, we carefully distinguish between the large and small copies of the distributive lattice of compact opens. In our presentation here, however, we abstract over this technical detail, as it makes the presentation rather cumbersome. We will hereafter make implicit use of the fact that the distributive lattice $\mathsf{K}(X)$ has a $\mathcal{U}$-small copy.

The upshot here is that the conditions (SP1)–(SP4) ensure, when combined, that the type $\mathsf{K}(X)$ forms a small distributive lattice. We denote by

$$\mathsf{K} : \mathbf{Spec}_{\mathcal{U}} \to \mathbf{DLat}_{\mathcal{U}}$$

the map sending a spectral locale to its small distributive lattice of compact opens, as constructed in the above lemma.

Technically, the above map is obtained as the composition:

$$\mathbf{Spec}_{\mathcal{U}} \longrightarrow \mathbf{DLat}_{\mathcal{U}^+} \longrightarrow \mathbf{DLat}_{\mathcal{U}},$$

where the first map $\mathbf{Spec}_{\mathcal{U}} \to \mathbf{DLat}_{\mathcal{U}^+}$ takes a spectral locale to its a priori *large* distributive lattice of compact opens, and the second one takes this distributive lattice to its small copy.

Remark 4.7.5. We will sometimes have to distinguish between the two maps above. When such technical precision is required, we will use the notations

- K^+ for the map $\mathbf{Spec}_{\mathcal{U}} \to \mathbf{DLat}_{\mathcal{U}^+}$, and
- K^- for the composition $\mathbf{Spec}_{\mathcal{U}} \to \mathbf{DLat}_{\mathcal{U}}$.

The reader is advised to bear this distinction in mind, which will play a particularly important rôle in Section 4.8.

4.7.3 Object part of Stone duality

So far, we have constructed functions:

$$\mathbf{Spec}_{\mathcal{U}} \underset{\mathsf{Spec}(\text{-})}{\overset{\mathsf{K}}{\rightleftarrows}} \mathbf{DLat}_{\mathcal{U}},$$

between the type of spectral locales and the type of distributive lattices. We now show that these form an equivalence of types.

The use of the term "equivalence" requires some caution here as we are constructing only the object part at this point. To be more specific, we will prove two results:

1. For every $\mathcal{U}$-lattice L, there is a distributive lattice isomorphism $L \cong \mathsf{K}(\mathsf{Spec}(L))$.
2. For every spectral locale X, there is a locale homeomorphism $X \cong \mathsf{Spec}(\mathsf{K}(X))$.

Under the assumption of univalence, the above isomorphisms can be turned into equalities using the structure identity principle, establishing that $\mathsf{K}(\text{-})$ is a retraction of $\mathsf{Spec}(\text{-})$ and vice versa. This allows us to extend the above to a *type equivalence*:

$$\mathbf{Spec}_{\mathcal{U}} \simeq \mathbf{DLat}_{\mathcal{U}}.$$

Lemma 4.7.6 (🖐). *Every small distributive lattice L is isomorphic to the lattice* $\mathsf{K}(\mathsf{Spec}(L))$.

Proof. Let L be a small distributive lattice. Recall that we have a map

$$\downarrow(\text{-}) : L \to \mathsf{K}(\mathsf{Spec}(L))$$

since principal ideals generated by the elements of L are compact in $\mathsf{Spec}(L)$ (by Lemma 4.4.34). We claim that this map forms a distributive lattice isomorphism. Recall that, in Lemma 4.4.38 we showed that for every compact open $K : \mathsf{Spec}(L)$, we have that there exists an element $x : L$ satisfying $K = \downarrow x$. Because the principal ideal map is an embedding by Lemma 4.4.16, we know that its fibres are propositions. This concludes that it is an equivalence as its fibres are contractible. By Lemma 4.4.9, it suffices to show that both sides of this equivalence are monotone to establish an isomorphism. We already showed in Lemma 4.4.18, that the principal ideal map preserves meets, which implies that it is monotone. The other direction is easy to see. □

Lemma 4.7.7 (🤌). *Every spectral locale X is homeomorphic to* $\mathsf{Spec}(\mathsf{K}(L))$.

Proof. Let X be a spectral locale over base universe $\mathcal{U}$. We know that there is a map $\phi : \mathcal{O}(X) \to \mathsf{Ideal}_{\mathcal{U}}(\mathsf{K}(X))$, defined (🤌) by

$$\phi(U) \quad :\equiv \quad \{C : \mathsf{K}(X) \mid C \leq U\},$$

which is a small collection thanks to the smallness of $\mathsf{K}(X)$. Note that this is the principal ideal restricted to the small collection $\mathsf{K}(X)$. We claim that this forms a frame isomorphism $\mathcal{O}(X) \cong \mathcal{O}(\mathsf{Spec}(L))$. The inverse of this is simply the map sending an ideal $I \subseteq \mathsf{K}(X)$ to the join of the family $\left(\Sigma_{(K:\mathsf{K}(X))} K \in I\right) \hookrightarrow \mathsf{K}(X)$. We denote this map by $\vartheta : \mathsf{Ideal}_{\mathcal{U}}(\mathsf{K}(X)) \to \mathcal{O}(X)$. It suffices to show that ϑ and ϕ are inverses.

Let $I \subseteq \mathsf{K}(X)$ be a $\mathcal{U}$-ideal over the distributive $\mathcal{U}$-lattice of compact opens of X. We first show that $I = \phi(\vartheta(I))$. We use antisymmetry. The nontrivial direction is $\phi(\vartheta(I)) \subseteq I$. Let K_1 be a compact open with $K_1 \leq \bigvee I$. Since ideals are directed, we know that there exists some compact $K_2 \in I$ with $K_1 \leq K_2$. By the downward closure of I, we have that $K_1 \in I$. For the other direction, consider an open $U : \mathcal{O}(X)$. The fact that $U = \vartheta(\phi(U))$ follows directly from Condition (4) of Theorem 4.2.11. □

It is easily verified that frame isomorphisms give a *standard notion of structure* for frames as discussed in Section 2.10.3. We thus obtain the following from the structure identity principle:

Lemma 4.7.8 (🤌). *For every pair of locales X and Y, there is an equivalence*

$$(X \cong Y) \simeq (X = Y),$$

where $X \cong Y$ denotes the type of homeomorphisms betwen X and Y.

Similarly for distributive lattice, we can show:

Lemma 4.7.9 (🤌). *For every pair of distributive lattices K and L, there is an equivalence*

$$(K \cong L) \simeq (K = L),$$

where $K \cong L$ denotes the type of isomorphisms betwen K and L.

From these, we get a type equivalence:

Theorem 4.7.10 (🤌). *The type* $\mathbf{Spec}_{\mathcal{U}}$ *is equivalent to the type* $\mathbf{DLat}_{\mathcal{U}}$.

Proof. We turn the isomorphism given by Lemma 4.7.6 and the homeomorphism given by Lemma 4.7.7 into equalities using Lemma 4.7.8 and Lemma 4.7.9. □

In the context of univalent foundations, the above is a rather strong statement. It implies that these two mathematical structures satisfy exactly the same properties.

4.8 Categorical part of Stone duality

So far, we have shown that the maps

$$\mathbf{Spec}_{\mathcal{U}} \underset{\text{Spec}}{\overset{\text{K}}{\rightleftarrows}} \mathbf{DLat}_{\mathcal{U}},$$

form an equivalence of *types*. In this section, we show that this can be extended to an equivalence of categories. Before we proceed to the proof, however, we first discuss the definition of the notion of categorical equivalence in HoTT/UF.

4.8.1 Categorical equivalences in HoTT/UF

Recall that the standard definition [Awo10, Definition 7.25] of an equivalence between categories $\mathbb{C}$ and $\mathbb{D}$ is as a pair of functors

$$\mathbb{C} \underset{F}{\overset{E}{\rightleftarrows}} \mathbb{D}$$

equipped with natural isomorphisms $\alpha : \mathbf{1}_{\mathbb{C}} \to F \circ E$ and $\beta : \mathbf{1}_{\mathbb{D}} \to E \circ F$. It is a well-known fact of category theory [Awo10, Proposition 7.26] that a functor $E : \mathbb{C} \to \mathbb{D}$ being part of an equivalence in the above sense is equivalent to it being fully faithful and essentially surjective.

In a constructive foundational setting, some caution must be exercised regarding whether to formulate the notion of an essentially surjective functor using specified or unspecified existence. In the HoTT Book, the formulation using specified existence is called a *split* essentially surjective functor:

Definition 4.8.1 (Split essentially surjective functor [Uni13, Definition 9.4.4])**.** A functor $F : \mathbb{C} \to \mathbb{D}$ is called *split essentially surjective* if, for every object $D : \mathbb{D}$, there is a *specified* object $C : \mathbb{C}$ such that $F(C) \cong D$.

For the sake of self-containment, we also provide the definitions of full and faithful functors.

Definition 4.8.2 ([Uni13, Definition 9.4.4])**.** Let $F : \mathbb{C} \to \mathbb{D}$ be a functor. If the mapping

$$F : \mathrm{Hom}(C_1, C_2) \to \mathrm{Hom}(F(C_1), F(C_2))$$

is an embedding for every pair of objects C_1 and C_2, functor F is called *faithful*. Similarly, F is called *full* if this mapping is surjective, for every pair of objects C_1 and C_2. If F is both full and faithful, it is called *fully faithful*.

Lemma 4.8.3 ([Uni13, Lemma 9.4.5]). *For every functor $E : \mathbb{C} \to \mathbb{D}$, the following are equivalent:*

1. *E is part of an equivalence of categories.*
2. *E is fully faithful and split essentially surjective.*

For our purposes, it will be convenient to make use of the above characterization of categorical equivalence when showing that the categories $\mathbf{Spec}_{\mathcal{U}}$ and $\mathbf{DLat}_{\mathcal{U}}$ are equivalent. More specifically, we will establish the categorical equivalence of interest by showing that the map $\mathsf{K} : \mathbf{Spec}_{\mathcal{U}} \to \mathbf{DLat}_{\mathcal{U}}$ extends to a fully faithful and split essentially surjective functor.

4.8.2 The functorial action of K^-

Recall that a spectral map $f : X \to Y$ is a continuous function whose defining frame homomorphism maps the compact opens of Y to the compact opens of X. More formally, the spectrality of a continuous map $f : X \to Y$ is witnessed by some function

$$\varphi : \Pi_{(V : \mathcal{O}(Y))} \, (V \text{ is compact} \Rightarrow f^*(V) \text{ is compact}).$$

We define the map $\mathsf{K}_f :\equiv (K, \kappa) \mapsto (f^*(K), \varphi_K(\kappa))$, where (K, κ) is a pair consisting of an open $K : \mathcal{O}(Y)$ equipped with some proof κ that it is compact. The function K_f maps such a pair to the pair $(f^*(K), \varphi_K(\kappa))$, where $\varphi_K(\kappa)$ is the witness that the result $f^*(K)$ is compact. This obviously gives a lattice homomorphism

$$\mathsf{K}_f : \mathsf{K}(Y) \to \mathsf{K}(X).$$

We thus have a mapping from the Hom-set of $\mathbf{Spec}_{\mathcal{U}}$ to the Hom-set of $\mathbf{DLat}_{\mathcal{U}^+}$:

$$\mathsf{K}_{(\text{-})} : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(\mathsf{K}(Y), \mathsf{K}(X)).$$

Although this is morally the definition of the functorial action of $\mathsf{K} : \mathbf{Spec}_{\mathcal{U}} \to \mathbf{DLat}_{\mathcal{U}}$ on the morphisms of $\mathbf{Spec}_{\mathcal{U}}$, we also need to account for the fact that we want to obtain a morphism in the category $\mathbf{DLat}_{\mathcal{U}}$ of *small* of distributive lattices. Recalling the distinction between K^+ and K^-, which we discussed Remark 4.7.5, the issue here is that the above definition is for the map $\mathsf{K}^+ : \mathbf{Spec}_{\mathcal{U}} \to \mathbf{DLat}_{\mathcal{U}^+}$, and not for the map $\mathsf{K}^- : \mathbf{Spec}_{\mathcal{U}} \to \mathbf{DLat}_{\mathcal{U}}$.

Given spectral locales X and Y, we know by the definition of spectrality that we have equivalences $e_X : \mathsf{K}^+(X) \simeq \mathsf{K}^-(X)$ and $e_Y : \mathsf{K}^+(Y) \simeq \mathsf{K}^-(Y)$, which are homomorphisms of distributive lattices as we showed in Lemma 4.7.3. We will use these isomorphisms to transport the functorial action from $\mathrm{Hom}(\mathsf{K}^+(Y), \mathsf{K}^+(X))$ to $\mathrm{Hom}(\mathsf{K}^-(Y), \mathsf{K}^-(X))$.

In the remainder of this section, we will use the notations

$$\mathsf{K}^+(X) \underset{r_X}{\overset{s_X}{\rightleftarrows}} \mathsf{K}^-(X) \qquad \mathsf{K}^+(Y) \underset{r_Y}{\overset{s_Y}{\rightleftarrows}} \mathsf{K}^-(Y),$$

to refer to the two directions of these equivalences. The diagram in Figure 4.1 summarizes this state of affairs.

Figure 4.1: Spectral locales X and Y and the resizing maps for their distributive lattices of compact opens.

$$\begin{array}{ccc}
\mathcal{O}(Y) & \xrightarrow{f^*} & \mathcal{O}(X) \\
{\scriptstyle \mathrm{pr}_1}\uparrow & & \uparrow{\scriptstyle \mathrm{pr}_1} \\
\mathsf{K}^+(Y) & \xrightarrow{\mathsf{K}^+_f} & \mathsf{K}^+(X) \\
{\scriptstyle r_Y}\uparrow\ \downarrow{\scriptstyle s_Y} & & {\scriptstyle r_X}\uparrow\ \downarrow{\scriptstyle s_X} \\
\mathsf{K}^-(Y) & \xrightarrow{\mathsf{K}^-_f} & \mathsf{K}^-(X)
\end{array}$$

Definition 4.8.4 (Functorial action of K^-)**.** Let $f : X \to Y$ be a spectral map, whose spectrality is formally witnessed by a proof

$$\varphi : \Pi_{(V : \mathcal{O}(Y))} \, (V \text{ is compact} \Rightarrow f^*(V) \text{ is compact}).$$

The functorial action $\mathsf{K}^-_f : \mathsf{K}^-(Y) \to \mathsf{K}^-(X)$ is defined as $\mathsf{K}^-_f :\equiv s_X \circ \mathsf{K}^+_f \circ r_Y$, which amounts to the map $K \mapsto s_X(f^*(\mathrm{pr}_1(r_Y(K))), \varphi(\mathrm{pr}_2(r_Y(K))))$.

Although we gave the fully formal definition above, we will generally not be as precise. For example, we will often omit applications of the projections for the sake of readability.

Lemma 4.8.5. *For every spectral map $f : X \to Y$ between spectral locales X and Y, the map $\mathsf{K}^-_f : \mathsf{K}^-(Y) \to \mathsf{K}^-(X)$ is a homomorphism of distributive lattices.*

Proof. In Lemma 4.7.3, we established that the equivalences $\mathsf{K}(Y) \simeq \mathsf{K}^-(Y)$ and $\mathsf{K}(X) \simeq \mathsf{K}^-(X)$ are homomorphisms of distributive lattices. The desired result follows from this combined with the fact that f^* is a frame homomorphism. □

We have thus completed the definition of the functor $\mathsf{K}^- : \mathbf{Spec}_{\mathcal{U}} \to \mathbf{DLat}_{\mathcal{U}}$.

4.8.3 The functor K^- gives a categorical equivalence

We are now ready to establish the desired categorical equivalence by showing that the functor K^- is (1) fully faithful and (2) split essentially surjective.

Lemma 4.8.6. *The functor $\mathsf{K}^- : \mathbf{Spec}_{\mathcal{U}} \to \mathbf{DLat}_{\mathcal{U}}$ is split essentially surjective.*

Proof. We need to show that, for every small distributive lattice L, there is a specified spectral locale X such that $\mathsf{K}^-(X) \cong L$. This was already established in Lemma 4.7.6, where we showed that the distributive lattice of compact opens of $\mathrm{Spec}(L)$, which is small, is isomorphic to L. □

Before we proceed to showing that the functor in consideration is fully faithful, we first record the following lemma in preparation.

Lemma 4.8.7 (🦚). *A monotone endomap $h : \mathcal{O}(X) \to \mathcal{O}(X)$ on a spectral locale X is Scott continuous whenever it satisfies the following condition: for every $U : \mathcal{O}(X)$ and compact $K : \mathcal{O}(X)$ with $K \leq h(U)$, there is some compact $K' \leq U$ such that $K \leq h(K')$.*

Proof. It suffices to show the relation $h\left(\bigvee_{i:I} U_i\right) \leq \bigvee_{i:I} h(U_i)$ holds. Since X is spectral, let $(K_j)_{j:J}$ be a small family of compact opens such that $\bigvee_{j:J} K_j = h\left(\bigvee_{i:I} U_i\right)$. For any $j : J$, we have $K_j \leq h\left(\bigvee_{i:I} U_i\right)$, so by assumption there is a compact open $K \leq \bigvee_{i:I} U_i$ such that $K_j \leq h(K)$. By compactness of K there is an $i : I$ such that $K \leq U_i$, and so $K_j \leq h(K) \leq h(U_i) \leq \bigvee_{i:I} h(U_i)$ and so we can take $K' :\equiv K_j$. □

Since the map (-) ∨ (-) is obviously Scott continuous on both arguments, we have the following useful corollary:

Corollary 4.8.8. *Let X be spectral locale and consider two opens $U, V : \mathcal{O}(X)$. For every compact open K with $K \leq U \vee V$, there exist compact opens K_1 and K_2 satisfying $K \leq K_1 \vee K_2$, and also $K_1 \leq U$ and $K_2 \leq V$.*

Lemma 4.8.9. *Let X and Y be two spectral locales and let $h : \mathsf{K}^-(Y) \to \mathsf{K}^-(X)$ be a homomorphism of distributive lattices. The map*

$$\mathsf{Spec}_h^*(V) :\equiv \bigvee \{r_X(h(K)) \mid K : \mathsf{K}^-(Y),\ r_Y(K) \leq V\}$$

is a frame homomorphism $\mathcal{O}(Y) \to \mathcal{O}(X)$.

Proof. The fact that the map Spec_h^* preserves finite meets is easy to see. We focus on (1) the preservation of joins and (2) spectrality.

(1) Let $(V_i)_{i:I}$ be a directed family of opens in Y. It is a well-known fact of lattice theory that a function preserves arbitrary joins if and only if it preserves finite joins as well as directed ones. If we assume that the family is directed, we can easily see that we have

$$\bigvee \left\{ r_X(h(K)) \mid K : \mathsf{K}^-(Y),\ r_Y(K) \leq \left(\bigvee_{i:I} V_i \right) \right\} = \bigvee_{i:I} \mathsf{Spec}_h^*(V_i)$$

by the compactness of $r_Y(K)$. For the case of binary joins, consider two opens V_1 and V_2. We need to show that $\mathsf{Spec}_h^*(V_1 \vee V_2) = \mathsf{Spec}_h^*(V_1) \vee \mathsf{Spec}_h^*(V_2)$. The

$$\mathsf{Spec}_h^*(V_1) \vee \mathsf{Spec}_h^*(V_2) \leq \mathsf{Spec}_h^*(V_1 \vee V_2)$$

direction is easy to see, so we focus on the other direction. We have to show that $\bigvee \{r_X(h(K)) \mid K : \mathsf{K}^-(Y),\ r_Y(K) \leq V_1 \vee V_2\}$ is below the join

$$\left(\bigvee \{r_X(h(K)) \mid K : \mathsf{K}^-(Y),\ r_Y(K) \leq V_1\} \right) \vee \left(\bigvee \{r_X(h(K)) \mid K : \mathsf{K}^-(Y),\ r_Y(K) \leq V_2\} \right).$$

First, observe that this is the same as the join

$$\left(\bigvee \{r_X(h(K_1 \vee K_1)) \mid K_1, K_2 : \mathsf{K}^-(Y),\ r_Y(K_1) \leq V_1,\ r_Y(K_2) \leq V_2\}\right)$$

by distributivity. Let $K : \mathsf{K}^-(Y)$ such that $r_Y(K) \leq V_1 \vee V_2$. We know by Corollary 4.8.8 that there exist compact K_1, K_2 such that $r_Y(K_1) \leq V_1$ and $r_Y(K_2) \leq V_2$ as well as $r_Y(K) \leq K_1 \vee K_2$. This implies that $r_X(h(K)) \leq r_X(h(K_1 \vee K_2))$ and also $r_Y(K_1) \leq V_1$ and $r_Y(K_2) \leq V_2$, which was exactly what we needed. □

Lemma 4.8.10. *Let X and Y be two spectral locales and let $h : \mathsf{K}^-(Y) \to \mathsf{K}^-(Y)$ be a distributive lattice homomorphism between them. We then have that*

$$\mathrm{Spec}_h^*(K) = r_X(h(s_Y(K))),$$

for every $K : \mathsf{K}^-(Y)$.

Proof. Let $h : \mathsf{K}^-(Y) \to \mathsf{K}^-(Y)$ be a distributive lattice homomorphism between the distributive lattices of compact opens of spectral locales X and Y, and consider some $K : \mathsf{K}^-(Y)$. We first show $\mathrm{Spec}_h^*(K) \leq r_X(h(s_Y(K)))$. Let $K' : \mathsf{K}^-(Y)$ such that $r_Y(K') \leq K$. Since we have $r_Y \dashv s_Y$, we know that $K' \leq s_Y(K)$, from which it follows that $r_X(h(K')) \leq r_X(h(s_Y(K)))$ since $r_X \circ h$ is monotone. For the

$$r_X(h(s_Y(K))) \leq \mathrm{Spec}_h^*(K)$$

direction, it suffices to observe that we have $r_Y(s_Y(K)) \leq K$, since $r_Y(s_Y(K)) = K$. □

Corollary 4.8.11. *For every distributive lattice homomorphism $h : \mathsf{K}^-(Y) \to \mathsf{K}^-(Y)$, where X and Y are spectral locales, the frame homomorphism Spec_h^* preserves compact opens, since $r_X(h(s_Y(K)))$ is a compact open for every compact open $K : \mathcal{O}(Y)$. This means that Spec_h is a spectral map.*

Lemma 4.8.12. *The map $\mathrm{Spec}_{(\text{-})} : \mathrm{Hom}(\mathsf{K}^-(Y), \mathsf{K}^-(X)) \to \mathrm{Hom}(X, Y)$ is the quasi-inverse of the map $\mathsf{K}^-_{(\text{-})} : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(\mathsf{K}^-(Y), \mathsf{K}^-(X))$*

Proof. We start by showing that $\mathrm{Spec}_{(\text{-})}$ is a retraction of $\mathsf{K}^-_{(\text{-})}$. Let $f : X \to Y$ be a spectral map and let $V : \mathcal{O}(Y)$. We need to show that $\mathrm{Spec}_{\mathsf{K}^-_f}(V) = f^*(V)$. First, observe that

$$\begin{aligned}
\mathrm{Spec}_{\mathsf{K}^-_f}(V) &\equiv \bigvee \left\{r_X(\mathsf{K}^-_f(K)) \mid K : \mathsf{K}^-(Y),\ r_Y(K) \leq V\right\} \\
&\equiv \bigvee \{r_X(s_X(f^*(r_Y(K)))) \mid K : \mathsf{K}^-(Y),\ r_Y(K) \leq V\} \\
&= \bigvee \{f^*(r_Y(K)) \mid K : \mathsf{K}^-(Y),\ r_Y(K) \leq V\}
\end{aligned}$$

We know that V can be decomposed into a join $\bigvee \{K \mid K : \mathcal{O}(Y),\ K \leq V\}$, and it is

easily verified that

$$f^*\left(\bigvee \{K \mid K : \mathcal{O}(Y),\, K \leq V\}\right) = \bigvee \{f^*(r_Y(K)) \mid K : \mathsf{K}^-(Y),\, r_Y(K) \leq V\}\,.$$

We now show that $\mathrm{Spec}_{(\text{-})}$ is a section of $\mathsf{K}^-_{(\text{-})}$. Let $h : \mathsf{K}^-(Y) \to \mathsf{K}^-(X)$ be a homomorphism of distributive lattices. We need to show that $\mathsf{K}^-_{\mathrm{Spec}_h}(K) = h(K)$, for every $K : \mathsf{K}^-(Y)$. First, observe the definitional equality

$$\begin{aligned} \mathsf{K}^-_{\mathrm{Spec}_h}(K) &\equiv s_X(\mathrm{Spec}^*_h(r_Y(K))) \\ &\equiv s_X\left(\bigvee \{r_X(h(K')) \mid K' : \mathsf{K}^-(Y),\, r_Y(K') \leq r_Y(K)\}\right) \qquad (\dagger) \end{aligned}$$

We first show that the join in $(\dagger)$ is below $h(K)$, for which it suffices to show

$$\left(\bigvee \{r_X(h(K')) \mid K' : \mathsf{K}^-(Y),\, r_Y(K') \leq r_Y(K)\}\right) \leq r_X(h(K)),$$

since r_X is the right adjoint of s_X. It suffices to show that $r_X(h(K')) \leq r_X(h(K))$, for every $K' : \mathsf{K}^-(Y)$ with $r_Y(K') \leq r_Y(K)$. Consider such a $K' : \mathsf{K}^-(Y)$ satisfying $r_Y(K') \leq r_Y(K)$. Since r_Y is an order embedding, we have that $K \leq K$ and it follows that $r_X(h(K')) \leq r_X(h(K))$ as $r_X \circ h$ is clearly a monotone map. We now address the other direction of the equality in consideration, namely,

$$h(K) \leq s_X\left(\bigvee \{r_X(h(K')) \mid K' : \mathsf{K}^-(Y),\, r_Y(K') \leq r_Y(K)\}\right).$$

As s_X is the right adjoint of r_X, it suffices to show

$$r_X(h(K)) \leq \bigvee \{r_X(h(K')) \mid K' : \mathsf{K}^-(Y),\, r_Y(K') \leq r_Y(K)\}\,,$$

which is immediate. □

Corollary 4.8.13. *The functor* $\mathsf{K}^- : \mathbf{Spec}_{\mathcal{U}} \to \mathbf{DLat}_{\mathcal{U}}$ *is fully faithful.*

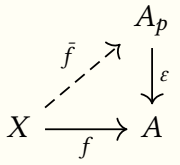
Chapter 5

The patch locale

We defined the categories **Spec** and **Stone** in Chapter 4, and explained that the latter is a full subcategory of the former. In this chapter, we delve deeper into the relationship between these two categories and examine how a spectral locale can be universally transformed into a Stone one in a predicative setting.

In an impredicative foundational setting, it is well known that the category **Stone** is a *coreflective subcategory* of the category **Spec** [Esc99; Esc01]. This coreflection amounts to the fact that every spectral locale can be universally transformed into a Stone one. The coreflector involved here is what is known as the patch locale of a spectral locale: the localic manifestation of the patch topology on a spectral space (also called the *constructible topology* in the literature [DST19, §1.3.11]). This can be thought of as the universal locale in which all compact opens of a spectral locale are transformed into clopens. Formally, this is captured by the universal property below.

Definition 5.0.1 (Universal property of the patch locale)**.** Let A be a spectral locale. A Stone locale A_p is said to be a *patch locale* of A if there is a specified spectral map $\varepsilon : A_p \to A$, such that, for every continuous map $f : X \to A$, from a Stone locale X into A, there exists a continuous map $\bar{f} : X \to A_p$, unique with the property that $f = \varepsilon \circ \bar{f}$. This is summarized by the diagram:



It is not a priori clear that the patch locale of every spectral locale can be constructed in a constructive and predicative foundational setting, since all proofs of this coreflection in the literature seem to use some form of impredicativity. The focus of this chapter is on showing that every spectral locale can be coreflected into a Stone one as given in

the above universal property. To achieve this, we follow an idea due to Escardó [Esc99; Esc01], that the patch locale can be defined by the frame of Scott continuous nuclei on a spectral locale. This description of the patch locale turns out to be amenable to constructive and predicative reasoning. Although our construction is mostly the same, Escardó's proof of the universal property [Esc99] of the patch locale relies on the existence of the frame of *all* nuclei, which can easily be constructed in an impredicative foundational setting. In our predicative setting, where the frame of all nuclei does not seem to be available, the proof of the universal property is new, and turns out to be more complicated.

The organization of this chapter is as follows:

Section 5.1: We present the poset of all nuclei, and show that it forms a meet-semilattice. Furthermore, we discuss the problem that this poset does not seem to be locally small without some form of propositional resizing.

Section 5.2: We explain that the meet-semilattice of all nuclei from Section 5.1 can be shown to be *complete* in an impredicative setting, and we survey the impredicative approaches to constructing joins in this lattice. Furthermore, we discuss the problem that the construction of these joins does not seem to be predicatively possible.

Section 5.3: We discuss a sub-meet-semilattice of the meet-semilattice of nuclei, namely the sublattice consisting of *Scott continuous* nuclei. Furthermore, we show that Escardó's construction of joins [Esc98] in this frame can easily be translated into our predicative setting. We thus conclude that this sublattice is small-complete *even in* a predicative setting, and that it forms a frame.

Section 5.4: We prove that the locale defined by the frame of Scott continuous nuclei from Section 5.3 satisfies the universal property of the patch locale from Definition 5.0.1, hence establishing that **Stone** is a coreflective subcategory of **Spec** in a predicative setting.

Section 5.5: We present several examples of the patch locale and discuss its lattice-theoretic interpretation through Stone duality.

5.1 Meet-semilattice of nuclei on a locale

5.1.1 Pointwise ordering of nuclei

Let X be a locale over base universe $\mathcal{U}$ and consider the type $\mathrm{Nucleus}(X)$, which lives in universe $\mathcal{U}^+$. This type forms a poset when ordered pointwise:

Lemma 5.1.1. *For every locale X (over base universe $\mathcal{U}$), the pointwise ordering,*

$$j \leq k \quad :\equiv \quad \Pi_{(U:\mathcal{O}(X))}\, j(U) \leq k(U),$$

is a partial order with large truth values i.e. truth values living in universe $\mathcal{U}$.

Definition 5.1.2. The *poset of nuclei* on a locale X is the type $\mathrm{Nucleus}(X)$ partially ordered by the above order.

Remark 5.1.3. Although the above order seems to have necessarily large truth values, we do not have a resizing taboo establishing this at the time of writing. We conjecture that the local smallness of this poset gives a form of resizing.

5.1.2 Meets of nuclei

Lemma 5.1.4. *For every locale X, the poset of nuclei on X has finite meets.*

Proof. Let X be a locale. The top nucleus is defined as the constant map with value $\mathbf{1}_X$ and the meet of two nuclei as $j \wedge k :\equiv U \mapsto j(U) \wedge k(U)$. It is easy to see that $j \wedge k$ is the greatest lower bound of the nuclei j and k, so it remains to show that $j \wedge k$ satisfies the conditions of being a nucleus.

Inflationarity can be seen to be satisfied from the inflationarity of j and k, combined with the fact that $j(U) \wedge k(U)$ is the greatest lower bound of $j(U)$ and $k(U)$. To see that meet preservation holds, let $U, V : \mathcal{O}(X)$.

$$\begin{aligned}
(j \wedge k)(U \wedge V) &\equiv j(U \wedge V) \wedge k(U \wedge V) \\
&= j(U) \wedge j(V) \wedge k(U) \wedge k(V) && \text{[meet-preservation]} \\
&= (j(U) \wedge k(U)) \wedge (j(V) \wedge k(V)) && \text{[commutativity]} \\
&\equiv (j \wedge k)(U) \wedge (j \wedge k)(V)
\end{aligned}$$

For idempotency, let $U : \mathcal{O}(X)$.

$$\begin{aligned}
& (j \wedge k)((j \wedge k)(U)) \\
\equiv\ & j(j(U) \wedge k(U)) \wedge k(j(U) \wedge k(U)) \\
=\ & j(j(U)) \wedge j(k(U)) \wedge k(j(U)) \wedge k(k(U)) && \text{[meet-preservation]} \\
\leq\ & j(j(U)) \wedge k(k(U)) \\
=\ & j(U) \wedge k(U) && \text{[idempotency]} \\
\equiv\ & (j \wedge k)(U)
\end{aligned}$$

□

5.2 Nuclei form a frame, impredicatively

In the traditional development of locale theory in impredicative foundations, the *frame of nuclei* is a standard construction. Johnstone [Joh82, §II.2.5] shows that the poset of nuclei forms a frame by constructing arbitrary meets and Heyting implications in it. He remarks in passing that "joins in [the frame of nuclei] are harder to describe explicitly" [Joh82, p. 51]. Let us first examine this difficulty involved in the explicit description of these joins, before looking at the details of Johnstone's construction.

Following the case of meets, one might attempt to define the join of a family of nuclei $(k_i)_{i:I}$ as the pointwise join, i.e.

$$\left(\bigvee_{i:I} k_i\right)(x) \quad :\equiv \quad \bigvee_{i:I} k_i(x).$$

However, this fails to be idempotent in general, and is not inflationary when the family in consideration is empty. There are several approaches to constructing these joins in the literature, but all of these involve some kind of impredicativity. We first discuss the aforementioned approach by Johnstone.

Suppose we were working with complete and small locales, the theory of which can be developed impredicatively as explained in Section 3.1.4. Let L be a complete and small frame. If L is meet-complete, then L is also join-complete since we could easily construct the joins as

$$\left(\bigvee_{i:I} x_i\right) \quad :\equiv \quad \bigwedge \{y : L \mid y \text{ is an upper bound of } (x_i)_{i:I}\},$$

which would be a small collection when the carrier set in consideration is small. Therefore, showing meet-completeness is sufficient to establish join-completeness in the context of complete, small lattices. Observe here that we must also have local smallness for this to be a small collection. As discussed in Lemma 3.1.3, local smallness follows automatically from the smallness of the carrier set in the context of complete, small lattices. In a predicative foundational setting, such a construction of joins from meets does not seem to be possible, since the above construction seems fundamentally reliant on the meet over the entire carrier set. Even if this collection were somehow small, we would still have the problem that the order does not have small truth values in general (see Remark 5.1.3).

Taking advantage of the definability of joins this way, Johnstone constructs meets to establish join-completeness. He defines [Joh82, §II.2.5] the meets pointwise:

$$\left(\bigwedge_{i:I} k_i\right) \quad :\equiv \quad x \mapsto \bigwedge_{i:I} k_i(x).$$

Moreover, to show that this join-complete lattice is a frame, Johnstone proceeds to construct Heyting implications. The existence of Heyting implications implies that the frame distributivity law holds, i.e.

$$x \wedge \left(\bigvee_{i:I} y_i\right) \quad = \quad \bigvee_{i:I} x \wedge y_i,$$

for every $x : L$ and every small family $(y_i)_{i:I}$. This is because $x \wedge (\text{-})$ is the left adjoint of $x \Rightarrow (\text{-})$, and left adjoints always preserve colimits. Therefore, the existence of the right adjoint of $x \wedge (\text{-})$ implies that it preserves the colimits of L. There is a second use of impredicativity at this point of the argument, since Johnstone's construction of Heyting implications [Joh82, p. 52] also uses a meet over a large collection. He defines Heyting implications as

$$(j \Rightarrow k)(x) \quad :\equiv \quad \bigwedge \{(j(y) \Rightarrow k(y)) \mid y : L, x \leq y\},$$

using the Heyting implications of the underlying frame L. In a predicative context, there is no reason for this collection to be small in general.

There are several other constructions of joins of nuclei in the literature. It seems to be the case, however, that all of these use some form of impredicativity (or classical principles):

- Banaschewski [Ban88] and Wilson [Wil94] both use constructions involving large meets, similar to Johnstone's construction discussed above.
- Simmons [Sim81] uses ordinals and transfinite induction, and his proof seems to be reliant on classical reasoning (though we have not carefully analyzed it).
- Escardó [Esc03] uses Pataraia's fixed point theorem, which also does not seem to be predicatively available. de Jong and Escardó show [dJE21, Theorem 52] that the main lemma used in the proof of Pataraia's fixed point theorem *implies* a form of propositional resizing.

In light of the fact that all of the above constructions involve some form of impredicativity, we conjecture that the meet-semilattice of nuclei cannot be shown to be small-complete in our setting, unless a form of propositional resizing holds.

In the next section, we will start working towards the predicative construction of the patch locale, the defining frame of which is the aforementioned frame of Scott continuous nuclei.

5.3 The frame of Scott continuous nuclei

We now restrict attention to those nuclei that are Scott continuous and show that, whilst joins in the lattice of *all* nuclei seem to involve an inherent impredicativity, joins in this sublattice can be constructed in a completely predicative manner. Furthermore, this sublattice consisting of Scott continuous nuclei also does not suffer from the problem of not being locally small, which we discussed in Remark 5.1.3.

As mentioned before, our motivation for constructing the frame of Scott continuous nuclei is to define the patch locale of a spectral locale, the universal property of which was given in the introduction to this chapter (Definition 5.0.1). The fact that the patch locale can be constructed this way is due to Escardó [Esc99; Esc01].

5.3.1 Meet-semilattice of Scott continuous nuclei

We now refine the meet-semilattice from Section 5.1 to those nuclei that are Scott continuous.

Definition 5.3.1. A nucleus $j : \mathcal{O}(X) \to \mathcal{O}(X)$ is called *Scott continuous* if it preserves joins of small directed families.

Example 5.3.2. For every open $U : \mathcal{O}(X)$ of a locale X, the closed nucleus $\mathbf{c}(U)$ (from Example 3.2.2) is Scott continuous since the map $U \vee (\text{-})$ preserves all joins, and in particular the directed ones.

Example 5.3.3 (). For every compact open $K : \mathsf{K}(X)$ of a spectral locale X, the open nucleus $\mathbf{o}(K)$ is Scott continuous.

Proof. Let $K : \mathsf{K}(X)$ be a compact open of some spectral locale X. We need to show that the map $K \Rightarrow (\text{-})$ is Scott continuous. As we know that this is a nucleus, we know that it is monotone since nuclei are always monotone (Lemma 3.2.7). We show that this map is Scott continuous by showing that it satisfies the sufficient condition from Lemma 4.8.7. Let V and K_1 be two opens of X and suppose that K_1 satisfies $K_1 \leq K \Rightarrow V$. We need to show that there exists some compact $K_2 \leq K$ such that $K_1 \leq K \Rightarrow K_2$. Pick $K_2 :\equiv K \wedge K_1$. We know that this is compact since compact opens are closed under binary meets in spectral locales. It remains to check $K_2 \leq V$ and $K_1 \leq K \Rightarrow K_2$. The former holds directly by the assumption that $K_1 \leq K \Rightarrow V$, since this implies $K_1 \wedge K \leq V$. The latter is immediate. □

Poset of Scott continuous nuclei

We now show that the order of the poset of Scott continuous nuclei on a spectral locale is small, in contrast with the case of all nuclei.

Lemma 5.3.4. *For every pair of nuclei $j, k : \mathcal{O}(X) \to \mathcal{O}(X)$ on some locale X, if*

1. *the locale X is spectral, and*
2. *the nuclei j and k are Scott continuous,*

then the pointwise ordering $\Pi_{(U:\mathcal{O}(X))} j(U) \leq k(U)$ is $\mathcal{U}$-small.

Proof. Let X be a spectral locale and let $j, k : \mathcal{O}(X) \to \mathcal{O}(X)$ be a pair of Scott continuous nuclei on X. We define the following small version of the pointwise order:

$$j \leq_s k \quad :\equiv \quad \Pi_{(K:\mathsf{K}(X))} j(K) \leq k(K),$$

which involves only a small quantification thanks to the smallness of the type of compact opens. We claim that $j \leq k$ if and only if $j \leq_s k$. The forward direction is trivially true, so it remains to show that $j \leq_s k$ implies $j \leq k$. Suppose $j \leq_s k$. To see that $j \leq k$, let $U : \mathcal{O}(X)$. As we have a base consisting of compact opens, we know that there is a directed basic covering family $(B_n)_{n:N}$ for U. Therefore, we just have to show

$$j\left(\bigvee_{n:N} B_n\right) \quad \leq \quad k\left(\bigvee_{n:N} B_n\right).$$

By the Scott continuity of the nuclei j and k, it suffices to show

$$\bigvee_{n:N} j(B_n) \quad \leq \quad \bigvee_{n:N} k(B_n).$$

It follows directly from our assumption of $j \leq_s k$ that $j(B_n) \leq k(B_n)$ for each $n : N$, so we are done. □

In summary, we have now established:

Corollary 5.3.5. *For every spectral locale X, the poset of Scott continuous nuclei on X is large and locally small.*

Meets of Scott continuous nuclei

Lemma 5.3.6. *The poset of Scott continuous nuclei on every locale forms a meet-semilattice.*

Proof. Let X be a locale. The construction is the same as the one from Lemma 5.1.4. The top element is the constant map with value $\mathbf{1}_X$, which is trivially Scott continuous, so it remains to show that the meet of two Scott continuous nuclei is Scott continuous. Let j and k be a pair of Scott continuous nuclei, and consider a small directed family $(U_n)_{n:N}$. We then have:

$$\begin{aligned}
(j \wedge k)\left(\bigvee_{i:I} U_i\right) &\equiv j\left(\bigvee_{n:N} U_n\right) \wedge k\left(\bigvee_{n:N} U_n\right) \\
&= \left(\bigvee_{n:N} j(U_n)\right) \wedge \left(\bigvee_{n:N} k(U_n)\right) && \text{[Scott continuity of } j \text{ and } k\text{]} \\
&= \bigvee_{(m,n):N\times N} j(U_m) \wedge k(U_n) && \text{[distributivity]} \\
&= \bigvee_{n:N} j(U_n) \wedge k(U_n) && [\dagger] \\
&\equiv \bigvee_{n:N} (j \wedge k)(U_n) && \text{[meet preservation]}
\end{aligned}$$

For the step (†) above, we use antisymmetry. The ($\geq$) direction is immediate. For the ($\leq$) direction, we need to show that

$$\bigvee_{(m,n):N\times N} j(U_m) \wedge k(U_n) \quad \leq \quad \bigvee_{n:N} j(U_n) \wedge k(U_n),$$

for which it suffices to show that $\bigvee_{n:N} j(U_n) \wedge k(U_n)$ is an upper bound of the family

$$\{j(U_m) \wedge k(U_n)\}_{(m,n):N\times N}.$$

Let $m, n : N$ be a pair of indices. As $(U_n)_{n:N}$ is a directed family, there must exist some $o : N$ such that U_o is an upper bound of U_m and U_n. Using the monotonicity of the nuclei j and k, we get $j(U_m) \wedge k(U_n) \leq j(U_o) \wedge k(U_o) \leq \bigvee_{n:N} j(U_n) \wedge k(U_n)$, which was what we needed. □

5.3.2 Joins

Unlike the frame of *all* nuclei, the meet-semilattice of Scott continuous nuclei does admit small joins in a predicative setting. We now look at the construction of these joins, which is the nontrivial component of the construction of the patch frame.

Let $(k_i)_{i:I}$ be a small family of Scott continuous nuclei. The problem that the pointwise join

$$U \mapsto \bigvee_{i:I} k_i(U)$$

is not idempotent, which we discussed in Section 5.2, is also present in the context of Scott continuous nuclei. When the family in consideration is *directed*, however, the pointwise join is idempotent, and it thus gives the join in the poset of Scott continuous nuclei:

Lemma 5.3.7 (). *For every directed family $(k_i)_{i:I}$ of Scott continuous nuclei, the join is computed pointwise, that is,*

$$\left(\bigvee_{i:I} k_i\right)(U) \quad = \quad \bigvee_{i:I} k_i(U),$$

for every open U.

Proof. The argument given in the paragraph preceding [Esc98, Lemma 3.1.8] works in our setting. □

The situation regarding arbitrary joins is more complicated.

Directification of a family of nuclei

A construction of the join, due to Escardó [Esc98], is based on the idea that, if we start with a family $(k_i)_{i:I}$ of nuclei, we can directify it by considering the family with index type $\mathsf{List}(I)$, defined by

$$(i_0 i_1 \cdots i_{n-1}) \mapsto k_{i_{n-1}} \circ \cdots \circ k_{i_1} \circ k_{i_0}.$$

This family is easily seen to be directed. We refer to this as the *family of finite compositions*, and define the following function $(\text{-})^*$ implementing it:

Definition 5.3.8 (Family of finite compositions)**.** Given a small family $(k_i)_{i:I}$ of nuclei on some locale X, its *family of finite compositions* is the family k^*, defined as

$$k^*(i_0 \cdots i_{n-1}) \quad :\equiv \quad k_{i_{n-1}} \circ \cdots \circ k_{i_0}.$$

A finite composition k_s^* is *not* a nucleus in general. It is, however, always a prenucleus:

Lemma 5.3.9 (🤌). *For every family $(k_i)_{i:I}$ of nuclei on a locale, the finite composition k_s^* is a prenucleus, for every $s : \mathsf{List}(I)$.*

Proof. Let $(k_i)_{i:I}$ be a family of nuclei on some locale X. We proceed by list induction.

Base case: the empty list ε. We are done since $k_\varepsilon \equiv \mathsf{id}$, and it is easy to see that the identity function id is a prenucleus.

Induction step: consider a list is, for some $i : I$, $s : \mathsf{List}(I)$, and assume inductively that the finite composition k_s is a prenucleus. We need to show that $k_s \circ k_i$ is a prenucleus. For meet preservation, let $U, V : \mathcal{O}(X)$.

$$\begin{aligned}
(k_s \circ k_i)(U \wedge V) &\equiv k_s(k_i(U \wedge V)) \\
&= k_s(k_i(U) \wedge k_i(V)) && [k_i \text{ preserves meets}] \\
&= k_s(k_i(U)) \wedge k_s(k_i(V)) && [\text{inductive hypothesis}] \\
&\equiv (k_s \circ k_i)(U) \wedge (k_s \circ k_i)(V)
\end{aligned}$$

For inflationarity, consider some $U : \mathcal{O}(X)$. We know that both k_i and k_s are prenuclei, and thus that they are inflationary. It follows that we have $U \leq k_i(U) \leq k_s(k_i(U))$. □

Lemma 5.3.10. *For every nucleus j and every small family $(k_i)_{i:I}$ of nuclei on a locale, if j is an upper bound of the family $(k_i)_{i:I}$, then it is also an upper bound of the family $(k_s^*)_{s:\mathsf{List}(I)}$ of its finite compositions.*

Proof. Let j and $(k_i)_{i:I}$ be, respectively, a nucleus and a family of nuclei on a locale, and suppose that j is an upper bound of $(k_i)_{i:I}$. We show by list induction that j is an upper bound of the family of finite compositions of $(k_i)_{i:I}$.

Base case: the empty list ε. We have that $k_\varepsilon^*(U) \equiv \mathsf{id}(U) \equiv U \leq j(U)$.

Induction step: consider a list is, for some $i : I$, $s : \mathsf{List}(I)$. Assume inductively that $k_s \leq j$. We need to show that $k_{is}^* \leq j$. We know that k_s^* is monotone since it is a prenucleus by Lemma 5.3.9 and prenuclei are monotone by Lemma 3.2.7. The result then follows since we have:

$$\begin{aligned}
k_s^*(k_i(U)) &\leq k_s^*(j(U)) && [\text{monotonicity of } k_s^*] \\
&\leq j(j(U)) && [\text{inductive hypothesis}] \\
&= j(U) && [\text{idempotency of } j]
\end{aligned}$$

□

Lemma 5.3.11 (🤌). *For every family $(k_i)_{i:I}$ of Scott continuous nuclei on a locale, the prenucleus k_s^* is Scott continuous for every $s : \mathsf{List}(I)$.*

Proof. Finite compositions of Scott continuous functions are Scott continuous. □

Lemma 5.3.12. *For every family $(k_i)_{i:I}$ of nuclei on a locale, the family of finite compositions is directed.*

Proof. The family $(k_s^*)_{s:\mathsf{List}(I)}$ of finite compositions is always inhabited by the identity nucleus, given by k_ϵ^*. For every pair of finite compositions k_s^* and k_t^*, the upper bound is given by the nucleus k_{st}^*, which is equal to $k_t^* \circ k_s^*$. The fact that this is an upper bound of k_s^* and k_t^* follows from the monotonicity of k_t^* and the inflationarity of k_s^*. □

Lemma 5.3.13 (🤌). *Let j and $(k_i)_{i:I}$ be, respectively, a nucleus and a family of nuclei on a locale X. Consider the family of finite compositions over the family $(j \wedge k_i)_{i:I}$. Each finite composition $(j \wedge k)_s^*$ is a lower bound of k_s and j, for every $s : \mathsf{List}(I)$.*

Construction of the joins

Having collected some key properties of the family of finite compositions, we are now ready to construct the join operation in the poset of Scott continuous nuclei. The construction here is due to [Esc98].

Theorem 5.3.14 (🤌 Join of Scott continuous nuclei)**.** *For every small family $(k_i)_{i:I}$ of Scott continuous nuclei on a spectral locale X, the join can be calculated as*

$$\left(\bigvee_{i:I} k_i\right)(U) \quad :\equiv \quad \bigvee_{s:\mathsf{List}(I)} k_s^*(U).$$

Proof. Let $(k_i)_{i:I}$ be a small family of nuclei on a locale X. It must be checked that this is (1) indeed the join, and (2) is a Scott continuous nucleus i.e. it is inflationary, binary-meet-preserving, idempotent, and Scott continuous.

We start by checking (2). The inflationarity property is direct. For meet preservation, consider some $U, V : \mathcal{O}(X)$. We have:

$$\begin{aligned}
\left(\bigvee_{i:I} k_i\right)(U \wedge V) &\equiv \bigvee_{s:\mathsf{List}(I)} k_s^*(U \wedge V) \\
&= \bigvee_{s:\mathsf{List}(I)} k_s^*(U) \wedge k_s^*(V) && \text{[Lemma 5.3.11]} \\
&= \bigvee_{s,\, t:\mathsf{List}(I)} k_s^*(U) \wedge k_t^*(V) && [\dagger] \\
&= \left(\bigvee_{s:\mathsf{List}(I)} k_s^*(U)\right) \wedge \left(\bigvee_{t:\mathsf{List}(I)} k_t^*(V)\right) && \text{[distributivity]} \\
&\equiv \left(\bigvee_{i:I} k_i\right)(U) \wedge \left(\bigvee_{i:I} k_i\right)(V)
\end{aligned}$$

The step (†) above uses antisymmetry. The ($\leq$) direction is direct, whereas for the

($\geq$) direction, we show that $\bigvee_{s:\mathsf{List}(I)} k_s^*(U) \wedge k_s^*(V)$ is an upper bound of the family $(k_s^*(U) \wedge k_t^*(V))_{s,t:\mathsf{List}(I)}$. Consider arbitrary $s, t : \mathsf{List}(I)$. By the directedness of the family of finite compositions we know that there exists some $u : \mathsf{List}(I)$ such that the prenucleus k_u^* is an upper bound of the prenuclei k_s^* and k_t^*. We then have:

$$k_s^*(U) \wedge k_t^*(V) \quad \leq \quad k_u^*(U) \wedge k_u^*(V) \quad \leq \quad \bigvee_{s:\mathsf{List}(I)} k_s^*(U) \wedge k_s^*(V).$$

For idempotency, let $U : \mathcal{O}(X)$.

$$\begin{aligned}
\left(\bigvee_i k_i\right)\left(\left(\bigvee_j k_j\right)(U)\right) &\equiv \bigvee_{s:\mathsf{List}(I)} k_s\left(\bigvee_{t:\mathsf{List}(I)} k_t(U)\right) \\
&= \bigvee_{s:\mathsf{List}(I)} \bigvee_{t:\mathsf{List}(I)} k_s^*(k_t^*(U)) && \text{[Lemma 5.3.11]} \\
&\leq \bigvee_{s,t:\mathsf{List}(I)} k_s^*(k_t^*(U)) \\
&\leq \bigvee_{s:\mathsf{List}(I)} k_s^*(U) && [\ddagger] \\
&\equiv \left(\bigvee_{i:I} k_i\right)(U)
\end{aligned}$$

For the step ($\ddagger$) above, it suffices to show that $\bigvee_{s:\mathsf{List}(I)} k_s^*(U)$ is an upper bound of the family $(k_s^*(k_t^*(U)))_{s,t:\mathsf{List}(I)}$. The prenucleus k_{ts}^* is an upper bound of k_s^* and k_t^* (as in Lemma 5.3.12). We have that $k_s(k_t(U)) \equiv k_{ts}(U) \leq \bigvee_{s:\mathsf{List}(I)} k_s(U)$.

For Scott continuity, let $(U_n)_{n:N}$ be a directed family of opens of X. We have:

$$\begin{aligned}
\left(\bigvee_{i:I} k_i\right)\left(\bigvee_{n:N} U_n\right) &\equiv \bigvee_{s:\mathsf{List}(I)} k_s^*\left(\bigvee_{n:N} U_n\right) \\
&= \bigvee_{s:\mathsf{List}(I)} \bigvee_{n:N} k_s^*(U_n) && \text{[Lemma 5.3.11]} \\
&= \bigvee_{n:N} \bigvee_{s:\mathsf{List}(I)} k_s^*(U_n) && \text{[joins commute]} \\
&\equiv \bigvee_{n:N} \left(\bigvee_{i:I} k_i\right)(U_n).
\end{aligned}$$

We now check that the above definition of the join gives the least upper bound of the family $(k_i)_{i:I}$. The fact that $\bigvee_i k_i$ is an upper bound of the family $(k_i)_{i:I}$ is easy to verify. To see that it is *the least* upper bound, consider a Scott continuous nucleus j, and assume that it is an upper bound of $(k_i)_{i:I}$. Let $U : \mathcal{O}(X)$. We need to show that $(\bigvee_i k_i)(U) \leq j(U)$. Since j is an upper bound of the family $(k_i)_{i:I}$, it follows from Lemma 5.3.10 that is also an upper bound of the family $(k_s^*)_{s:\mathsf{List}(I)}$ of finite compositions. This is to say that $k_s^*(U) \leq j(U)$ for every $s : \mathsf{List}(I)$. It follows that $\left(\bigvee_{i:I} k_i\right)(U) \equiv \bigvee_{s:\mathsf{List}(I)} k_s^*(U) \leq j(U)$. □

We have thus established that the meet-semilattice of nuclei over a spectral locale is small-complete. To conclude that it is a frame, it remains just to show that the distributivity law holds.

The distributivity law

We appeal to Lemma 5.3.13 to prove the distributivity law.

Lemma 5.3.15 (Distributivity)**.** *For every nucleus j and every small family $(k_n)_{n:N}$ of Scott continuous nuclei, we have the equality:*

$$j \wedge \left(\bigvee_{n:N} k_n \right) \quad = \quad \bigvee_{n:N} j \wedge k_n.$$

Proof. Let j and $(k_n)_{n:N}$ be, respectively, a nucleus and a small family of Scott continuous nuclei. To see that $j \wedge \bigvee_{n:N} k_n$ is extensionally equal to the join $\bigvee_{n:N} j \wedge k_n$, let $U : \mathcal{O}(X)$.

$$\begin{aligned}
\left(j \wedge \left(\bigvee_{i:I} k_i \right) \right)(U) &\equiv j(U) \wedge \left(\bigvee_{i:I} k_i \right)(U) \\
&\equiv j(U) \wedge \left(\bigvee_{s:\mathsf{List}(I)} k_s^*(U) \right) \\
&= \bigvee_{s:\mathsf{List}(I)} j(U) \wedge k_s^*(U) && \text{[distributivity]} \\
&\equiv \bigvee_{s:\mathsf{List}(I)} (j \wedge k_s^*)(U) \\
&\equiv \left(\bigvee_{i:I} j \wedge k_i \right)(U) && [\dagger]
\end{aligned}$$

For the step labelled (†), we need to show that join of the family $s \mapsto j(U) \wedge k_s^*(U)$ is equal to the join of the family of finite compositions of the family

$$s \mapsto (j \wedge k_{(\text{-})})_s^*(U).$$

One direction of this follows from Lemma 5.3.13 whereas the other can be shown by straightforward induction. □

We have thus established:

Lemma 5.3.16 ( ). *For every spectral locale X, the poset of Scott continuous nuclei on X forms a frame.*

Proof. The construction of finite meets was given in Lemma 5.3.6, and local smallness in Corollary 5.3.5. Joins were constructed in Theorem 5.3.14. That finite meets distribute over small joins was established in Lemma 5.3.15 above. □

Remark 5.3.17. The construction of this frame could in fact be generalized from spectral locales to *all* locales, except the resulting frame of Scott continuous nuclei would not always be locally small. To ensure local smallness, a *specified* small base seems to be necessary. Since we will be restricting attention to the case of spectral locales anyway, we refrain from presenting these results at the highest level of generality for the sake of brevity. The point where the assumption of spectrality becomes absolutely necessary is when proving that the frame of Scott continuous nuclei satisfies the universal property of the patch frame, which we address in the next section.

5.4 The patch locale

In the previous section, we presented the construction of the frame of Scott continuous nuclei over a spectral locale X. We now proceed to show that the locale defined by this frame satisfies the universal property of the patch locale on X.

Definition 5.4.1 (Patch locale of a spectral locale)**.** Let X be a spectral locale. We denote by $\mathsf{Patch}(X)$ the locale defined by the frame of Scott continuous nuclei on X.

Although our proof of the universal property will have to to wait until Section 5.4.3, we already adopt the notation of $\mathsf{Patch}(X)$ for the locale defined by the frame of Scott continuous nuclei. Moreover, we also remind the reader at this point that the patch locale is large, locally small, and small-complete, like every locale that we work with in this thesis. This is to say that, for every spectral locale X in the category $\mathbf{Loc}_{\mathcal{U}}$, the locale $\mathsf{Patch}(X)$ is also an object falling in $\mathbf{Loc}_{\mathcal{U}}$.

5.4.1 Base construction for the patch locale

We now construct a small base for the patch locale using the notions of closed and open nuclei, which we first briefly recall. Let U be an open of a locale.

1. The closed nucleus induced by U is the map $V \mapsto U \vee V$ (Example 3.2.2);
2. The open nucleus induced by U is the map $V \mapsto U \Rightarrow V$ (Definition 3.7.7).

Lemma 5.4.2. *The top nucleus on a locale X, which we defined in Lemma 5.1.4 as the map $U \mapsto \mathbf{1}_X$, is equal to each of the following:*

- *the closed nucleus* $\mathbf{c}(\mathbf{1}_X)$,
- *the open nucleus* $\mathbf{o}(\mathbf{0}_X)$.

Proof. Follows directly from the identities $\mathbf{1}_X \vee U = \mathbf{1}_X$ and $(\mathbf{0}_X \Rightarrow U) = \mathbf{1}_X$, which hold for every open $U : \mathcal{O}(X)$. □

Using the closed and open nuclei, we construct a base for $\mathsf{Patch}(X)$ by considering the family

$$\mathsf{K}(X) \times \mathsf{K}(X) \xrightarrow{\gamma} \mathcal{O}(\mathsf{Patch}(X)),$$

defined by

$$\gamma(K_1, K_2) :\equiv \mathbf{c}(K_1) \wedge \mathbf{o}(K_2).$$

It is easy to see that this family is small. We now do some preparation towards showing that this forms a base.

Lemma 5.4.3 (🤌). *For every Scott continuous nucleus $j : \mathcal{O}(X) \to \mathcal{O}(X)$ on a locale X, we have that*

$$j \quad = \quad \bigvee_{K:\mathsf{K}(X)} \mathbf{c}(j(K)) \wedge \mathbf{o}(K) \quad = \quad \bigvee_{\substack{K_1,K_2:\mathsf{K}(X) \\ K_1 \leq j(K_2)}} \mathbf{c}(K_1) \wedge \mathbf{o}(K_2).$$

Proof. The second equality in the statement is clear, so let us show the first one. We use the fact, proved in [Joh82, Lemma II.2.7], that for every nucleus j on any locale,

$$j = \bigvee_{U:\mathcal{O}(X)} \mathbf{c}(j(U)) \wedge \mathbf{o}(U).$$

Suppose additionally that X is spectral and the nucleus j is Scott continuous. The inequality $\bigvee_{K:\mathsf{K}(X)} \mathbf{c}(j(K)) \wedge \mathbf{o}(K) \leq \bigvee_{U:\mathcal{O}(X)} \mathbf{c}(j(U)) \wedge \mathbf{o}(U) = j$ is trivial, so let us show the reverse one. Let $K : \mathsf{K}(X)$ and notice that

$$(\mathbf{c}(j(K)) \wedge \mathbf{o}(K))\,(K) = (j(K) \vee K) \wedge (K \Rightarrow K) = j(K).$$

Therefore

$$j(K) = (\mathbf{c}(j(K)) \wedge \mathbf{o}(K))\,(K) \leq \left(\bigvee_{K':\mathsf{K}(X)} \mathbf{c}(j(K')) \wedge \mathbf{o}(K') \right)(K),$$

so the required relation follows from Lemma 5.3.4. □

Lemma 5.4.4 (🤌). *For every spectral locale X, the above family*

$$\mathsf{K}(X) \times \mathsf{K}(X) \xrightarrow{\gamma} \mathcal{O}(\mathsf{Patch}(X)),$$

forms a small, strong base.

Proof. Let j be a Scott continuous nucleus over a locale X. For the basic covering

family for j, we pick the family

$$\left(\Sigma_{(K_1,K_2:\mathrm{K}(X))}(K_1 \leq j(K_2))\right) \xrightarrow{\gamma\circ\mathrm{pr}_1} \mathcal{O}(\mathsf{Patch}(X)),$$

which is easily seen to be small. The fact that $j = \bigvee\{\mathbf{c}(K_1) \wedge \mathbf{o}(K_2) \mid K_1 \leq j(K_2)\}$ was already established in Lemma 5.4.3. These basic covering families do not seem to be directed, so we work with their directifications as explained in Section 3.5.3. □

5.4.2 Patch is Stone

We prove that the patch locale is Stone by showing (1) that it is compact, and (2) that the base constructed in Lemma 5.4.4 is a zero-dimensional one.

Lemma 5.4.5 (🤌). *The open nucleus $\mathbf{o}(U)$ is the Boolean complement of the closed nucleus $\mathbf{c}(U)$, for every open U of a spectral locale X.*

Proof. The argument from [Esc99, p. 6] works in our setting, once it is ensured that Heyting implications always exist as discussed in Definition 3.7.5. We provide the full-proof here for the sake of self-containment.

Let X be locale and consider an open $U : \mathcal{O}(X)$. We claim that $\mathbf{o}(U)$ is both the meet-complement and the join-complement of $\mathbf{c}(U)$. To show that it is the meet-complement, we need to show that $(\mathbf{c}(U)\wedge\mathbf{o}(U))(V) = \mathbf{0}(V) \equiv V$, for every $V : \mathcal{O}(X)$. Observe that

$$\begin{aligned}
(\mathbf{c}(U) \wedge \mathbf{o}(U))(V) &\equiv \mathbf{c}(U)(V) \wedge \mathbf{o}(U)(V) \\
&= (U \vee V) \wedge (U \Rightarrow V) \\
&= (U \wedge (U \Rightarrow V)) \vee (V \wedge (U \Rightarrow V)) \\
&\leq V \vee V \\
&= V.
\end{aligned}$$

To show that it is the join-complement, we need to show

$$(\mathbf{o}(U) \vee \mathbf{c}(U))(V) = \mathbf{1}(V) \equiv \mathbf{1}_X,$$

for every $V : \mathcal{O}(X)$. Let $V : \mathcal{O}(X)$. It follows from the construction of the join from Theorem 5.3.14 that the pointwise binary join satisfies

$$\mathbf{o}(U)(V) \vee \mathbf{c}(U)(V) \leq (\mathbf{o}(U) \vee \mathbf{c}(U))(V).$$

Therefore, it suffices to show $\mathbf{1}_X \leq \mathbf{o}(U)(V) \vee \mathbf{c}(U)(V)$, which we derive below.

$$\begin{aligned}
\mathbf{1}_X &\leq U \Rightarrow (U \vee V) \\
&= \mathbf{o}(U)(U) \wedge \mathbf{o}(U)(V) \\
&= \mathbf{1}_X \wedge \mathbf{o}(U)(V) \\
&\leq \mathbf{o}(U)(V) \\
&\leq \mathbf{o}(U)(V) \vee \mathbf{c}(U)(V).
\end{aligned}$$

□

Corollary 5.4.6. *For every compact open $K : \mathcal{O}(X)$ of a spectral locale X, the closed nucleus $\mathbf{c}(K)$ and the open nucleus $\mathbf{o}(K)$ are* clopens *as they complement each other.*

In Lemma 5.4.8, we prove that the map whose inverse image sends an open U to the closed nucleus $\mathbf{c}(U)$ is perfect. Before Lemma 5.4.8, we record an auxiliary result:

Lemma 5.4.7. *Let X be a spectral locale. The right adjoint $\varepsilon_* : \mathcal{O}(\mathsf{Patch}(X)) \to \mathcal{O}(X)$ to the closed nucleus formation operation $\mathbf{c}(\text{-})$ is given by $\varepsilon_*(j) = j(\mathbf{0})$, for every Scott continuous nucleus j on X.*

Lemma 5.4.8 (🤌). *The closed nucleus formation operation is a perfect frame homomorphism*

$$\mathbf{c}(\text{-}) : \mathcal{O}(X) \to \mathcal{O}(\mathsf{Patch}(X)).$$

Proof. It is easy to see that this is a frame homomorphism, so we show that it is perfect. We have to show that the right adjoint ε_* of $\mathbf{c}(\text{-})$ is Scott continuous. Let $(k_i)_{i:I}$ be a directed family of Scott continuous nuclei. Thanks to Lemma 5.4.7, it suffices to show

$$\left(\bigvee_{i:I} k_i\right)(\mathbf{0}) = \bigvee_{i:I} \varepsilon_*(k_i).$$

By Lemma 5.3.7, we have that $\left(\bigvee_{i:I} k_i\right)(\mathbf{0}) = \bigvee_{i:I} k_i(\mathbf{0})$. □

Lemma 5.4.9 (🤌). *The locale $\mathsf{Patch}(X)$ is compact, for every spectral locale X.*

Proof. Recall that the top open $\mathbf{1}$ of $\mathsf{Patch}(X)$ is the top nucleus defined as

$$\mathbf{1} :\equiv U \mapsto \mathbf{1}_X.$$

Since ε^* is a frame homomorphism, we know that $\mathbf{1} = \varepsilon^*(\mathbf{1}_X)$, meaning it suffices to show $\varepsilon^*(\mathbf{1}_X) \ll \varepsilon^*(\mathbf{1}_X)$. By Lemma 4.2.15, it suffices to show $\mathbf{1}_X \ll \mathbf{1}_X$ which is immediate by the assumption that X is spectral. □

Lemma 5.4.10 (🤌). *For every spectral locale X, the locale $\mathsf{Patch}(X)$ is Stone.*

Proof. Let X be a spectral locale. We showed in Lemma 5.4.9 that $\mathsf{Patch}(X)$ is compact. It suffices to show that it is zero-dimensional, for which we show that the base constructed in Lemma 5.4.4 consists of clopens. For every pair K_1, K_2 of compact opens of X, we know by Corollary 5.4.6 that the closed nucleus $\mathbf{c}(K_1)$ and the open nucleus $\mathbf{o}(K_2)$ are both clopens. It follows from Lemma 3.4.4 that the meet $\mathbf{c}(K_1) \wedge \mathbf{o}(K_2)$ is clopen, so we are done. □

5.4.3 The universal property of the patch locale

We are finally ready to show that the locale $\mathsf{Patch}(X)$ satisfies the universal property of the patch locale from Definition 5.0.1.

Lemma 5.4.11. *For every spectral map $f : X \to A$ from a Stone locale X into a spectral locale A, define a map $\bar{f}^* : \mathcal{O}(\mathsf{Patch}(A)) \to \mathcal{O}(X)$ by*

$$\bar{f}^*(j) \quad :\equiv \quad \bigvee_{K:\mathsf{K}(A)} f^*(j(K)) \wedge \neg f^*(K).$$

Then, the map $\bar{f}_* : \mathcal{O}(X) \to \mathcal{O}(\mathsf{Patch}(A))$ defined by

$$\bar{f}_*(V) \quad :\equiv \quad f_* \circ \mathbf{c}(V) \circ f^*$$

is the right adjoint of $\bar{f}^*$.

Proof. First, note that $f_* \circ \mathbf{c}(V) \circ f^*$ is indeed a Scott continuous nucleus, and both $\bar{f}^*$ and $\bar{f}_*$ are clearly monotone. Let us first prove the forward implication. Assume that for $j : \mathcal{O}(\mathsf{Patch}(A))$ and $V : \mathcal{O}(X)$ the relation $\bar{f}^*(j) \leq V$ holds. By Lemma 5.3.4, in order to show that $j \leq \bar{f}_*(V)$, it suffices to show that for every compact open $K : \mathsf{K}(A)$, the inequality $j(K) \leq \bar{f}_*(V)(K)$ holds. Hence, let $K : \mathsf{K}(A)$, and note that

$$f^*(j(K)) \wedge \neg f^*(K) \leq \bar{f}^*(j) \leq V$$

Notice that $f^*(K)$ is clopen, by the fact that f^* is a spectral map and Lemma 4.5.8. It is therefore complemented in the lattice $\mathcal{O}(X)$, and so we have

$$f^*(j(K)) \leq V \vee f^*(K) = \mathbf{c}(V)(f^*(K)),$$

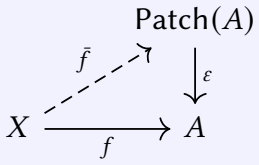
which by adjunction yields

$$j(K) \leq f_*(\mathbf{c}(V)(f^*(K))) = \bar{f}_*(K),$$

as required.

Let us now show the reverse implication. Let $j : \mathcal{O}(\mathsf{Patch}(A))$ and $V : \mathcal{O}(X)$ and assume that $j \leq \bar{f}_*(V)$. Once again, by the definition of the ordering on $\mathcal{O}(\mathsf{Patch}(A))$, for all $K : \mathsf{K}(A)$ we have $j(K) \leq f_*(V \vee f^*(K))$, which by adjunction equivalently yields $f^*(j(K)) \leq V \vee f^*(K)$. Since $f^*(K)$ is clopen, and hence complemented in the lattice $\mathcal{O}(X)$ it follows that $f^*(j(K)) \wedge \neg f^*(K) \leq V$. Hence, $\bar{f}^*(j) \leq V$. □

Theorem 5.4.12 (🤌)**.** *For every spectral map $f : X \to A$ from a Stone locale into a spectral locale, there exists a unique continuous map $\bar{f} : X \to \mathsf{Patch}(A)$ satisfying $\varepsilon \circ \bar{f} = f$, as illustrated in the following diagram in* **Spec***:*



Proof. Assume that a locale map $\bar{f} : X \to \mathrm{Patch}(A)$ satisfies the condition in the theorem. Then, for every $j : \mathcal{O}(\mathrm{Patch}(A))$, we have

$$\begin{aligned}
\bar{f}^*(j) &= \bar{f}^*\left(\bigvee_{K:\mathrm{K}(A)} \mathbf{c}(j(K)) \wedge \mathbf{o}(K)\right) \\
&\qquad\qquad [\text{Lemma 5.4.3}] \\
&= \bigvee_{K:\mathrm{K}(A)} \bar{f}^*\left(\mathbf{c}(j(K)) \wedge \mathbf{o}(K)\right) \\
&\qquad\qquad [\bar{f}^* \text{ preserves small joins}] \\
&= \bigvee_{K:\mathrm{K}(A)} \bar{f}^*\left(\mathbf{c}(j(K))\right) \wedge \neg \bar{f}^*\left(\mathbf{c}(K)\right) \\
&\qquad\qquad [\bar{f}^* \text{ preserves meets and complements}] \\
&= \bigvee_{K:\mathrm{K}(A)} f^*\left(j(K)\right) \wedge \neg f^*\left(K\right) \\
&\qquad\qquad [\text{commutativity of the diagram}]
\end{aligned}$$

and hence $\bar{f}$ is uniquely determined. If we now define a monotone map $\overline{f}^* : \mathcal{O}(\mathrm{Patch}(A)) \to \mathcal{O}(X)$ by

$$\bar{f}^*(j) \quad :\equiv \quad \bigvee_{K:\mathrm{K}(A)} f^*(j(K)) \wedge \neg f^*(K),$$

it is easy to show it preserves the top element (namely the top nucleus with constant

value $\mathbf{1}_A$) because $\mathbf{0}_A$ is compact. It also preserves binary (pointwise) meets as

$$
\begin{aligned}
& \bar{f}^*(j_1) \wedge \bar{f}^*(j_2) \\
= & \bigvee_{K_1,K_2:\mathsf{K}(A)} f^*\left(j_1(K_1)\right) \wedge \neg f^*\left(K_1\right) \wedge f^*\left(j_2(K_2)\right) \wedge \neg f^*\left(K_2\right) \\
& \text{[distributivity]} \\
= & \bigvee_{K_1,K_2:\mathsf{K}(A)} f^*\left(j_1(K_1) \wedge j_2(K_2)\right) \wedge \neg f^*\left(K_1\right) \wedge \neg f^*\left(K_2\right) \\
& [f^* \text{ preserves binary meets}] \\
\leq & \bigvee_{K_1,K_2:\mathsf{K}(A)} f^*\left(j_1(K_1 \vee K_2) \wedge j_2(K_1 \vee K_2)\right) \wedge \neg f^*\left(K_1\right) \wedge \neg f^*\left(K_2\right) \\
& \text{[monotonicity]} \\
= & \bigvee_{K_1,K_2:\mathsf{K}(A)} f^*\left((j_1 \wedge j_2)(K_1 \vee K_2)\right) \wedge \neg f^*\left(K_1\right) \wedge \neg f^*\left(K_2\right) \\
= & \bigvee_{K_1,K_2:\mathsf{K}(A)} f^*\left((j_1 \wedge j_2)(K_1 \vee K_2)\right) \wedge \neg f^*\left(K_1 \vee K_2\right) \\
& \text{[De Morgan law]} \\
= & \bar{f}^*(j_1 \wedge j_2) \\
& [\mathsf{K}(X) \text{ closed under } (\text{-}) \vee (\text{-})]
\end{aligned}
$$

Moreover, Lemma 5.4.11 and Theorem 3.7.3 ensure that $\bar{f}^*$ preserves small joins and so it is a frame homomorphism.

Let us finally show that $\bar{f}$ makes the diagram commute. Since compact opens form a small base of A, it suffices to show that $\bar{f}^*(\mathbf{c}(K)) = f^*(K)$ for every $K : \mathsf{K}(A)$. Let $K : \mathsf{K}(A)$ and note that

$$
\begin{aligned}
\bar{f}^*(\mathbf{c}(K)) &= \bigvee_{K':\mathsf{K}(A)} f^*\left(K \vee K'\right) \wedge \neg f^*\left(K'\right) \\
&= \bigvee_{K':\mathsf{K}(A)} f^*\left(K\right) \wedge \neg f^*\left(K'\right) && [f^* \text{ preserves binary joins}] \\
&= f^*(K) \wedge \bigvee_{K':\mathsf{K}(A)} \neg f^*\left(K'\right) && \text{[distributivity]} \\
&= f^*(K) \wedge \mathbf{1}_X && [\mathbf{0}_X \text{ is compact}]
\end{aligned}
$$

as required. □

5.5 Examples of patch

To make things a bit more concrete, we now look at some examples of the patch transformation.

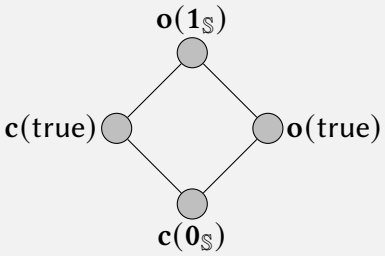
Patch of the terminal locale

Example 5.5.1. The patch locale $\mathsf{Patch}(\mathbf{1}_{\mathcal{U}})$ of the terminal locale is homeomorphic to the terminal locale itself, since $\mathbf{1}_{\mathcal{U}}$ is Stone.

Patch of Sierpiński

Example 5.5.2. The locale $\mathsf{Patch}(\mathbb{S}_{\mathcal{U}})$ is homeomorphic the discrete locale on type $\mathbf{2}_{\mathcal{U}}$.

Proof. It was given in Example 4.4.31 that the frame $\mathcal{P}_{\mathcal{U}}(\mathbf{2})$ is the frame of ideals over the lattice $\mathbf{M}_2$. It is therefore sufficient by duality (Theorem 4.7.10) to show that the lattice $\mathsf{K}(\mathsf{Patch}(\mathbb{S}_{\mathcal{U}}))$ is isomorphic to $\mathbf{M}_2$. From the base construction for $\mathsf{Patch}(\mathbb{S}_{\mathcal{U}})$, and the fact that that the compact opens coincide with the basic opens in spectral locales (Corollary 4.2.5), it can be seen that $\mathsf{K}(\mathsf{Patch}(\mathbb{S}_{\mathcal{U}}))$ is the following distributive lattice:



where $\mathsf{true} : \mathcal{O}(\mathbb{S})$ is the nontrivial open of Sierpiński constructed in Proposition 3.1.25. This is clearly the same lattice as $\mathbf{M}_2$ from Example 4.4.31. □

Chapter 6

Topology of Scott domains

In this chapter, we investigate the point-free topology of domains [GHK+03], with a particular focus on the class of domains known as *Scott domains*. Our motivation for this is twofold:

1. Scott domains provide a rich source of examples of spectral locales through their *Scott topologies*, which are always spectral.
2. The patch topology on the Scott topology of a Scott domain is a particularly illustrative example of the patch construction, giving a *computational intuition* on the patch topology through its relationship to the so-called *Lawson topology* in domain theory [GHK+03, Definition III-1.5].

In addition to these, whether the topology of domains can be carried out predicatively is also a natural question in light of the recent developments of de Jong and Escardó [dJon21b; dJon23], showing that many nontrivial constructions of domain theory can be developed in constructive and predicative univalent foundations.

The organization of this chapter is as follows:

Section 6.1: We provide a brief summary of de Jong and Escardó's constructive and predicative development of domain theory in univalent foundations.

Section 6.2: We define the Scott topology of a DCPO and show that it is a large, locally small, and small-complete frame when the DCPO in consideration is algebraic.

Section 6.3: We summarize de Jong's notion of *sharp element* in the context of Scott domains, and we then give a topological characterization of it in terms of the Scott topology of the Scott domain in consideration.

Section 6.4: We present and discuss the notion of *spectral point* in the context of the Scott topology.

Section 6.5: We show that the spectral points of the Scott locale of a Scott domain D are in bijective correspondence with the sharp elements of D.

Section 6.6: Extending the result from Section 6.5, we show that the sharp elements of a Scott domain D are in bijective correspondence with the points

of the patch locale of the Scott locale of D. This is because the points of the patch of a spectral locale X are exactly the spectral points of X.

6.1 Primer on domain theory in UF

As the work we present in this chapter builds on de Jong and Escardó's [dJon21b; dJE24] development of constructive and predicative domain theory in univalent foundations, we provide a brief summary of their work.

6.1.1 Definition of DCPO

In Definition 3.1.1, we gave a definition of the notion of frame with all universes generalized. After this, we explained de Jong and Escardó's result (Theorem 3.1.11) that complete and small frames cannot be constructed without a form of propositional resizing, and we accordingly adopted the convention of working *only* with large, locally small, and small-complete frames over a fixed base universe. Since exactly the same situation applies to DCPOs, we omit their definition with generalized universes altogether, and directly provide the definition for the well-behaved case of large, locally small, and directed small-complete DCPOs.

Definition 6.1.1 (*cf.* [dJon23, Definition 3.2.7])**.** A DCPO (over base universe $\mathcal{U}$) consists of

- a type $|D| : \mathcal{U}^+$,
- a partial order $(\text{-}) \sqsubseteq (\text{-}) : |D| \to |D| \to \Omega_{\mathcal{U}}$,
- an operation $\bigsqcup(\text{-}) : \mathsf{Fam}^{\uparrow}_{\mathcal{U}}(D) \to D$, giving the least upper bound $\bigsqcup_{i:I} x_i$ of every directed small family $(x_i)_{i:I}$ of elements of D.

As in the case of frames, the sethood of the carrier type follows automatically from the partial order thanks to Lemma 3.1.2, and it hence does not have to be explicitly stipulated.

We adopt the convention of using the distinct order symbol ($\sqsubseteq$) to differentiate the underlying poset of a DCPO than that of a frame. We sometimes use the term *information order* to refer to the order of a DCPO. Furthermore, we adopt the usual abuse of notation of denoting the carrier set $|D|$ by D.

6.1.2 Pointed DCPOs

Definition 6.1.2 (Pointed DCPO [dJon23, Definition 3.2.11])**.** A DCPO is called *pointed* if it has a bottom element.

We denote the bottom element of a pointed DCPO D by $\bot_D$, often dropping the subscript and writing simply $\bot$.

The archetypical example of a pointed DCPO is the *flat domain* of natural numbers, the classical picture of which is given in the Hasse diagram in Figure 6.1. There is a bottom element $\bot$, and all other elements besides $\bot$ are incomparable. It is helpful to

think of the elements of this DCPO as possibly undefined computations. The bottom element represents a computation failing to produce a value e.g. due to nontermination. An element above $\bot$ is said to be *defined*, and it is thought of as a computation successfully terminating with a value in $\mathbb{N}$.

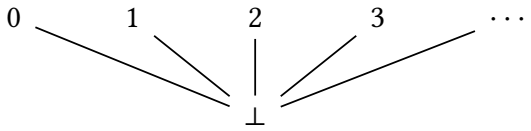


Figure 6.1: The **classical** picture of the flat domain of natural numbers.

More generally, the flat DCPO obtained by adjoining a bottom element to a set X is called the *lifting of the set* X. The construction of the lifting of a set X as $X + \mathbf{1}$, pictured in Figure 6.1, cannot be shown to be directed-complete unless the Limited Principle of Omniscience holds [dJon23, Proposition 3.4.1]. Escardó and Knapp [EK17] gave a construction of the lifting DCPO that lends itself to a constructive (and predicative) development:

Example 6.1.3 (*cf.* [dJon23, Proposition 3.4.14])**.** The lifting of a set $X : \mathcal{U}$ is the type

$$\mathcal{L}_{\mathcal{U}}(X) \quad :\equiv \quad \Sigma_{(P:\Omega_{\mathcal{U}})}(P \to X),$$

equipped with the partial order $(P, \varphi) \sqsubseteq (Q, \psi) :\equiv \Sigma_{(f:P\Rightarrow Q)}(\varphi \sim \psi \circ f)$. This forms a pointed DCPO. The bottom element is the false proposition $\bot_{\mathcal{U}}$ with the unique function $\bot \to X$.

For each inhabitant $x : X$, its lifting into $\mathcal{L}_{\mathcal{U}}(X)$ is given by the tuple $(\top, \star \mapsto x)$. This defines a map, denoted $\eta_X : X \to \mathcal{L}_{\mathcal{U}}(X)$.

The flat domain of natural numbers can then be simply defined as:

Example 6.1.4 (Flat domain of $\mathbb{N}$)**.** The *flat domain of natural numbers* is the DCPO $\mathcal{L}_{\mathcal{U}_0}(\mathbb{N})$.

Example 6.1.5. The flat domain $\mathcal{L}_{\mathcal{U}}(\mathbf{1})$ is isomorphic to $\Omega_{\mathcal{U}}$ considered as a DCPO. It is easy to see that the carrier type is just $\Omega_{\mathcal{U}}$ since $(P \to \mathbf{1}) \simeq \mathbf{1}$, for every proposition $P : \Omega_{\mathcal{U}}$. Moreover, since every pair of functions $\varphi, \vartheta : P \to \mathbf{1}$ are equal to each other, the order $(P, \varphi) \sqsubseteq (Q, \psi)$ amounts to $P \Rightarrow Q$.

6.1.3 Finite elements of domains

The way-below relation from Definition 3.3.9 can be transferred directly to the context of DCPOs. In fact, this relation was first formulated in the context of domains and was subsequently adapted to point-free topology (see [GHK+03, p. 76] for an encyclopedic history).

Definition 6.1.6 ([dJon23, Definition 4.2.1]). An element x of a DCPO is said to be *way below* an element y if for every small, directed family $(z_i)_{i:I}$ of elements with $y \sqsubseteq \bigsqcup_{i:I} z_i$, there is some $i : I$ such that $x \sqsubseteq z_i$.

Similar to the notion of compact open in a frame, which is defined as an open that is way below itself, we have a notion of compact element in a DCPO. We will follow the domain-theoretic convention of using the term *finite element* in the context of domains. The formulation of the notion of finite element is exactly the same as Condition (3) from Lemma 3.3.3:

Definition 6.1.7 ([dJon23, Definition 4.2.3]). An element x of a DCPO is called *finite* if it is way below itself.

We write $\mathsf{K}(D)$ for the type of finite elements of a DCPO D, overloading the notation we use for locales. This slight abuse of notation is justified by the fact that an open U of a locale X is compact if and only if it is a finite element of the underlying DCPO of the frame $\mathcal{O}(X)$.

It is important to observe here that the characterization of compactness in frames from Lemma 3.3.3 does not apply to DCPOs. Some of the equivalent formulations from Lemma 3.3.3 rely on the existence of all small joins even to be formulated, and even in the presence of all joins, some of the equivalences rely crucially on the frame distributivity law.

Lemma 6.1.8 ([dJon23, Example 4.2.4]). *For every pointed DCPO D, the bottom element $\bot_D$ is finite.*

Proof. The argument from Lemma 3.3.8 works here as well. □

Example 6.1.9 (*cf.* [dJon23, Example 4.2.6]). Every element $\eta(x)$ is finite in the lifting DCPO $\mathcal{L}_{\mathcal{U}}(X)$ over some small set X.

The class of *continuous DCPOs* [GHK+03, Theorem I-4.17] is of special interest in domain theory. The so-called continuity condition on a DCPO D captures the idea that every element $x : D$ is determined by the elements that are finite relative to x. The class of *algebraic DCPOs* are a subclass of continuous DCPOs where the continuity condition is stricter. The algebraicity condition states that every computation in a DCPO D is expressible as the join of the (globally) finite computations below it.

For the purposes of our work, we will only be interested in algebraic domains, and particularly in the further special case of Scott domains. We thus refrain from presenting continuous DCPOs here and restrict our attention to algebraic ones. The reader interested in the details of the predicative treatment of continuous DCPOs is referred to [dJon23, §4.4].

6.1.4 Algebraic DCPOs

The definition of algebraic DCPO that we provide below differs from those in [dJon23; dJon21b; dJE24]. We work with an equivalent reformulation of de Jong and Escardó's definition, our motivation for which is twofold:

1. to remain close to our approach to spectral locales,
2. for technical convenience whilst constructing the Scott locale of a Scott domain in Section 6.2.

In more recent work [dJE24, §5.3], de Jong and Escardó discuss the fact that the split support result we gave in Lemma 4.2.9 applies also to the case of algebraic DCPOs (formalized in TypeTopology by de Jong and Tosun [dJT23]). In fact, de Jong and Escardó prove a lemma [dJE24, Lemma 5.23] showing that a formulation analogous to our Condition (4) from Theorem 4.2.11 is possible through this split support result, but they do not give an explicit definition based on this. We turn de Jong and Escardó's Lemma 5.23 from *op. cit.* into an explicit definition here:

Definition 6.1.10 (Algebraic DCPO)**.** A DCPO D (over base universe $\mathcal{U}$) is called *algebraic DCPO* if it satisfies the following two conditions:

(AD1) For every $x : D$, the family $\left(\Sigma_{(c:\mathsf{K}(D))} c \sqsubseteq x\right) \xhookrightarrow{\mathrm{pr}_1} D$ is directed and gives x as its join.
(AD2) The type $\mathsf{K}(D)$ is $\mathcal{U}$-small.

Condition (AD1) above says that the type $\mathsf{K}(D)$ forms a basis (with certain kinds of basic covering families), which we could make precise by formulating an auxiliary notion of basis for DCPOs. However, as our aim here is merely to summarize predicative domain theory to the extent that we need it, we refrain from addressing such questions in detail. Definition 6.1.10 above is sufficient for the purposes of our work.

Remark 6.1.11. The above definition amounts to Definition 4.8.1 from de Jong's thesis [dJon23], combined with the assertion that the basic family in consideration is the inclusion

$$\mathsf{K}(D) \xhookrightarrow{\mathrm{pr}_1} D.$$

Note, however, that de Jong [dJon23] uses a more general definition of DCPO which permits the underlying poset not to be locally small. Accordingly, the definition from *op. cit.* also stipulates local smallness for basic elements (Condition (iii) of [dJon23, Definition 4.8.1]). We do not have such a condition here as that is built into Definition 6.1.1.

Example 6.1.12 (*cf.* [dJon23, Example 4.9.4])**.** The lifting DCPO $\mathcal{L}_{\mathcal{U}}(X)$ is algebraic, for every set $X : \mathcal{U}$.

Before we give the proof of this example, we prove some auxiliary lemmas.

Lemma 6.1.13 (*cf.* [dJon23, Example 4.9.2])**.** *For every small set X, every element of $\mathcal{L}_{\mathcal{U}}(X)$ is decomposable into a directed join of elements in the family $\beta : \mathbf{1} + X \to \mathcal{L}_{\mathcal{U}}(X)$, defined as*

$$\beta(\mathsf{inl}(\star)) :\equiv \bot,$$
$$\beta(\mathsf{inr}(x)) :\equiv \eta(x).$$

Lemma 6.1.14. *The map $\beta : \mathbf{1} + X \to \mathsf{K}(\mathcal{L}_{\mathcal{U}}(X))$ from Lemma 6.1.13 is an equivalence, meaning we have $\mathbf{1} + X \simeq \mathsf{K}(\mathcal{L}_{\mathcal{U}}(X))$.*

Corollary 6.1.15. *The type $\mathsf{K}(\mathcal{L}_{\mathcal{U}}(X))$ of finite elements of the lifting is small, for every small set X.*

Proof of Example 6.1.12. Let X be a small set.

(AD1) Follows directly from Lemma 6.1.13.

(AD2) Given in Corollary 6.1.15.

□

6.1.5 Scott domains

In the classical development of domain theory, a *Scott domain* is defined as an algebraic pointed DCPO that is bounded-complete. We now develop the notion of Scott domain in the context of de Jong and Escardó's constructive and predicative development. To ensure that our definition of Scott domain is amenable to constructive reasoning, we include an additional *decidability condition* that holds in all examples of interest.

For every pair of elements $x, y : D$ of a domain D, we define the proposition

$$x \uparrow y \quad :\equiv \quad \exists_{(u:D)}(x \sqsubseteq u \times y \sqsubseteq u),$$

expressing that elements x and y are *bounded above* (or *upper bounded*).

Definition 6.1.16 (Scott domain)**.** A *Scott domain* is a pointed algebraic DCPO D that satisfies the following two conditions:

(SD1) *bounded-completeness*: every small family that has an upper bound, also has a least upper bound,

(SD2) *decidability of upper boundedness for compact elements*: the type $c \uparrow d$ is decidable, for every pair of compact elements $c, d : \mathsf{K}(D)$.

Lemma 6.1.17 ()**.** *For every pair of elements x and y of a Scott domain D, if there does not exist an upper bound for x and y, then we have $\uparrow x \wedge \uparrow y = \emptyset$.*

Proof. Let $x, y : D$ be a pair of elements of some Scott domain D and assume that they are not bounded above. Consider some $z : D$ with $z \in \uparrow x \cap \uparrow y$. This is to say $x \sqsubseteq z$ and $y \sqsubseteq z$, meaning z is an upper bound of x and y. This contradicts the assumption that x and y are not bounded above. □

We will see that, in quite a few examples, these conditions satisfy the decidability condition by virtue of being trivially true.

Lemma 6.1.18. *Every small-complete poset automatically satisfies the conditions* (SD1) *and* (SD2).

Proof. Let L be a small-complete lattice. Condition (SD1) holds trivially in the presence of least upper bounds for all small families. Furthermore, small-completeness implies that every pair of elements has a least upper bound, meaning for every pair of elements c and d the proposition $c \uparrow d$ is trivially true and hence decidable. □

Examples of Scott domains

Example 6.1.19. The lifting $\mathcal{L}_{\mathcal{U}}(\mathbf{1})$ is a Scott domain since we know by Example 6.1.5 that $\mathcal{L}_{\mathcal{U}}(\mathbf{1})$ is the initial frame $\Omega_{\mathcal{U}}$ qua DCPO, and is thus small-complete. Lemma 6.1.18 then implies that it is a Scott domain.

Example 6.1.20. The domain $\mathcal{P}(\mathbb{N})$ is a Scott domain, as we know that it is small-complete by Example 3.1.20.

6.2 Scott locale of an algebraic DCPO

Having summarized the preliminaries of constructive and predicative domain theory in univalent foundations, we now begin to develop the topology of domains. We start by defining the *Scott topology* over a DCPO.

Definition 6.2.1 (Scott open)**.** Let D be DCPO (over base universe $\mathcal{U}$) and consider a subset $S : D \to \Omega_{\mathcal{U}}$. The subset S is called *Scott open* if

(SO1) it is upward closed, and
(SO2) it is inaccessible by small, directed families, that is,

$$\bigsqcup_{i:I} x_i \in S \Rightarrow \exists_{(i:I)} x_i \in S,$$

for every small and directed family $(x_i)_{i:I}$ of elements of D.

Remark 6.2.2. It is possible to generalize the above definition as to define the Scott open $\mathcal{V}$-subsets of a $\mathcal{U}$-DCPO. However, this generalization is not necessary for our purposes, and nor is its mathematical meaning clear. The reader interested in the generalization of universes is referred to the Agda formalization where some of the results we present here are carried out at a higher level of universe generality.

As explained by Smyth [Smy93, §2.2], it is often useful to think of the Scott open subsets of a DCPO D as the *finitely verifiable* properties of the elements of D, which

can be thought of as representing computations. Under this interpretation,

- Condition (SO1) says that, once a computation x has been verified to satisfy a finitely verifiable property U, then every further refinement of the computation x will also satisfy the property U.
- Condition (SO2) says that, if the limit of a sequence of increasingly more refined computations satisfies a finitely verifiable property U, then there must be some computation in the sequence that *already* satisfies the property U. In other words, it should always be possible to judge that the limit computation satisfies the property on the basis of its finite approximations.

6.2.1 Frame structure

We now show that the poset of Scott opens has finite meets and small joins.

Lemma 6.2.3. *Scott open subsets are closed under finite meets. In other words, the full subset $x \mapsto \top$ is Scott open and the intersection of two Scott open subsets is Scott open.*

Proof. Trivial and well-known fact of domain theory. The proof is unchanged in our setting. □

Lemma 6.2.4. *The union of a small family of Scott open subsets is Scott open.*

Proof. Let $(U_i)_{i:I}$ be a small family of Scott open subsets over some DCPO D.

(SO1): Let $x \in \bigcup_{i:I} U_i$ and consider some $y \sqsupseteq x$. By the definition of subset union, there is some U_i with $x \in U_i$. It follows from the upward closedness of U_i that $y \in U_i$, and we thus have $y \in \bigcup_{i:I} U_i$.

(SO2): Let $(x_j)_{j:J}$ be a directed family of elements of D such that

$$\left(\bigsqcup_{j:J} x_j\right) \in \bigcup_{i:I} U_i.$$

We know that there is some U_i with $\left(\bigsqcup_{j:J} x_j\right) \in U_i$, and it thus follows from the inaccessibility of U_i by directed joins that there is some $x_j \in U_i$. This implies that $x_j \in \bigcup_{i:I} U_i$ by definition, which was what we needed.

□

We have thus established that the poset of Scott open subsets over every DCPO has finite meets and is small-complete.

6.2.2 Local smallness

In Example 3.1.20, we defined the discrete locale over a set X and discussed the issue that it is not locally small unless the set X in consideration is small. We formally showed this with the resizing taboo we gave in Corollary 3.1.22, establishing that the local smallness of the powerset frame does not hold in general without a form of propositional resizing.

A similar situation is present in the poset of Scott open subsets over a DCPO. Although this poset is small-complete, it does not seem to be locally small in general. It *is*, however, locally small when the DCPO in consideration is algebraic.

Lemma 6.2.5 (🤌). *For every DCPO D (over some base universe $\mathcal{U}$), if D is algebraic then the inclusion order has $\mathcal{U}$-small truth values.*

Proof. Let D be an algebraic DCPO and consider two Scott opens $U, V : D \to \Omega_{\mathcal{U}}$. We define the following small version of the inclusion order:

$$U \subseteq_s V \quad :\equiv \quad \Pi_{(c : \mathsf{K}(D))}\, c \in U \Rightarrow c \in V.$$

We need to show that $U \subseteq_s V$ implies $U \subseteq V$. Suppose $U \subseteq_s V$ and let $x \in U$. Since the DCPO D is algebraic, we know that $x = \bigsqcup \{c : \mathsf{K}(D) \mid c \sqsubseteq x\}$. The fact that $(\bigsqcup \{c : \mathsf{K}(D) \mid c \sqsubseteq x\}) \in U$ implies, thanks to (SO2), that there is some $c \in U$ with $c \sqsubseteq x$. We thus have $c \in V$ by our assumption of $S \subseteq_s V$, which implies $x \in V$ by the upward closedness of Scott open subsets. □

Lemma 6.2.6 (🤌). *For every algebraic DCPO D, the poset of Scott open subsets of D forms a frame.*

Proof. The existence of finite meets and small joins was established in Lemmas 6.2.3 and 6.2.4. We have just shown local smallness in Lemma 6.2.5 above. The frame distributivity law holds automatically since this is a subframe of the powerset frame. □

Definition 6.2.7 (🤌). The frame constructed in the above lemma is called the *frame of Scott open subsets* of an algebraic DCPO D and is denoted $\sigma(D)$. We refer to the locale that this defines as the *Scott locale* of an algebraic DCPO D and denote it $\mathrm{Scott}(D)$.

6.2.3 Elementary properties of the Scott locale

In this section, we collect some key properties of the Scott locale.

Lemma 6.2.8. *For every pointed DCPO D, the principal filter $\uparrow \bot_D$ is the top element of the frame $\sigma(D)$.*

Proof. It is easy to see that every element of D falls in $\uparrow \bot_D$ by virtue of $\bot_D$ being the bottom element. □

Lemma 6.2.9 (🤌). *For every Scott open subset $U : D \to \Omega_{\mathcal{U}}$ of a pointed DCPO D, the subset U contains the bottom element $\bot_D$ if and only if it is the top element of the frame $\sigma(D)$.*

Proof. For every pointed DCPO D and Scott open subset U, it is easy to see that, if U contains $\bot_D$ then U contains every element of x by upward closedness. □

Lemma 6.2.10 (🤌). *For every element $x : D$ of an algebraic DCPO D, the element x is finite if and only if the principal filter $\uparrow x$ is Scott open.*

Proof. Let $x : D$ be an element of a DCPO D. It is easy to see that $\uparrow x$ being inaccessible by directed joins is exactly the finiteness of x, since it says, for every directed family $(y_i)_{i:I}$ with $\bigsqcup_{i:I} y_i \in \uparrow x$, there is some $i : I$ with $y_i \in \uparrow x$. The ($\Leftarrow$) direction immediately follows from this. For the ($\Rightarrow$) direction, observe that $\uparrow x$ always satisfies Condition (SO1) by construction. □

Lemma 6.2.11 (🤌). *Every Scott open subset $U : D \to \Omega_{\mathcal{U}}$, of an algebraic DCPO D, is expressible as the union of the principal filters generated by the finite elements that it contains.*

Proof. Let D be an algebraic DCPO, and consider a Scott open subset $U : D \to \Omega_{\mathcal{U}}$. We claim that

$$U \quad = \quad \bigcup \{\uparrow c \mid c \in U, c \text{ finite}\}.$$

It is easy to see that the ($\supseteq$) direction holds since the upward closedness of Scott open subsets implies that $\uparrow x \subseteq U$, for every $x \in U$.

For the ($\subseteq$) direction, let $x \in U$. We need to show that there is a finite $c \in U$ with $x \in \uparrow c$. As the DCPO in consideration is algebraic, we know that

$$x = \bigsqcup \{c : \mathrm{K}(D) \mid c \sqsubseteq x\}.$$

The inaccessibility of U by directed joins then implies that there is some finite $c \in U$ such that $c \sqsubseteq x$, which is to say $x \in \uparrow c$. □

Lemma 6.2.12 (🤌). *For every finite element c of a DCPO, the principal filter $\uparrow c$ is a compact open of the frame $\sigma(D)$.*

Proof. Let D be a DCPO and consider a finite element $c : D$. To see that the principal filter $\uparrow c$ is compact, let $(U_i)_{i:I}$ be a directed family of Scott open subsets such that $\uparrow c \subseteq \bigcup_{i:I} U_i$. As $c \in \uparrow c$, we have that $c \in \bigcup_{i:I} U_i$ i.e. $c \in U_i$ for some $i : I$. It follows from the upward closedness of U_i that $\uparrow c \subseteq U_i$, which was what we needed. □

When the DCPO in consideration is pointed, the principal filter $\uparrow \bot$ gives the top Scott open as we explained in Lemma 6.2.8. Combining this with the above lemma, we obtain:

Lemma 6.2.13 (🤌). *For every pointed DCPO D, the locale $\mathrm{Scott}(D)$ is compact.*

Proof. By Lemma 6.2.8, the top open of $\mathrm{Scott}(D)$ is $\uparrow \bot_D$, which is compact by Lemma 6.2.12 since $\bot_D$ is always a finite element (Lemma 6.1.8). □

6.2.4 Base construction for the Scott locale over a Scott domain

We showed in Lemma 6.2.11 that every Scott open subset can be decomposed as the union of the principal filters it contains. We now extend this result and show that the principal filters on the compact elements in fact form a basis for the Scott locale.

Lemma 6.2.11 suggests that the base of $\mathrm{Scott}(D)$ should be given by the family

$$\uparrow(\text{-}) : \mathsf{K}(D) \to \sigma(D)$$

of principal filters on finite elements of D. Our primary motivation for this base construction is to establish that $\mathrm{Scott}(D)$ is a spectral locale for every Scott domain D, and it turns out to be more convenient to work with all finite unions of such principal filters for this purpose.

Lemma 6.2.14 (🤌). *For every Scott domain D, the family $\gamma : \mathsf{List}(\mathsf{K}(D)) \to D$, defined by*

$$\gamma(c_0 \dots c_{n-1}) \quad :\equiv \quad \uparrow c_0 \cup \dots \cup \uparrow c_{n-1},$$

forms a strong, directed base for the locale $\mathrm{Scott}(D)$.

Proof. Let D be a Scott domain and consider a Scott open subset $U : D \to \Omega_{\mathcal{U}}$. As the basic covering family for U, we pick the basic covering index family

$$\left\{(c_0 \dots c_{n-1}) : \mathsf{List}(\mathsf{K}(D)) \mid \Pi_{(i:\mathbb{N})}\, 0 \leq i < n \Rightarrow c_i \in U\right\},$$

of all lists of compact elements contained in U. Formally, this gives the basic covering:

$$\left(\Sigma_{(s:\mathsf{List}(\mathsf{K}(D)))} \Pi_{(c:\mathsf{K}(D))}\, c \in s \Rightarrow \uparrow c \in U\right) \xhookrightarrow{\gamma \circ \mathrm{pr}_1} \mathcal{O}(\mathrm{Scott}(D))$$

It easily follows from Lemma 6.2.11 that this family gives U as its join.

It is also easy to see that this family is directed. It is always inhabited by the empty list ε which gives $\gamma(\varepsilon) = \emptyset$. Furthermore, given a pair of lists $s :\equiv c_0 \dots c_{m-1}$ and $t :\equiv d_0 \dots d_{n-1}$ of finite elements of D, the least upper bound of the basic Scott opens $\gamma(s)$ and $\gamma(t)$ is in the family since it is give by $\gamma(sl)$. □

Observe that the base constructed above is small, since (1) the type $\mathsf{K}(D)$ is a small type, and (2) the type of lists over a small type is small.

By appealing to Lemma 4.2.4 on the above base, we obtain:

Corollary 6.2.15. *Every compact Scott open K is equal to a union $\uparrow c_0 \cup \dots \uparrow c_{n-1}$ for some list $c_0 \dots c_{n-1}$ of finite elements of D, for every Scott domain D.*

6.2.5 Spectrality of the Scott locale over a Scott domain

As we have already constructed a base for the Scott locale, we can easily establish its spectrality by showing that this base is spectral as in Definition 3.6.1.

Lemma 6.2.16 (✍). *The base constructed in Lemma 6.2.14 consists of compact opens. In other words, for every list $c_0 \dots c_{n-1}$ of finite elements of an algebraic DCPO D, the Scott open subset given by the union $\uparrow c_0 \cup \dots \cup \uparrow c_{n-1}$ is compact.*

Proof. We know by Lemma 3.3.8 that compact opens are closed under finite joins. □

We now would like to show that the base $\gamma : \mathsf{List}(\mathsf{K}(D)) \to \sigma(D)$ is closed under binary intersections, for which it is crucial that the DCPO in consideration is a Scott domain.

Lemma 6.2.17. *For every Scott domain D, the principal filter map*

$$\uparrow(\text{-}) : \mathsf{K}(D) \to \mathcal{O}(\mathsf{Scott}(D))$$

is (1) antitone, and (2) reverses joins, to the extent that they exist.

Lemma 6.2.18 (✍). *For every Scott domain D, the base for $\mathsf{Scott}(D)$, constructed in Lemma 6.2.14, is closed under binary intersections.*

Proof sketch. Let D be a Scott domain, and consider two lists $c_0 \dots c_{m-1}$ and $d_0 \dots d_{m-1}$ of finite elements. We need to show that $\gamma(c_0 \dots c_{m-1}) \cap \gamma(d_0 \dots d_{n-1})$ falls in the image of γ. It can be shown by straightforward induction that

$$\gamma(c_0 \dots c_{m-1}) \cap \gamma(d_0 \dots d_{n-1}) = \gamma(c_0 \dots c_{m-1} d_0 \dots d_{n-1}).$$

□

Theorem 6.2.19 (✍). *For every Scott domain D, its Scott locale $\mathsf{Scott}(D)$ is spectral.*

Proof. Let D be a Scott domain.

- ($\mathrm{SP_b}1$) Given in Lemma 6.2.16.
- ($\mathrm{SP_b}2$) We know that the top Scott open is $\uparrow \bot_D$. It is thus easy to see that the base contains it, since $\bot_D$ is always compact and thus $\gamma(\bot) = \uparrow \bot = \mathbf{1}_{\sigma(D)}$.
- ($\mathrm{SP_b}3$) Given in Lemma 6.2.18.

□

6.2.6 The universal property of Sierpiński

In Section 3.1.5, we defined the Sierpiński locale and postponed the proof of its universal property. Now that we have the necessary topological framework for algebraic DCPOs in place, we shall now revisit this question and prove the universal property

of the Sierpiński locale. This also serves as an illustrative example of the technique of presenting a locale through a DCPO.

Recall that in Example 3.1.23 we defined the Sierpiński locale $\mathbb{S}_{\mathcal{U}}$ as the locale given by the frame of Scott continuous endomaps on the initial frame $\Omega_{\mathcal{U}}$. An alternative description of this frame is as the Scott locale of the lifting DCPO $\mathcal{L}_{\mathcal{U}}(\mathbf{1})$.

Lemma 6.2.20. *The frame $\sigma(\mathcal{L}_{\mathcal{U}}(\mathbf{1}))$ is isomorphic to the frame of Scott continuous maps on the initial frame $\Omega_{\mathcal{U}}$.*

Proof. The type $\mathcal{L}_{\mathcal{U}}(\mathbf{1})$ is defined as $\Sigma_{(P:\Omega_{\mathcal{U}})} P \to \mathbf{1}$, which is clearly equivalent to $\Omega_{\mathcal{U}}$. We now show that a function $\Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}}$ is Scott continuous if and only if it is Scott open (in the sense of Definition 6.2.1). Let $S : \Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}}$ and let $(P_i)_{i:I}$ be a directed family of propositions.

($\Rightarrow$) Suppose S is Scott continuous. It is a well-known fact of domain theory that Scott continuous maps are monotone, which is exactly the upward closedness of S, viewed as a subset of $\Omega_{\mathcal{U}}$. To see that S is inaccessible by directed joins, suppose $S\left(\bigvee_{i:I} P_i\right)$ holds. This implies, by the Scott continuity of S, that $\bigvee_{i:I} S(P_i)$ holds, which says exactly that some P_k holds.

($\Leftarrow$) Conversely, suppose S is a Scott open of $\Omega_{\mathcal{U}}$ qua DCPO. We need to show that

$$S\left(\bigvee_{i:I} P_i\right) = \bigvee_{i:I} S(P_i).$$

The upward closedness of S is exactly the fact that it is monotone, which immediately gives $\bigvee_{i:I} S(P_i) \leq S\left(\bigvee_{i:I} P_i\right)$. For the other direction, suppose we know $S\left(\bigvee_{i:I} P_i\right)$. By the inaccessibility of S by directed joins, we have that there exists some $k : I$ such that $S(P_k)$, which is to say that we have $\bigvee_{i:I} S(P_i)$.

Finally, it is verified easily that the orders of the two frames are the same, from which we conclude that the two are isomorphic frames. □

Recall that we denote by truth the singleton open of the Sierpiński locale, which is simply the identity map $\mathsf{truth}(P) :\equiv P$, asserting a given proposition to be true.

Lemma 6.2.21. *The map* truth *is the principal filter* $\uparrow\top$.

Proof. It is an easy observation that a proposition P holds if and only if it is below the true proposition $\top$. □

Since principal filters on finite elements of DCPOs are compact opens of the Scott locale (as shown in Lemma 6.2.12), it follows that:

Corollary 6.2.22. *The singleton open* truth *is compact.*

Lemma 6.2.23. *An open of Sierpiński is compact if and only if it is either the top open $\mathbf{1}_{\mathbb{S}}$, the singleton open* truth*, or the bottom open $\mathbf{0}_{\mathbb{S}}$.*

Proof. We know that all three of these are compact: the bottom open $\mathbf{0}_{\mathbb{S}}$ is compact in every locale, the fact that $\mathbf{1}_{\mathbb{S}}$ is compact was established in in Example 3.3.7, and we showed in Corollary 6.2.22 that truth is compact. We now show that *every* compact open of the Sierpiński locale is one of the above. Firstly, we already know by Corollary 4.2.5 that the compact opens are exactly the opens falling in the base. It thus suffices to show that every basic open is one of the above three opens. Recall the base construction for the Scott locale from Lemma 6.2.14: the base of $\mathrm{Scott}(\mathcal{L}_{\mathcal{U}}(\mathbf{1}))$ consists of finite unions of principal filters on the basic elements of the DCPO in consideration. We know that $\mathcal{L}_{\mathcal{U}}(\mathbf{1})$ has exactly two finite elements: $\top_{\mathcal{U}}$ and $\bot_{\mathcal{U}}$. We now proceed by induction to show that each finite union of principal filters on these basic elements is one of the above three opens.

- Base case: the empty list. The empty union is $\mathbf{0}_{\mathbb{S}}$.
- Inductive step, case (1): $\uparrow\top \cup \uparrow c_1 \cup \cdots \cup \uparrow c_{n-1}$. The principal filter $\uparrow\top$ is exactly the open truth, as established in Corollary 6.2.22. By the inductive hypothesis, we know that $\uparrow c_1 \cup \cdots \cup \uparrow c_{n-1}$ is one of the three opens of interest. If it is $\mathbf{0}_{\mathbb{S}}$, the union is $\mathsf{truth} \cup \mathbf{0}_{\mathbb{S}} = \mathsf{truth}$. If it is $\mathbf{1}_{\mathbb{S}}$, the union is simply $\mathsf{truth} \cup \mathbf{1}_{\mathbb{S}} = \mathbf{1}_{\mathbb{S}}$. If it is truth, the union is $\mathsf{truth} \cup \mathsf{truth} = \mathsf{truth}$.
- Inductive step, case (2): $\uparrow\bot \cup \uparrow c_1 \cup \cdots \cup \uparrow c_{n-1}$. We know by Lemma 6.2.8 that $\uparrow\bot = \mathbf{1}_{\mathbb{S}}$, so the union is simply $\mathbf{1}_{\mathbb{S}}$.

□

We are now ready to give the proof of Proposition 3.1.25, in which we stated that our construction of the Sierpiński locale has the universal property of Sierpiński (from Definition 3.1.24).

Proof of Proposition 3.1.25. Let X be a locale and let $U : \mathcal{O}(X)$. We need to show that there is a continuous map $f : X \to \mathbb{S}$ unique with the property that $U = f^*(\mathsf{truth})$.

We define this map as follows: first, we define a family

$$\alpha : \Pi_{(S:\mathcal{O}(\mathbb{S}))}\, S(\top) + S(\bot) \to \mathcal{O}(X),$$

which gives U if $\top \in S$ and $\mathbf{1}_X$ if $\bot \in S$. Formally, this is encoded as the function:

$$\alpha_S(\mathsf{inl}(p)) :\equiv U$$
$$\alpha_S(\mathsf{inr}(q)) :\equiv \mathbf{1}_X.$$

We then define the desired map $f : X \to \mathbb{S}$ as $f(S) :\equiv \bigvee \{\alpha_S(p) \mid p : S(\top) + S(\bot)\}$.

The fact that f satisfies the desired property of $U = f^*(\mathsf{truth})$ is obvious: $\mathsf{truth}(\top)$ always holds, so we have $\alpha_{\mathsf{truth}}(\mathsf{inl}(\star)) = U$.

Moreover, it is easy to see that f^* preserves the top element of $\mathcal{O}(\mathbb{S})$ as well as the joins. We focus on the preservation of binary meets: let $S, T : \Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}}$ be two Scott continuous maps. We need to show that $f^*(S \cap T) = f^*(S) \wedge f^*(T)$. The $f^*(S \cap T) \leq f^*(S) \wedge f^*(T)$ direction is trivial by monotonicity so we focus on the $f^*(S) \wedge f^*(T) \leq f^*(S \cap T)$ direction. First, observe that we have the following

equality thanks to distributivity:

$$\begin{aligned} & f^*(S) \wedge f^*(T) \\ \equiv \quad & \left(\bigvee \{\alpha_S(p) \mid p : S(\top) + S(\bot)\}\right) \wedge \left(\bigvee \{\alpha_S(p) \mid p : T(\top) + T(\bot)\}\right) \\ = \quad & \bigvee \{\alpha_S(p) \wedge \alpha_T(q) \mid p : S(\top) + S(\bot),\ q : T(\top) + T(\bot)\} \qquad (\dagger) \end{aligned}$$

It suffices to show that this is below $f(S \cap T)$, which expands to

$$\bigvee \left\{\alpha_{(S\cap T)}(p) \mid p : (S \cap T)(\top) + (S \cap T)(\bot)\right\} \qquad (\ddagger)$$

We show that each element of the family in ($\dagger$) is below some element of the family in ($\ddagger$). Let $p : S(\top) + S(\bot)$ and $q : T(\top) + T(\bot)$. We proceed by case analysis on the proofs p and q.

- Case: $p = \mathsf{inl}(p')$ and $q = \mathsf{inl}(q')$. This means that have $\top \in S$ and $\top \in T$, which implies $\alpha_S(\mathsf{inl}(p')) \wedge \alpha_S(\mathsf{inl}(q')) = U \wedge U = U$. In this context, it is easy to show that U is below ($\ddagger$) since $r :\equiv (p', q')$ constitutes proof that $\top \in S \cap T$, meaning we have $U \leq \alpha_{(S\cap T)}(\mathsf{inl}(r)) \equiv U$.
- Case: $p = \mathsf{inl}(p')$ and $q = \mathsf{inr}(q')$. This means that we have $\top \in S$ and $\bot \in T$, which implies $\alpha_S(\mathsf{inl}(p')) \wedge \alpha_T(\mathsf{inr}(q')) = U \wedge \mathbf{1}_X = U$. By Lemma 6.2.9, we know that a Scott open subset containing $\bot$ must also contain $\top$. Pairing this with $p' : S(\top)$, we obtain a proof $r : \top \in S \cap T$, and thus we can easily conclude $U \leq \alpha_{(S\cap T)}(\mathsf{inl}(r)) \equiv U$.
- Case: $p = \mathsf{inr}(p')$ and $q = \mathsf{inl}(q')$. The same argument as in the previous case applies here as well.
- Case: $p = \mathsf{inr}(p')$ and $q = \mathsf{inr}(q')$. This means that we have $\bot \in S$ and $\bot \in T$, which implies $\alpha_S(\mathsf{inr}(p')) \wedge \alpha_T(\mathsf{inr}(q')) = \mathbf{1}_X \wedge \mathbf{1}_X = \mathbf{1}_X$. Observe that (p', q') constitutes a proof that $\bot \in S \cap T$. The desired result follows from this since we have $\mathbf{1}_X \leq \alpha_{(S\cap T)}(\mathsf{inr}(p', q')) \equiv \mathbf{1}_X$.

We now proceed to show that the continuous map f is unique. Let $g : X \to \mathbb{S}$ be some other continuous map satisfying the property that $g^*(\mathsf{truth}) = U$. We show that $f^*(S) = g^*(S)$, for every $S : \mathcal{O}(\mathbb{S})$. The $f^*(S) \leq g^*(S)$ direction is easy to see, so we focus on the $g^*(S) \leq f^*(S)$ direction. We established in Lemma 6.2.20 that the Sierpiński locale is $\mathrm{Scott}(\mathcal{L}_{\mathcal{U}}(\mathbf{1}))$, which we know to be always spectral when the DCPO in consideration is a Scott domain. Thanks to this, we know that S can be decomposed as $S = \bigvee_{i:I} B_i$, where $(B_i)_{i:I}$ is a family of compact Scott opens. Because the map g^* is a frame homomorphism and thus monotone, it remains to show that $g(B_i) \leq f^*(S)$, for each $i : I$. We appeal to Lemma 6.2.23, which implies that each basic open B_i is either the top open $\mathbf{1}_{\mathbb{S}}$, the singleton open truth, or the bottom open $\mathbf{0}_{\mathbb{S}}$.

- Case: $B_i = \mathbf{1}_{\mathbb{S}}$. This is to say that $S = \mathbf{1}_{\mathbb{S}}$. The desired result follows, since we have $g^*(B_i) = \mathbf{1}_{\mathbb{S}} = f^*(\mathbf{1}_X) = f^*(S)$.
- Case: $B_i = \mathsf{truth}$. We know $g^*(\mathsf{truth}) = U$, and we already showed that $U = f^*(\mathsf{truth})$. This gives $g^*(B_i) = g^*(\mathsf{truth}) = f^*(\mathsf{truth}) = f^*(B_i) \leq f^*(S)$.
- Case: $B_i = \mathbf{0}_{\mathbb{S}}$. Observe that we have $g^*(\mathbf{0}_{\mathbb{S}}) = \mathbf{0}_X \leq f^*(S)$.

□

6.3 Sharp elements

de Jong [dJon21a] formulated a notion of *sharp element* in the context of his constructive development of domain theory. A sharp element [dJon21a, Definition 5.8] of an algebraic DCPO D is an element $x : D$ whose principal ideal $\downarrow x$ admits decidable membership for finite elements.

Although this notion can be formulated for DCPOs in general, its description in the context of algebraic DCPOs is much more concrete and straightforward. As algebraic DCPOs are our primary focus here, we will study sharp elements directly in the context of algebraic DCPOs.

6.3.1 Sharp elements of algebraic DCPOs

Definition 6.3.1 (*cf.* [dJon21a, Proposition 5.11])**.** An element $x : D$ of an algebraic DCPO D is called *sharp* if $c \sqsubseteq x$ is a decidable proposition for every finite element $c : D$. In other words, an element is sharp if it admits decidable membership in principal filters generated by finite elements.

Lemma 6.3.2. *For every element $x : D$ of an algebraic DCPO D, the proposition expressing that x is sharp is small.*

Proof. Directly follows from local smallness and the stipulation that $\mathsf{K}(D)$ is a small type □

Examples of sharp elements

Example 6.3.3. The sharp elements of $\mathcal{P}(\mathbb{N})$ qua DCPO are exactly the decidable subsets of $\mathbb{N}$.

Example 6.3.4. The sharp elements of the Sierpiński DCPO $\mathcal{L}_{\mathcal{U}}(\mathbf{1})$ are exactly the trivial propositions $\bot$ and $\top$.

Proof. Let P be a sharp element of $\Omega_{\mathcal{U}}$ qua DCPO. Since $\top_{\mathcal{U}}$ is a finite element of $\Omega_{\mathcal{U}}$, we have that $\top_{\mathcal{U}} \Rightarrow P$ is decidable. If $\top_{\mathcal{U}} \Rightarrow P$, then $P = \top$. Otherwise, we have $\neg(P = \top)$, which means $P = \bot$. □

Example 6.3.5. The sharp elements of the flat domain $\mathcal{L}_{\mathcal{U}}(\mathbb{N})$ are those elements (P, φ) such that $\eta(n) \sqsubseteq (P, \varphi)$ is decidable, for every $n : \mathbb{N}$.

Proof. We know by Lemma 6.1.14 that the finite elements of $\mathcal{L}(\mathbb{N})$ are $\bot$ along with those of the form $\eta(n)$ for $n : \mathbb{N}$. This is to say sharp elements of $\mathcal{L}(\mathbb{N})$ are those elements x such that $\bot \sqsubseteq x$ and $\eta(n) \sqsubseteq x$ are decidable. The former is trivially true meaning x is sharp if and only if $\eta(n) \sqsubseteq x$ is decidable, for every $n : \mathbb{N}$. □

6.3.2 Topological characterization of sharp elements

We now give a new characterization of the sharp elements of an algebraic DCPO D in terms of the Scott opens of D. We first observe that every element $x : D$ that admits decidable membership in compact Scott opens is sharp:

Lemma 6.3.6. *For every element $x : D$ of an algebraic DCPO D, we have that x is sharp whenever the proposition $x \in K$ is decidable for every compact Scott open $K : D \to \Omega_{\mathcal{U}}$.*

Proof. Let $x : D$ be an element that admits decidable membership in compact Scott opens. Let c be a finite element of D. Since we know by Lemma 6.2.12 that $\uparrow c$ is a compact Scott open, the proposition $x \in \uparrow c$ is decidable which is to say that x is a sharp element. □

In fact, it turns out that the converse is also true:

Lemma 6.3.7 (🤌). *An element $x : D$ of an algebraic DCPO is sharp if and only if it admits decidable membership in compact Scott opens.*

Proof. We established the ($\Leftarrow$) direction in Lemma 6.3.6. To see that the ($\Rightarrow$) direction also holds, let $x : D$ be a sharp element and let $K : D \to \Omega_{\mathcal{U}}$ be a compact Scott open In Lemma 6.2.14, we showed that the family

$$(c_0 \ldots\ldots c_{n-1}) \mapsto \uparrow c_0 \cup \cdots \cup \uparrow c_{n-1}$$

forms a base for the locale $\mathrm{Scott}(D)$. We thus know by Corollary 6.2.15 that there must be a list $c_0 \ldots c_{n-1}$ such that $K = \uparrow c_0 \cup \cdots \cup \uparrow c_{n-1}$. Since the proposition $x \in \uparrow c_i$ is decidable for each i with $0 \leq i < n$, it follows that $x \in K$ is decidable, since decidable propositions are closed under finite joins. □

6.4 Spectral points of the Scott locale

The notion of a space in point-set topology is defined as a set X of points equipped with a topology i.e. a subframe of the powerset frame $\mathcal{P}(X)$. In other words, the opens are defined in terms of the points. We previously discussed that, in point-free topology, this conceptual state of affairs is reversed: we start with the opens, and define points in their terms when the need arises. Up until this section, we have not had any motivation to work with the *points of a locale*. We now start discussing points, as they arise naturally in our study of the Lawson topology of a Scott domain in a constructive setting.

Before we delve into the technical details, we provide a conceptual summary of the notion of point in the context of point-free topology. Consider a point $x \in X$ of a topological space X. We can collect together all open neighbourhoods of x, which can be thought of as the class of all *observable properties* of x. We denote this by $\mathcal{F}_x$:

$$\mathcal{F}_x \quad :\equiv \quad \{U \in \mathcal{O}(X) \mid x \in U\}.$$

When we view $\mathcal{F}_x$ as a lattice under the inclusion order, we can think of $\mathcal{F}_x$ as the class of observations about x that get increasingly better, eventually converging around x. This satisfies three properties:

1. *Upward closedness*: for every $U \in \mathcal{F}_x$ and every $V \supseteq U$, we have that $V \in \mathcal{F}_x$.
2. *Closure under binary intersections*: for every pair of opens $U, V \in \mathcal{F}_x$, the intersection $U \cap V$ is also in $\mathcal{F}_x$.
3. *Inaccessibility by unions*: if we have $(\bigcup_{i:I} U_i) \in \mathcal{F}_x$ for a family $(U_i)_{i:I}$ of opens, then there is some $i : I$ such that $U_i \in \mathcal{F}_x$.

These three properties define what is called a *completely prime filter* of the frame $\mathcal{O}(X)$ of open subsets of X. If there is a bijection between the set X of points and the class of completely prime filters of the frame $\mathcal{O}(X)$, then the space X is said to be *sober*. In the point-free setting where one works with an abstract frame of opens, one takes a completely prime filter as the definition of a point of a locale. For further reading on points in point-free topology, we refer the reader to [PP11, §II.3.1].

6.4.1 The notion of point in locale theory

In the literature on locale theory, there are three common definitions of the notion of point of a locale X [Joh82, §II.1.3]:

1. as a completely prime filter of the frame $\mathcal{O}(X)$, on which we elaborated above,
2. as a continuous map $\mathbf{1}_{\mathcal{U}} \to X$, and
3. as a prime element of the frame $\mathcal{O}(X)$.

Whilst Conditions (2) and (1) are equivalent constructively, the equivalence of Condition (3) to the other two is fundamentally classical. We take (2) as our primary formulation of the notion of point:

Definition 6.4.1. A *point* of a locale X (over base universe $\mathcal{U}$) is a continuous map

$$\mathbf{1}_{\mathcal{U}} \to X,$$

which is the same thing as a frame homomorphism $\mathcal{O}(X) \to \Omega_{\mathcal{U}}$.

We denote the type of points of a locale by $\mathbf{Pt}(X)$.

Points are exactly the completely prime filters

Definition 6.4.2 (Filter)**.** A subset $F \subseteq L$ of a frame L is called a *filter* if it is

- upward closed, and
- closed under finite meets.

Definition 6.4.3 (Completely prime filter)**.** A filter $F \subseteq L$ in a frame L is said to be *completely prime* or (*small-completely prime* for extra clarity) if for every small family $(x_i)_{i:I}$ with $(\bigvee_{i:I} x_i) \in F$, there is some $i : I$ such that $x_i \in F$.

Lemma 6.4.4. *Points of a locale X amount to the completely prime filters of $\mathcal{O}(X)$.*

Proof. Let X be a locale over base universe $\mathcal{U}$. For every point $p : \mathbf{1}_{\mathcal{U}} \to X$, it is easy to see that its defining frame homomorphism $p^* : \mathcal{O}(X) \to \Omega_{\mathcal{U}}$ is a completely prime filter, as it preserves finite meets and small joins. Conversely, every completely prime filter $F : \mathcal{O}(X) \to \Omega_{\mathcal{U}}$ is a frame homomorphism. Since F contains the top element, we have $F(\mathbf{1}_X) = \top$. Furthermore, the upward closedness of F says exactly that it is monotone when viewed as a frame homomorphism. That F is closed under binary meets means $F(U) \wedge F(U) \leq F(U \wedge V)$, which implies by monotonicity that $F(U \wedge V) = F(U) \wedge F(V)$. Finally, the complete primality condition says exactly that

$$F\left(\bigvee_{i:I} U_i\right) \leq \bigvee_{i:I} F(U_i),$$

from which it follows by monotonicity that

$$F\left(\bigvee_{i:I} U_i\right) = \bigvee_{i:I} F(U_i).$$

□

We will often use the letter p for variables ranging over the set $\mathbf{1} \to X$ of points of a locale X. In light of the equivalence above, however, we will also use the variable letter F, when we are taking the view of a point simply as a subset $F : \mathcal{O}(X) \to \Omega$.

6.4.2 Spectral points

In Definition 4.2.14, we defined a spectral map as a map that reflects compact opens. We now look at points that are spectral when considered as continuous maps.

Definition 6.4.5. A point $\mathbf{1} \to X$ is called *spectral* if it is a spectral map of locales.

We denote by $\mathbf{Pt}_{\mathsf{sp}}(X)$ the type of spectral points of a locale X.

Lemma 6.4.6. *A point $p : \mathbf{1}_{\mathcal{U}} \to X$ is spectral if and only if the frame homomorphism $p^* : \mathcal{O}(X) \to \Omega_{\mathcal{U}}$ is a subset admitting decidable membership of compact opens of X.*

Proof. Let $p : \mathbf{1} \to X$ be a point. Recall that the map p being spectral amounts to the statement that $p^*(K)$ is a compact open of the initial frame $\Omega_{\mathcal{U}}$ whenever K is a compact open of X. We know by Example 4.6.13 that the terminal locale $\mathbf{1}_{\mathcal{U}}$ is Stone, and we thus know that its compact opens coincide with its clopens since Lemma 4.6.2 says that the compact opens coincide with the clopens in Stone locales. It is therefore easy to see that p^* preserves compact opens if and only if it maps the compact opens of X to the clopens of $\mathbf{1}_{\mathcal{U}}$, which are exactly the decidable propositions. □

6.5 Sharp elements and spectral points

We now turn to the relationship between the two notions we have discussed so far: sharp elements and spectral points. It turns out that the sharp elements of a Scott domain D coincide with the spectral points $\mathbf{1} \to \mathrm{Scott}(D)$. Before we get to the proof of this, however, we will need to do some preparation.

6.5.1 Points of the Scott locale of Scott domain

We will need the following auxiliary lemma.

Lemma 6.5.1 (🤌). *Let D be a Scott domain. For every point F of $\mathrm{Scott}(D)$, the family*

$$\{c : \mathsf{K}(D) \mid \uparrow c \in F\}$$

is small and directed.

Proof. Let D be a Scott domain and consider a point F of $\mathrm{Scott}(D)$. It is easy to see that the family $\{c : \mathsf{K}(D) \mid \uparrow c \in F\}$ is small, as the quantification over $\mathsf{K}(D)$ and the proposition $\uparrow c \in F$ are both small.

We now show that this family is directed. The bottom element $\bot_D$ of the domain is a finite element (Lemma 6.1.8) and always inhabits the above family since we always have that $\uparrow \bot_D = \mathbf{1}_{\sigma(D)}$, and we also know $\mathbf{1}_{\sigma(D)} \in F$ since F is a filter. To see that this family is directed, let $b, c : \mathsf{K}(D)$ satisfying $\uparrow b \in F$ and $\uparrow c \in F$.

We need to show that there is some compact d that is above both b and c and satisfies $\uparrow d \in F$. We appeal to our assumption that $b \uparrow c$ is a decidable proposition (Condition (SD2) from Definition 6.1.16) and proceed by case analysis.

(**Case 1**) There exists an upper bound u of b and c. By bounded-completeness, the least upper bound of b and c must then exist, for which we write $b \vee c$. This is finite as finite elements are always closed under binary joins. It remains to show that $\uparrow(b \vee c) \in F$. The principal filter operation $\uparrow(\text{-})$ reverses joins so we have $\uparrow(b \vee c) = \uparrow b \wedge \uparrow c$. It then remains to show $(\uparrow b \wedge \uparrow c) \in F$, which holds if and only if $F(\uparrow b) \wedge F(\uparrow c)$, by the fact that F preserves meets. Notice that this means we are done because we already know that both conjuncts $F(\uparrow b)$ and $F(\uparrow c)$ hold.

(**Case 2**) There does not exist an upper bound of b and c. We then know by Lemma 6.1.17 that $\uparrow b \wedge \uparrow c = \mathbf{0}_{\sigma(D)}$. Because we know $\uparrow b \in F$ and $\uparrow c \in F$, it must be the case that $F(\uparrow b \wedge \uparrow c)$ holds since $F(\uparrow b \wedge \uparrow c) = F(\uparrow b) \wedge F(\uparrow c)$ by meet preservation. This means that $F(\mathbf{0}_{\sigma(D)})$ holds, which is a contradiction since we know that $F(\mathbf{0}_{\sigma(D)}) = \bot$ as F is a frame homomorphism. □

The idea above is that, by collecting together all the finite elements consistent with the property expressed by the filter F, we obtain a directed family that gives the specified computation in the limit.

Lemma 6.5.2. *We have an equivalence $D \simeq \mathbf{Pt}(\mathrm{Scott}(D))$, for every Scott domain D.*

Proof. Let D be a Scott domain. We define maps

$$D \underset{v}{\overset{\mathrm{pt}_{(\text{-})}}{\rightleftarrows}} \mathbf{Pt}(\mathrm{Scott}(D))$$

as follows:

$$\begin{aligned} \mathrm{pt}_x(U) &:\equiv x \in U \\ v(F) &:\equiv \bigsqcup \{c : \mathsf{K}(D) \mid \uparrow c \in F\}. \end{aligned}$$

We start by checking that these functions are well-defined. The fact that the directed join above exists follows from the directedness of the family $\{c : \mathsf{K}(D) \mid \uparrow c \in F\}$, which was established in Lemma 6.5.1. The fact that $\mathrm{pt}_x : \mathcal{O}(\mathrm{Scott}(D)) \to \Omega$ defines a frame homomorphism for every $x : D$ is easy to see. Let $x : D$. It is immediate that it preserves the top element since $\mathrm{pt}_x(\mathbf{1}) = x \in \mathbf{1} = \top$. Given two Scott open subsets U and V of D, we have that $\mathrm{pt}_x(U \cap V) \equiv x \in U \cap V$ if and only if $\mathrm{pt}_x(U)$ and $\mathrm{pt}_x(V)$ hold, which concludes that this map also preserves meets. It is similarly easy to see that it preserves joins. Let $(U_i)_{i:I}$ be a small family of Scott open subsets. We have that $\mathrm{pt}_x(\bigvee_{i:I} U_i) \equiv x \in \bigvee_{i:I} U_i$ if and only if $\exists i : I.\ x \in U_i$ if and only if $\exists i : I.\ \mathrm{pt}_x(U_i)$, which is the same thing as $\bigvee_{i:I} \mathrm{pt}_x(U_i)$.

We now show that v is a retraction of $\mathrm{pt}(\text{-})$ and vice versa. Let $x : D$. We need to show that $v \circ \mathrm{pt}_x = x$. This follows directly from the fact that D is an algebraic domain:

$$\begin{aligned} v(\mathrm{pt}_x) &= \bigsqcup \{c : \mathsf{K}(D) \mid \uparrow c \in \mathrm{pt}_x\} \\ &= \bigsqcup \{c : \mathsf{K}(D) \mid x \in \uparrow c\} \\ &= \bigsqcup \{c : \mathsf{K}(D) \mid c \sqsubseteq x\} \\ &= x. \end{aligned}$$

To see that $\mathrm{pt}(\text{-})$ is a retract of v, let F be point of $\mathrm{Scott}(D)$. We need to show that $\mathrm{pt}_{v(F)}(U) = F(U)$. By Lemma 6.2.11, it suffices to show

$$\mathrm{pt}_{v(F)}(U) = F\left(\bigvee \{\uparrow c \mid c \in U, c \text{ finite}\}\right),$$

and because F is a frame homomorphism, it remains to show

$$\left(\bigvee \{c : \mathsf{K}(D) \mid \uparrow c \in F\}\right) \in U \quad \Leftrightarrow \quad \bigvee \{F(\uparrow c) \mid c \in U, c \text{ finite}\}.$$

For the ($\Leftarrow$) direction, suppose there is some finite $c \in U$ with $\uparrow c \in F$. Since

$$c \sqsubseteq \left(\bigvee \{c : \mathsf{K}(D) \mid \uparrow c \in F\}\right),$$

we know that $(\bigvee \{c : \mathsf{K}(D) \mid \uparrow c \in F\}) \in U$ by the upward closedness of U. For the ($\Rightarrow$) direction, suppose we know $(\bigvee (c : \mathsf{K}(D) \mid \uparrow c \in F)) \in U$. Because U is inaccessible by directed joins, there must be some finite $c \in U$ with $\uparrow c \in F$, which means the proposition $\bigvee \{F(\uparrow c) \mid c \in U, c \text{ finite}\}$ holds. □

6.5.2 Spectral points of a Scott locale

We now refine the above equivalence to an equivalence between sharp elements and spectral points.

Lemma 6.5.3. *Let D be a Scott domain. For every finite $c : D$ and every spectral point $F : \mathcal{O}(\mathrm{Scott}(D)) \to \Omega$, we have that*

$$\uparrow c \in F \quad \Leftrightarrow \quad c \sqsubseteq \nu(F).$$

Proof. Let c be a finite element of D, and $F : \mathcal{O}(\mathrm{Scott}(D)) \to \Omega$, a spectral point of $\mathrm{Scott}(D)$. The ($\Leftarrow$) direction is immediate: if $\uparrow c \in F$, then

$$c \sqsubseteq \nu(F) \equiv \bigsqcup \{b : \mathsf{K}(D) \mid \uparrow b \in F\}.$$

For the ($\Rightarrow$) direction, suppose that $c \sqsubseteq \bigvee \{b : \mathsf{K}(D) \mid \uparrow b \in F\}$ holds, which is to say

$$\left(\bigsqcup \{b : \mathsf{K}(D) \mid \uparrow b \in F\}\right) \in \uparrow c.$$

Because $\uparrow c$ is a Scott open and is hence inaccessible by directed joins, we know that there is some compact $b : D$ with $\uparrow b \in F$ and $b \in \uparrow c$. The point F is monotone (as it is a frame homomorphism), which means $F(\uparrow b) \Rightarrow F(\uparrow c)$, so it suffices to show $\uparrow b \leq \uparrow c$. Since we already know $c \sqsubseteq b$, it follows that $\uparrow b \leq \uparrow c$ holds since the principal filter operation $\uparrow(\text{-})$ is antitone (Lemma 6.2.17). □

Theorem 6.5.4 (🦃)**.** *For every Scott domain D, we have a type equivalence*

$$\mathbf{Pt}_{\mathrm{sp}}(\mathrm{Scott}(D)) \simeq \#D.$$

Proof. The equivalence is given by the bijection from the construction of Lemma 6.5.2. It remains to show:

1. that pt_x is spectral point for every sharp element $x : D$, and
2. that $\nu(F)$ is a sharp element, for every spectral point F of $\mathrm{Scott}(D)$.

For (1), let $x : D$ be a sharp element. We need to show that pt_x is a spectral point. By Lemma 6.4.6, it suffices to show that the frame homomorphism pt_x^* is a subset admitting decidable membership for compact opens. Let K be a compact Scott open of D. We know that $\mathrm{pt}_x(K) \equiv x \in K$ is decidable by our topological characterization of sharp elements (Lemma 6.3.7), stating that sharp elements are exactly the elements admitting decidable membership in compact Scott opens.

For (2), let F be a spectral point of $\mathrm{Scott}(D)$. To see that $\nu(F)$ is a sharp element, let $c : \mathsf{K}(D)$. We know by Lemma 6.5.3 that $c \sqsubseteq \nu(F)$ if and only if $\uparrow c \in F$, which we know to be a decidable proposition by Lemma 6.4.6. □

6.6 Sharp elements and the points of patch

In this section, we extend the equivalence from Theorem 6.5.4 as to prove that the set $\#D$ of sharp elements of a Scott domain D is in bijection with the set

$$\mathbf{1} \to \mathsf{Patch}(\mathsf{Scott}(D))$$

of points of its patch locale.

6.6.1 Spectral points coincide with the points of patch

Lemma 6.6.1. *For every spectral locale X, we have a type equivalence*

$$\mathbf{Pt}_{\mathrm{sp}}(X) \simeq \mathbf{Pt}_{\mathrm{sp}}(\mathsf{Patch}(X)).$$

Proof. Let X be a spectral locale. For every spectral point $p : \mathbf{1} \to X$, we know by the universal property of the patch locale (Theorem 5.4.12) that there exists a point $\bar{p} : \mathbf{1} \to \mathsf{Patch}(X)$ of patch, unique with the property that $p = \varepsilon \circ \bar{p}$. Conversely, for every point $p : \mathbf{1} \to \mathsf{Patch}(X)$ of the patch locale, the map $\varepsilon \circ p$ is a spectral point of X, since we know that ε is spectral by Lemma 5.4.8. We thus have maps

$$\mathbf{Pt}_{\mathrm{sp}}(X) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathbf{Pt}_{\mathrm{sp}}(\mathsf{Patch}(X))$$

defined as $\alpha :\equiv p \mapsto \bar{p}$ and $\beta :\equiv p \mapsto \varepsilon \circ p$.

The universal property of $\mathsf{Patch}(X)$ directly implies that these maps form a bijection. For every spectral point $p : \mathbf{1} \to X$, the existence part of the universal property gives exactly that $p = \varepsilon \circ \bar{p} \equiv \beta(\alpha(p))$. Conversely, let $p : \mathbf{1} \to \mathsf{Patch}(X)$ be a spectral point of the patch of X. We need to show $p = \overline{\varepsilon \circ p}$, which follows immediately from the uniqueness of $\overline{\varepsilon \circ p}$. □

Recall Corollary 4.6.11 stating that every continuous map of Stone locales is spectral. When combined with the above, this gives the corollary below, since the terminal locale $\mathbf{1}$ is Stone by Example 4.6.13.

Corollary 6.6.2. *For every spectral locale X, we have a type equivalence*

$$\mathbf{Pt}_{\mathrm{sp}}(X) \simeq \mathbf{Pt}(\mathsf{Patch}(X)).$$

6.6.2 Summary of characterizations of sharp elements

We finally combine all of the above equivalences to derive the main result of interest: the sharp elements of a Scott domain D are in bijection with the points of $\mathsf{Patch}(\mathsf{Scott}(D))$.

Theorem 6.6.3 (🦔). *For every Scott domain D, we have equivalences*

$$\#D \simeq \mathbf{Pt}_{\mathrm{sp}}(\mathsf{Scott}(D)) \simeq \mathbf{Pt}_{\mathrm{sp}}(\mathsf{Patch}(\mathsf{Scott}(D))) \simeq \mathbf{Pt}(\mathsf{Patch}(\mathsf{Scott}(D))).$$

Proof. We established the first equivalence in Theorem 6.5.4, and the other two in Lemma 6.6.1 and Corollary 6.6.2 above. □

Chapter 7

Conclusion

We conclude by providing a brief summary of our contributions as well as a discussion of avenues for further research. Furthermore, we discuss related work in Section 7.2, which, for us, mostly consists of the subject known as *formal topology*. Accordingly, we provide a brief introduction to formal topology and compare it with the approach that we have presented in this thesis.

7.1 Summary

We have presented a new approach to the constructive and predicative study of point-free topology in univalent foundations. We argued that, if one works under suitable size assumptions, a considerable amount of locale theory can be carried out without resorting to the use of impredicative principles. To support this case, we have developed several nontrivial constructions of locale theory. To be more specific, we have presented the following developments:

1. We developed the basic properties of salient classes of locales, including spectral, Stone, zero-dimensional, and regular locales. We have focused particularly on the first two, and demonstrated that their theory is well behaved in the constructive and predicative setting of univalent foundations.
2. We showed that various constructions of locale theory, which are traditionally constructed by impredicative means, can be carried out in a predicative setting under the assumption of a small base. The primary example of this has been our construction of the right adjoint of a frame homomorphism, which has served a crucial rôle in our development. The two main applications of this have been Heyting implications and the notion of perfect map. The former allowed us to work with the notion of open nucleus whereas we used the latter in several proofs.
3. We gave a refined, predicative version of the Stone duality for spectral locales,

in terms of the category of small distributive lattices and the category of large, locally small, and small-complete spectral locales.

4. We exhibited the category **Stone** of Stone locales as a coreflective subcategory of the category **Spec** of spectral locales and spectral maps, by constructing the patch locale of a spectral locale and proving its universal property. We explained how the proof of the universal property becomes more complicated in a predicative setting, where one does not have access to the frame of all nuclei — a construction that seems to be inherently impredicative.
5. Finally, we investigated the point-free topology of Scott domains, and used our predicative framework to establish a new result in constructive domain theory: the set $\#D$ of sharp elements of a Scott domain D is in bijection with the set of points of $\mathsf{Patch}(\mathsf{Scott}(D))$ i.e. the patch locale of the Scott locale of D.

In the work that we have presented, we worked only with locales and introduced the assumption of a base only when it became necessary. This is in contrast with the standard approach to the predicative treatment of point-free topology in type theory, where one restricts attention to small presentations of locales in terms of their basic opens, called *formal topologies*. In Section 7.2.1, we discuss and compare this with our approach.

7.2 Related work

In this section, we provide a brief survey of related work on the constructive development of point-free topology. For us, the most relevant line of research in this direction is the subject known as *formal topology* [Sam87; Neg02; CSSV03].

7.2.1 Formal topology

As explained by Negri [Neg02]: "formal topology was introduced in a series of lectures by Per Martin-Löf in 1985 as an answer to the question of whether it is possible to generalize the concept of Scott domain so as to comprise all thinkable topological spaces". This idea was then developed further through the collaboration of Giovanni Sambin with Martin-Löf; see the introduction of [Sam87] for a detailed history.

Sambin explicitly remarks [Sam00, p. 1] that the development of topology in type theory was one of the original motivations for the subject:

> *What is formal topology? A good approximation to the correct answer is: formal topology is topology as developed in (Martin-Löf's) type theory.*

For the sake of self-containment, we provide a brief introduction to the key ideas of formal topology.

Brief summary of formal topology

It is an idea familiar from Stone duality that certain classes of spaces are captured completely by their bases consisting of certain classes of opens e.g. compact opens or clopens. For spectral spaces, for example, this allows one to obtain a description of the

space in consideration in terms of the finitary algebra of compact opens, which is always a small collection in practice. This technique is highly convenient for a predicative development; when one works with the small algebra of basic opens generating the spectral space in consideration, the theory becomes much more amenable to predicative reasoning. For example, although describing the product of two locales predicatively is an open problem, the product readily admits an easy description for spectral locales as one can simply construct the coproduct of the distributive lattices of compact opens of the spectral locales in consideration.

Formal topology arises as a generalization of this technique: one axiomatizes a notion of a small algebra of basic opens, using which one presents a locale. Although there are various definitions in the literature, this is usually a meet-semilattice B equipped with a cover relation

$$(\text{-}) \lhd (\text{-}) \subseteq B \times \mathcal{P}(B),$$

subject to certain axioms (see [Neg02, Definition 2.1] for a complete definition). Given a basic open $b \in B$, the subsets $S \subseteq B$ such that $b \lhd S$ are intended to be the collections of basic opens with $b \leq \bigvee S$ in the presented frame. This way, the cover relation governs the manner in which the basic opens are to be combined as to generate a topology.

More formally, the frame *presented by* a formal topology is the so-called frame of *saturated subsets* [Neg02, p. 23] of the formal topology in consideration. One takes the class $\downarrow B$ of downward closed subsets of B, and considers the subclass of sets that are fixed points with respect to the nucleus $\mathcal{C} : \downarrow B \to \downarrow B$, defined by

$$\mathcal{C}(U) :\equiv \{b \in B \mid b \lhd U\}.$$

From a predicative point of view, the topology here is a large one since the subsets of B in consideration form a large collection. Since this large collection is generated by a *small collection of basic opens*, however, this allows one to circumvent a great deal of size problems. In summary, formal topology is the branch of point-free topology where one works with small presentations of frames, given by a generalized notion of an algebra of basic opens.

Inductively generated formal topologies

Although formal topologies enable a predicative development of point-free topology to a considerable degree of sophistication, they do not solve all size problems. Historically, one of the most salient open problems in formal topology has been the predicative construction of the product of two formal topologies. To address this issue (among other motivations), Coquand, Sambin, Smith, and Valentini [CSSV03] have developed the notion of an *inductively generated formal topology*. In the literature, the term "formal topology" is used to refer to both inductively generated formal topologies as well as formal topologies that are not necessarily inductively generated.

The key idea of inductively generated formal topologies is to require the cover relation $(\text{-}) \lhd (\text{-})$ to be an inductive type family. Although there are various technical mechanisms to achieve this, this stipulation is usually made using a variant of *indexed* W*-types* (which we briefly mentioned in Section 2.3).

When working with inductively generated formal topologies, for example, it becomes easy to describe the product of two formal topologies. However, this comes at the cost of restricting the class of spaces that can be presented by one's notion of formal topology. In [CSSV03, §4.7], the authors give an example of a formal topology that is *not* inductively generated.

At the time of writing, the problem of defining the product of two arbitrary formal topologies remains open according to Coquand (personal communication). This construction might be possible to carry out in our foundational setting through the use of suitably powerful higher inductive types, and should be investigated further.

Comparison of our approach with formal topology

In this thesis, we took an alternative approach to the predicative development of point-free topology that is slightly different from the strand of research on formal topology. Instead of delineating a category of formal topologies to embody a predicative manifestation of the notion locale, we worked directly with locales under certain size requirements ensuring that it is predicatively well behaved. To be more specific, we worked with large, locally small, and small-complete frames under the assumption of a small base (subject of Section 3.5), drawing inspiration from the work of de Jong and Escardó [dJon21b; dJE24; dJon23].

Instead of requiring all locales to be equipped with a small base, however, we assumed the existence of a suitable small base *only when* necessary. In other words, we introduced small bases into our theory to the extent that we needed them, rather than replace our notion of space with some presentation given by an algebra of basic opens. It is easy to see that every frame presented by a small formal topology will meet these size requirements (see [Tos20, §4.4]).

This approach has several benefits and drawbacks compared to inductively generated formal topologies:

- Our theory is not as predicatively well behaved as that of inductively generated formal topologies. Most saliently, our notion of locale with a small base suffers from the same problem that it is unclear if the product of two locales can be predicatively defined.
- As we work with the general notion of a locale, we can investigate the general constructions that *do work well* without any restrictions.
- In our development of the theory of spectral locales, we gave a definition of spectrality (Definition 4.2.1), using the the univalence axiom, where a notion of small base arises naturally. We thus obtained a definition that is equivalent to working with a formal topology (it is easy to see that the cover from Definition 4.4.20 is an inductively generated cover in the sense of formal topology), but this was well motivated starting from the impredicative definition.
- Finally, we also chose to start a predicative investigation of point-free topology from first principles due to the novelties provided by univalent type theory. It is conceivable that some of the problems motivating formal topology can be circumvented in other ways in univalent foundations, such as through the use of suitable higher inductive types giving certain resizing principles. The work of Coquand and the author [Tos20; CT21] showed that, in univalent foundations,

one needs certain higher inductive types to work with inductively generated formal topologies in the first place. So the ramifications of higher inductive types for the type-theoretic, predicative development of locale theory merit further study, and prematurely restricting one's attention to formal topologies might narrow the scope of such an investigation.

Review of relevant work in formal topology

Type-theoretic formalization of Cederquist One of the earliest formalizations of formal topology was developed by Cederquist in his PhD thesis [Ced97]. Cederquist's work is focused particularly on the applications of point-free topology to constructive analysis [CN96; CCN98]. In his licentiate thesis [Ced94] (which is part of the PhD thesis [Ced97]), Cederquist studies the points of the Scott topology of a Scott domain in a formal-topological framework, as part of his formalized library of formal topology. This has parallels with our investigation from Section 6.5.1, and the exact relationship to the formal-topological framework should be studied further. Cederquist's formalization is developed in the ALF proof assistant [Mag94].

Formal topology in UF In the context of univalent foundations, the first formalization of formal topology was undertaken by the author's MSc work [Tos20; CT21]. A completely predicative account of inductively generated formal topologies was developed, and was fully formalized by the author [Tos21].

Coquand and Zhang's description of patch Coquand and Zhang [CZ03] give a predicative description of the patch topology of a stably compact locale in terms of the entailment relation presenting the locale. Their approach to presentations is closely related to formal topology, although they do not explicitly work with formal topologies.

In our work, we exhibit the patch construction only as the coreflector of Stone locales into the category of spectral locales (and spectral maps). We leave it as further work to extend this to the coreflection of compact regular locales into the category of stably compact locales. Coquand and Zhang, on the other hand, describe the patch locale for the more general case of stably compact locales, and use this to derive the patch of a spectral locale as a special case.

Their work is an alternative to Escardó's predicative description of the patch locale that we used in Section 5.4. Adapting this to our setting would be a promising direction of future research. We believe that it will be possible to use the approach presented in this thesis instead of the formal-topological approach of [CZ03].

7.2.2 Other related works

Constructive but impredicative developments of locale theory

Johnstone's development of point-free topology in [Joh82] is classical. Although he carefully marks his uses of the axiom of choice with an asterisk, he freely uses the law of excluded middle in his development. Unlike [Joh82], however, Johnstone develops some locale theory in [Joh02, §C] in a completely constructive way. Other noteworthy

developments of locale theory that are constructive in the sense of the internal language of a topos are Manuell's PhD thesis [Man19] and his lecture notes [Man22], and the PhD thesis of Wrigley [Wri24, §A.II].

Other constructive and predicative developments

In Section 7.2.1, we explained that formal topology is the standard approach to the development of constructive and predicative locale theory in the context of type theory. In the context of constructive set theory, however, there is a line of work on constructive and predicative[1] topology, originating in the work of Aczel [Acz06]; we do not attempt to provide a survey here. In this context, one works with a notion of *set-generated* locale [Acz06, Definition 11], which is very similar to what we called large and small-complete locales. The precise relationship of set-generated locales to the predicative notions of locales that we worked with deserves further investigation.

7.3 Future work

7.3.1 Extension to stably compact locales

In our work, we exhibited the patch construction only as the coreflector of the category of Stone locales into the category of spectral locales (and spectral maps). As shown by Escardó [Esc99; Esc01], the same patch construction also exhibits the category **CReg** of compact regular locales (and continuous maps) into the category **SC** of stably compact locales (and perfect maps). A promising direction for further research is to develop this coreflection in the framework that we have described in this thesis. In fact, we already gave a candidate for the definition of the notion of regular locale in Definition 3.6.5.

We believe that this extension of the coreflection will be predicatively possible, although it might present further difficulties compared to the case for spectral locales. For a stably compact locale X, the basic covering families for an open U consist of opens that are not globally compact, but only compact *relative to* U. This means that there is no single type, such as the type of all compact opens, giving the base. So the situation in this direction will be similar to de Jong's investigation of continuous lattices [dJon23, §4.4]

7.3.2 The patch locale and the Lawson topology

Our investigation of the topology of domains in Chapter 6 focused mostly on the Scott topology of a Scott domain. In the literature on domain theory, there is another crucial topology considered on domains: the *Lawson topology* [GHK+03, Definition III-1.1].

In a classical setting, the relationship between the Lawson topology and the patch of the Scott topology is straightforward: the space $\mathsf{Lawson}(D)$ is homeomorphic to $\mathsf{Patch}(\mathsf{Scott}(D))$. In a constructive setting, however, the situation here is more compli-

[1] We remind the reader once again that set theorists use the term "constructive" to refer to what we call "constructive *and* predicative" in this thesis.

cated. Constructively, it cannot be the case in general that

$$\mathsf{Lawson}(D) \cong \mathsf{Patch}(\mathsf{Scott}(D)) \tag{※}$$

This is because $\mathsf{Lawson}(D)$ is defined as a topological space, given by the set of points D equipped with a topology given as a certain class of subsets of D. The above homeomorphism would thus imply that $\mathsf{Patch}(\mathsf{Scott}(D))$ is always a *spatial locale*, since it asserts that it is homeomorphic to a point-set topological space.

Consider the powerset DCPO $\mathcal{P}(\mathbb{N})$, which is a Scott domain. By taking the patch of its Scott locale, we get the Cantor locale $2^{\mathbb{N}}$:

$$2^{\mathbb{N}} = \mathsf{Patch}(\mathsf{Scott}(\mathcal{P}(\mathbb{N}))).$$

So the homeomorphism in (※) immediately gives the spatiality of the Cantor space as a special case. Unfortunately, however, there is a well known result in constructive mathematics due to Fourman and Grayson [FG82] stating that the *spatiality of the Cantor space* is equivalent to the Fan Theorem, which is known to be independent of constructive mathematics. We thus know that (※) cannot hold in general.

We conjecture that an analogue to the above homeomorphism can obtained constructively (and predicatively) if the Lawson topology is defined on the set $\#D$ of sharp elements of D, instead of all its elements. The result we proved in Theorem 6.6.3 is the first step in this direction, since it shows that the points of $\mathsf{Patch}(\mathsf{Scott}(D))$ are exactly the sharp elements. We believe it will be possible to show that this equivalence of points is a homeomorphism with respect to a suitable definition of the Lawson topology on the set $\#D$. However, the construction of the Lawson topology is rather involved and we have not formulated a suitable definition at the time of writing. We thus leave this as further work.

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Index of symbols

$U \ll V$	U is way below V, 59
$U \lessdot V$	U is well inside V, 60
ap_f	action of function f on paths (from Lemma 2.5.3), 15
$x \uparrow y$	x and y are bounded above, 133
$\mathsf{K}(X)$	the type of compact opens of locale X, 59
$\mathbf{DLat}_{\mathcal{U}}$	category of distributive $\mathcal{U}$-lattices, 82
$\bot_D$	bottom element of DCPO D, 129
$a \equiv b : A$	equality judgement expressing that term a and b are definitionally equal inhabitants of type A, 8
$\mathsf{Sub}_{\mathcal{V}}(X)$	the type of $\mathcal{V}$-subtypes of type X, 39
$\mathbf{0}$	empty type with universe left implicit, 9
$\mathbf{0}_{\mathcal{U}}$	empty type in universe $\mathcal{U}$, 11
$X \simeq Y$	type of equivalences between X and Y, 25
$f \sim g$	extensional equality of functions f and g, 23
$\mathsf{K}(D)$	type of finite elements of DCPO D, 131
$\bigvee_{i:I} x_i$	join of a family $(x_i)_{i:I}$, 44
$\mathbf{Frm}_{\mathcal{U}}$	the category of large, locally small, and small-complete frames over base universe $\mathcal{U}$, 48
idtoeqv	the canonical map from $X =_{\mathcal{U}} Y$ into $X \simeq Y$, 30
$\mathsf{Id}_A(a, b)$	type of identifications (or equalities) of inhabitants a and b of type A, 9
$\mathsf{Ideal}_{\mathcal{U}}(L)$	type of $\mathcal{U}$-ideals over lattice L, 83

is-contractible(X)	type X is contractible, 18
is-equiv(f)	function f is an equivalence, 25
is-$\mathcal{V}$-small(X)	the type expressing that type X is $\mathcal{V}$–small, 37
J	eliminator of the identity type, 15
List(A)	The type of lists (or words) over type A, 9
$\mathbb{S}$	Sierpiński locale, 53
$\mathbf{1}_{\mathcal{U}}^{\neg\neg}$	the locale defined by the frame of ¬¬-stable propositions., 54
$\mathbf{Loc}_{\mathcal{U}}$	the category of large, locally small, and small-complete locales over base universe $\mathcal{U}$, 49
$\mathbb{N}$	The type of natural numbers, 9
Nucleus(X)	the type of nuclei on locale X, 109
Nucleus(X)	type of nuclei over locale X, 55
Patch(X)	patch locale on a spectral locale X, 120
$p \bullet q$	composition of paths p and q, 17
$\Pi_{(a:A)} B(a)$	Dependent product type of an A-indexed family of types B, 9
$\Pi_{(x:X)} Y_x$	dependent product type, 9
$\mathbf{Pt}(X)$	type of points of locale X, 145
$\mathcal{P}_{\mathcal{V}}(X)$	type of $\mathcal{V}$-valued subsets of X, 36
$A \times B$	product type of types A and B, 9
QInv(f)	the type of quasi-inverses of function f, 24
refl$_a$	construction of the identity type $a =_A a$, 10
$\sigma(D)$	the frame of Scott open subsets of algebraic DCPO D, 136
Scott(D)	Scott locale of an algebraic DCPO D, 136
$\Sigma_{(a:A)} B(a)$	Dependent sum type of an A-indexed family of types B, 9
$\mathbf{Pt}_{\mathsf{sp}}(X)$	type of spectral points of locale X, 146

Spec	category of spectral locale with base universe left implicit, 77
$\mathbf{Spec}_{\mathcal{U}}$	category of spectral locales over base universe $\mathcal{U}$, 77
$\mathsf{Spec}(L)$	spectrum of small distributive lattice L, 86
$\mathbf{Stone}_{\mathcal{U}}$	the category of Stone locales over base universe $\mathcal{U}$, 96
Stone	the category of Stone locales with base universe left implicit, 96
$A + B$	The sum type of types A and B, 9
$\mathsf{transport}_Y$	inhabitant of the transport lemma (from Lemma 2.5.2, 15
$\lvert(\text{-})\rvert$	constructor of the propositional truncation type, 34
$\lVert X \rVert$	propositional truncation of type X, 34
$\mathcal{U} \sqcup \mathcal{V}$	Maximum of universes $\mathcal{U}$ and $\mathcal{V}$, 11
$\mathcal{U}^+$	Successor universe of universe $\mathcal{U}$, 11
$\mathcal{U}_0$	ground universe, 11
$\mathbf{1}$	The unit type, 9
$\mathbf{1}_{\mathcal{U}}$	Unit type in universe $\mathcal{U}$, 11
$\downarrow x$	principal ideal generated by x, 83
$\Omega_{\mathcal{U}}$	type of propositions in universe $\mathcal{U}$, 18
$\mathsf{Fam}_{\mathcal{V}}(X)$	the type of $\mathcal{V}$-families on type X, 38
$\mathsf{Fin}(n)$	the finite type with exactly n inhabitants, 37
$\mathsf{KFin}(X)$	the type of Kuratowski finite subsets of X, 37
wrec	eliminator of the W-type, 12
$a : A$	Typing judgement expressing that term a is of type A, 8

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