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Pointfree bispaces and pointfree bisubspaces

A thesis presented to the University of Birmingham for the degree of Doctor of Philosophy in Computer Science

by

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Pointfree bispaces and pointfree bisubspaces

Anna Laura Suarez

Abstract This thesis is concerned with the study of pointfree bispaces, and in particular with the pointfree notion of inclusion of bisubspaces. We mostly work in the context of d-frames. We study quotients of d-frames as pointfree analogues of the topological notion of bisubspace. We show that for every d-frame $\mathcal{L}$ there is a d-frame $\mathsf{A}(\mathcal{L})$ such that it plays the role of the assembly of a frame, in the sense that it has the analogue of the universal property of the assembly and that its spectrum is a bitopological version of the Skula space of the bispace $\mathsf{dpt}(\mathcal{L})$, the spectrum of $\mathcal{L}$. Furthermore, we show that this bitopological version of the Skula space of $\mathsf{dpt}(\mathcal{L})$ is the coarsest topology in which the d-sober bisubspaces of $\mathsf{dpt}(\mathcal{L})$ are closed. We also show that there are two free constructions in the category of d-frames $\mathsf{A}_{ct}(\mathcal{L})$ and $\mathsf{A}_{\pm}(\mathcal{L})$, such that they represent two variations of the bitopological version of the Skula topology. In particular, we show that in $\mathsf{dpt}(\mathsf{A}_{ct})$ the positive closed sets are exactly those d-sober subspaces of $\mathsf{dpt}(\mathcal{L})$ that are spectra of quotients coming from an increase in the con component, and that the negative closed ones are those that come from increases in the tot component. For $\mathsf{dpt}(\mathsf{A}_{\pm}(\mathcal{L}))$, we show that the positive closed sets are exactly those bisubspaces of $\mathsf{dpt}(\mathcal{L})$ that are spectra of quotients coming from a quotient of L^{+}, and that the negative closed sets come in the same way from quotients of L^{-}. We thus show that the collection of all d-sober bisubspaces of a d-sober space is bitopological in more than one natural way.

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Introduction

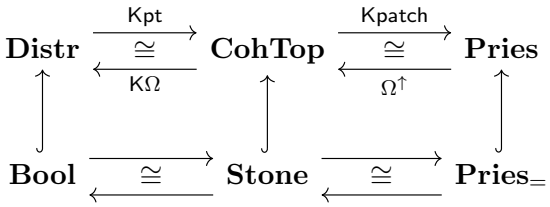
Why study bispaces?

With his famous representation theorem, Stone proved in [51] a result which, expressed in contemporary mathematical language, establishes a dual equivalence of categories between Boolean algebras and certain topological spaces. Nowadays, these topological spaces – namely the compact, Hausdorff, zero dimensional spaces – are called *Stone spaces* in honour of this result. Stone's representation theorem was only the first in a series of results connecting lattice theory and topology. For example, Priestley proved (see [42], [43]) that the category of distributive lattices is dually equivalent to a category whose objects are Stone spaces equipped with an order relation on the set of points. One usually calls *Stone-type dualities* those results establishing dual equivalences between categories of lattices and categories of spaces. There is by now a large family of these results. Stone-type dualities continue to receive a great deal of attention, among mathematicians and computer scientists alike. For instance, the connections with the theory of formal languages and automata theory have been explored recently (see [19]). There has also been a significant amount of work towards generalizing the theory in various directions: for example see [18] for an order theoretical approach, or [14] for a topos theoretical one, or [13] for a generalization of the duality which does not need the Axiom of Choice.

Let us see some fundamental notions about bitopological spaces, without which we could not explain certain upcoming concepts in this introduction. A *bitopological space* is a triple (X, τ^+, τ^-) in which X is a set and τ^+ and τ^- are both topologies on that set. A morphism between bispaces is a function between their underlying sets such that it is *bicontinuous*: continuous, that is, with respect to both topologies. The category **BiTop** has bitopological spaces as objects and bicontinuous maps as morphisms. There is a functor $\mathsf{patch} : \mathbf{BiTop} \to \mathbf{Top}$ assigning to each bitopological space (X, τ^+, τ^-) its *patch*, which is the topological space $(X, \tau^+ \vee \tau^-)$, in which the join $\tau^+ \vee \tau^-$ is computed in the lattice of all possible topologies

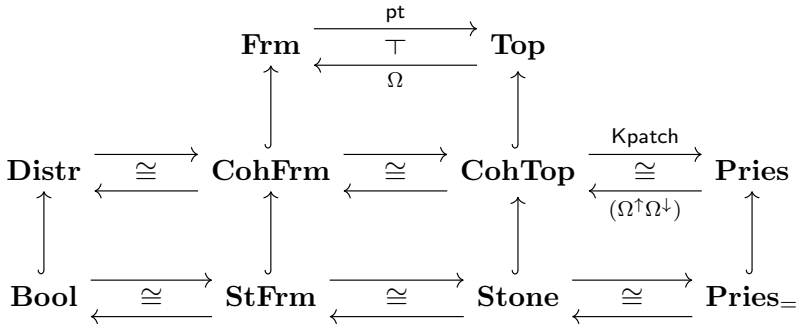
on X, and so it is the topology generated by $\tau^+ \cup \tau^-$. The patch functor has a right adjoint, and it is the functor mapping each space (X, τ) to the bispace (X, τ, τ).

The view that we now wish to express is that there can be no full understanding of Stone and Priestley duality if there is no understanding of pointfree bispaces. Let us see why we hold this view. Priestley's result is a generalization of Stone's, in the sense that we have the following picture.



The category $\mathbf{Pries}_=$ is the full subcategory of the category **Pries** of Priestley spaces in which the order on the points is the identity. This is equivalent to the category **Stone** of Stone spaces. The same functors which give this equivalence (whose definition is now not important), when extended to **Pries**, give us an equivalence between **Pries** and the category **CohTop** of *coherent* topological spaces. The crucial thing to remember about coherent spaces is that the morphisms between them are required to be not only continuous, but also well-behaved with respect to compactness, and so the category **CohTop** is not a full subcategory of **Top**. The functor $\mathsf{K}\Omega : \mathbf{CohTop} \to \mathbf{Distr}$ takes a space and maps it to the ordered collection of its compact open set. Its restriction to **Stone** is the clopen set functor giving the equivalence in classical Stone duality.

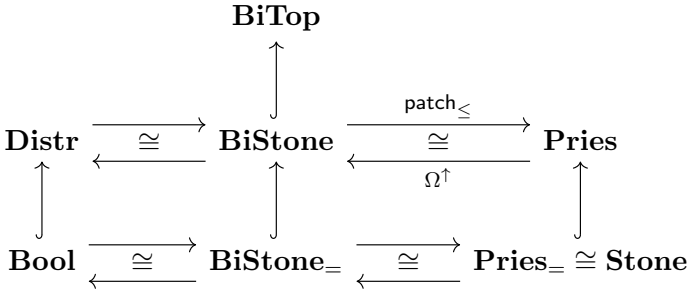
However, by now, when we mention the "pointfree dual" of a space, we do not have its Boolean algebra of clopens in mind anymore, but rather, its frame of opens. When we integrate frames into the picture, it becomes as follows.



Here, **CohFrm** is the category of the so called *coherent frames*. This is a subcategory of **Frm** which is not full, and which is equivalent to the category of distributive lattices. We

have omitted the definition of the functors that witness this equivalence. Recall now that the inclusion **CohTop** $\hookrightarrow$ **Top** is not full, either. This picture is then somewhat unnatural. Bitopological spaces provide us with the tool to get a much more complete and elegant picture of the situation.

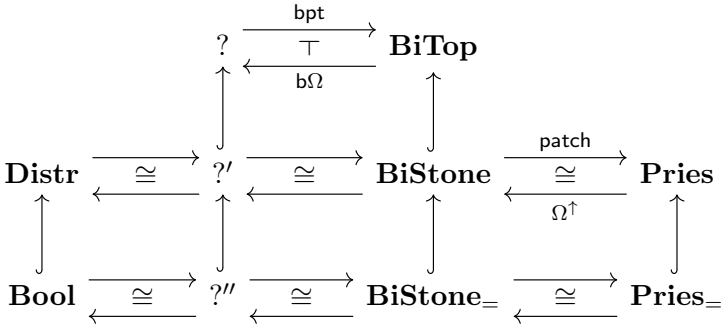
In [29] and in [12] it is explained that Priestley spaces are nothing but pairwise Stone spaces, and that Stone dualities are bitopological in nature. We recall that a Stone space is a topological space which is compact, Hausdorff, and zero dimensional. Using very natural bitopological generalizations of these three properties one can define *pairwise Stone* bispaces. In [12] it is shown that the category of Priestley spaces is equivalent to the full subcategory **BiStone** of **BiTop** consisting of the pairwise Stone spaces. The new, updated picture is as follows. Here the functor $\mathsf{patch}_{\leq}$ takes the patch of a pairwise Stone space and equips it with a certain natural order. Its inverse takes a Priestley space X and maps it to the bispace with the same points, whose positive opens are the upper opens, and whose negative opens are the lower opens. The category $\mathbf{BiStone}_{=}$ is the full subcategory of **BiStone** in which the two topologies are the same.



All inclusions appearing in this diagram are full. Priestley spaces, then, are nothing but bitopological generalizations of Stone spaces, and Priestley duality is the most natural bitopological generalization of Stone duality. Notice that here we have replaced coherent spaces with pairwise Stone spaces. Coherent spaces are missing half of the picture: whenever we have a coherent space $(X, \Omega(X))$, the pairwise Stone space associated with it is $(X, \Omega(X), \Omega^{\mathsf{K}^c}(X))$, for some topology $\Omega^{\mathsf{K}^c}(X)$ on X. This means that every coherent space is only half of a pairwise Stone space. Working with the category of pairwise Stone bispaces is much more natural than working in **CohTop**: the condition that maps are well-behaved with respect to compactness translates simply to bicontinuity of the map, eliminating the need for extra constraints in order to get our duality. Not only do coherent spaces miss half of a pairwise Stone bispace, but them missing this half also means that in their category we need an extra constraint on morphisms.

It is difficult to argue that one can have a full understanding of Stone-type dualities without

understanding what a pointfree bispace is. If coherent spaces are missing half of the picture because they forget half a bispace, then so are coherent frames. To complete this elegant and comprehensive picture, one needs to generalize frames in the same way as we have generalized spaces with bispaces. It would offer us an elegant and full view of the situation to have some adjunction $\mathsf{b}\Omega : \mathbf{BiTop} \leftrightarrows ?^{op} : \mathsf{bpt}$ mimicking the adjunction $\Omega \dashv \mathsf{pt}$ for frames and spaces, and appropriate restrictions of our category "?" of pointfree bispaces falling into the picture as follows.



In order to do this, we need to understand what is the correct notion of pointfree bispace or, more modestly, to understand what is the notion of pointfree bispace which best suits our purpose of understanding Stone dualities. There is no consensus on what is the most appropriate notion of pointfree bispace. Until now, two possible answers to this question are the theory of biframes (as in [8]) and that of d-frames (as in [29]). Both theories have their advantages and disadvantages.

- The advantage of biframes is that they are the most literal translation into pointfree language of the concept of bispace. Taking the biframe of opens of a bispace simply means taking the frame of the opens of its patch, which we call the *main component* of the biframe, while keeping the information on which opens are in the positive and in the negative topology. The parts of the theory of biframes which are still unsatisfactory all revolve around one fact: because taking the biframe of opens of a space means taking the frame of the patch topology, many notions in the theory of biframes collapse to frame theoretical notions, so that many aspects of the theory are not genuinely bitopological. Let us see some instances of this. The bispaces which are fixpoints of the adjunction between bispaces and biframes are exactly the bispaces whose patch is sober. This means that "bisobriety" collapses to classical sobriety. A pointfree bisubspace of a biframe is a quotient. The lattice of quotients of a biframe is the same as the lattice of frame quotients of its main component. So, the whole theory of pointfree bisubspaces – in a certain sense – collapses to the theory of pointfree spaces in the biframe setting.

- The main advantage of the theory of d-frames is that, apart from being suitable pointfree duals of bispaces for the end of expressing Stone dualities (see Chapter 2 of [22]), these objects also represent four valued logical theories and, as such, they have applications in logic and computation, too. They then highlight possible connections between the theory of bispaces and these fields. The disadvantage of this theory is that pointfree bisubspaces are not well-behaved here. The lattice of quotients of a d-frame is not in general a distributive lattice. This is very much in contrast with the frame case. The lattice of pointfree subspaces of a frame L is always a frame, meaning that for each space we have the "space of all its subspaces", in the sense that its elements are precisely the pointfree subspaces of L. This is a central feature of pointfree topology and so it is desirable to preserve this in a theory of pointfree bispaces.

Why study pointfree spaces?

The equivalence of categories given by Stone's theorem is given on one side by a functor $\mathsf{K}\Omega : \mathbf{Stone} \to \mathbf{Bool}^{op}$, assigning to each space the ordered collection of its clopen sets (or, equivalently, its compact open sets). On the other side, we have a functor $\mathsf{Spt} : \mathbf{Bool}^{op} \to \mathbf{Stone}$, assigning to each Boolean algebra the set of its prime filters, topologized in a certain natural way. After Stone, the task of the algebraic, order-theoretical study of topological spaces was undertaken by McKinsey and Tarski (see [33]), Ehresmann (see [17]), Papert (see [36]), Isbell (see [21]), Banaschewski (see [7]), Simmons (see [50]), Pultr (see [44], [45]) and Picado (see [37]). Ehresmann and Bénabou were the first to consider frames as abstract representations of topological spaces (see for instance [**?**]). A frame is a complete lattice L in which we have the following distributivity law for every $x, y_i \in L$.

$$(\bigvee_i y_i) \wedge x = \bigvee_i (y_i \wedge x).$$

This distributivity law captures abstractly the fact that in topological spaces arbitrary unions of opens are well-behaved, but only finite intersections are. Since Ehresmann and Bénabou put this view forward, frames are considered to be the standard order theoretical representation of topological spaces. The category of frames is called **Frm**, and we have an adjunction $\Omega : \mathbf{Top} \leftrightarrows \mathbf{Frm}^{op} : \mathsf{pt}$ with $\Omega \vdash \mathsf{pt}$ in which Ω is a functor assigning to each space its ordered collection of open sets. It is by virtue of this adjunction, and because of a logical discussion on the concept of observability (which we discuss below), that we consider frames to be pointfree versions of spaces.

Boolean algebras and Stone spaces are classical logical theories. In classical topological semantics, we take a Stone space as being a classical logical theory. The points of the space are models, and every clopen set of the space is identified with an equivalence class of logically equivalent propositions in the theory. This is done by considering certain opens as the *positive extents* of formulae, that is, the set of models in which it holds. Since in the classical setting we have negation, positive extents are closed under set-theoretical complementation: the positive extent of the negation $\neg\varphi$ of a formula φ is just the set-theoretical complement of the positive extent of φ.

Instead, frames and topological spaces provide semantics for logical theories of *verifiability*, or of *observability*. We follow Vickers' approach in [53]. In a frame L, propositions are ordered under interprovability, just like in an ordinary Lindenbaum algebra. A formula is verifiable in a certain system iff within this system we can establish its truhhood. It could be that it is impossible for us to ever establish its falsehood, e.g. because it would take us infinitely many steps. Think for instance of the assertion *there exists at least one black swan*: it suffices to present one instance of a black swan to verify this; but to establish its falsehood we would need to present all of the existing examples of swan. It is not an issue of how many steps we need to actually confirm the statement: it might be difficult to find a black swan. The crucial thing here is that a confirmation of the assertion consists of much less information compared to a refutation of it. Verifiable assertions find their natural counterpart in *refutable* ones. Think of an assertion such as *all swans are white*. In order to refute it, it suffices to provide just one counterexample; but to verify it a very large amount of information is needed. For a family φ_i of verifiable assertions, to verify the disjunction $\bigvee_i \varphi_i$ it suffices to verify just one of the assertions. If it is possible in our system to verify assertion φ_i by performing observation O_i, then any of the observations in $\{O_i : i \in I\}$ serve to verify $\bigvee_i \varphi_i$. So, indeed, there exists a possible observation in our system which verifies $\bigvee_i \varphi_i$. What one may not be able to do is to verify the assertion $\bigwedge_i \varphi_i$: in order to do this, we need to verify all of the assertions φ_i, for which we need to perform all the observations $\{O_i : i \in I\}$; and this set might be infinite. If the index set I is finite, however, the set of observations to make in order to verify the conjunctions is finite. It is sensible, then, that an algebraic theory of a logic of verifiability should have infinite disjunctions, but only finite conjunctions. It remains to justify the frame distributivity law. Suppose that we have verifiable assertions φ_i, ψ. For this it suffices to notice that an observation O verifies $\bigvee_i(\varphi_i \wedge \psi)$ precisely when it confirms both ψ and some φ_j, and this is exactly the condition under which an observation verifies $(\bigvee_i \varphi_i) \wedge \psi$. The predicate $\bigvee_i(\varphi_i \wedge \psi)$, then, is verified precisely when $(\bigvee_i \varphi_i) \wedge \psi$ is.

Why study d-frames?

Regardless of how suitable a candidate d-frames are as pointfree duals of bispaces, their study is important above all from a logical and a computational point of view. D-frames offer an algebraic representation of *four-valued* logical systems, paraconsistent logical systems particularly suitable for modeling computation as well as information. Let us first explain the classical interpretation of a frame as a logical theory. Usually one takes a frame L to be the "Lindenbaum algebra" of a certain logical theory. This means that the elements of L are considered to be equivalence classes of logically equivalent propositions, and the order on the frame represents logical implication inside that theory. In the frame L, joins are the same as logical disjunctions and meets the same as logical conjunctions. The bottom element 0_L represents contradictions (reflecting the logical law that everything follows logically from a contradiction), and the top element 1_L tautologies (reflecting the law that everything implies any tautology). Every frame L has an associated topological space $\mathsf{pt}(L)$, and every element $a \in L$ determines an open $\varphi_L(a) \subseteq \mathsf{pt}(L)$ of this space. The points of the space $\mathsf{pt}(L)$ are seen to be models, and for every equivalence class of propositions $a \in L$ the open $\varphi_L(a)$ is the set of all models in which these propositions hold. The way that propositions induce opens obeys the following rules

- $\phi_L(0_L) = \emptyset$,
- $\phi_L(1_L) = \mathsf{pt}(L)$,
- $\phi_L(a_1 \wedge ... \wedge a_n) = \phi_L(a_1) \cap ... \cap \phi_L(a_n)$
- $\phi_L(\bigvee_i a_i) = \bigcup_i \phi_L(a_i)$.

All these rules reflect our intuitive notions of how logical connective behaves: in no models can we have that a contradiction is true, the models in which a logical conjunction is true are precisely the models in which all of the conjuncts are true, and so on. The fact that $\phi_L(-)$ is well-behaved with respect to arbitrary disjunctions but only finite conjunctions is a subtlety which need not concern us at this stage.

A *d-frame* is a quadruple $\mathcal{L} = (L^+, L^-, \mathsf{con}_L, \mathsf{tot}_L)$ in which L^+ and L^- are frames, and $\mathsf{con}_L, \mathsf{tot}_L \subseteq L^+ \times L^-$ satisfy certain axioms. Within the logical interpretation of a d-frame, the elements of $L^+ \times L^-$ represent propositions. The con_L component represents *consistent* propositions, while the tot_L component represents the *total* ones. Let us see what this means. A d-frame $\mathcal{L}$ has an associated bispace $\mathsf{dpt}(\mathcal{L})$, in the same way as a frame has an associated space. Furthermore, every proposition (a^+, a^-) has *two* opens associated with it: the positive

open $\varphi^+_{\mathcal{L}}(a^+)$ and the negative open $\varphi^-_{\mathcal{L}}(a^-)$. The points of the bispace $\mathsf{dpt}(\mathcal{L})$ represent models, and for every proposition (a^+, a^-) the positive open $\varphi^+_{\mathcal{L}}(a^+)$ represents all the models that verify the proposition, and the negative open $\varphi^-_{\mathcal{L}}(a^-)$ represents the models that refute it. The propositions which are in con_L are the propositions which are consistent: they are those propositions (a^+, a^-) with $\varphi^+_{\mathcal{L}}(a^+) \cap \varphi^-_{\mathcal{L}}(a^-) = \emptyset$, i.e. those such that there are no models that both verify them and refute them. The propositions in tot_L are the ones which are total: they are those such that $\varphi^+_{\mathcal{L}}(a^+) \cup \varphi^-_{\mathcal{L}}(a^-) = \mathsf{dpt}(\mathcal{L})$, such that every model either verifies them or refutes them. This means that the logic of d-frames is *paraconsistent*: we may have that the same proposition is both refuted and verified in certain models, and this in fact happens whenever a proposition $(a^+, a^-) \in L^+ \times L^-$ is such that $\varphi^+_{\mathcal{L}}(a^+) \cap \varphi^-_{\mathcal{L}}(a^-) \neq \emptyset$.

Paraconsistent logic has found applications both in computer science and in philosophy of science. For applications to computer science, consider the following. The Stanford Encyclopedia of Philosophy entry for "Paraconsistent logic" (see [41]) reads as follows.

> One of the applications is automated reasoning (information processing). Consider a computer which stores a large amount of information, as in [5]. While the computer stores the information, it is also used to operate on it, and, crucially, to infer from it. Now it is quite common for the computer to contain inconsistent information, because of mistakes by the data entry operators or because of multiple sourcing. This is certainly a problem for database operations with theorem-provers, and so has drawn much attention from computer scientists. Techniques for removing inconsistent information have been investigated. Yet all have limited applicability, and, in any case, are not guaranteed to produce consistency. (There is no algorithm for logical falsehood.) Hence, even if steps are taken to get rid of contradictions when they are found, an underlying paraconsistent logic is desirable if hidden contradictions are not to generate spurious answers to queries.
>
> Nelson's paraconsistent (four-valued) logic N4 has been specifically studied for applications in computer science ([30]). Annotated logics were proposed by Subrahmanian ([52]) and then by da Costa, Subrahmanian, and Vago ([15]); these tools are now being extended to robotics, expert systems for medical diagnosis, and engineering, with recent work gathered in the volumes edited by Abe, Akama, and Nakamatsu ([3]) and Akama ([4]).

For a survey of paraconsistent logic in philosophy, see [40].

The theory of d-frames brings together these two different approaches. On one side we have a logical theory of verifiability: the opens of the form $\varphi^+_{\mathcal{L}}(a^+)$ are positive extents of propositions, i.e. sets of models in which a certain proposition is verified to hold. On the other side we have a logic of falsifiability: the opens of the form $\varphi^-_{\mathcal{L}}(a^-)$ are sets of models in which a certain proposition is refuted. The two logical systems are connected harmoniously by the con_L and the tot_L components. D-frames, therefore, allow us to model settings in which certain propositions are refutable, and certain others are verifiable. More than this, they allow

us to model situations in which the same proposition has a verifiable and a refutable part. On the topic of the logical interpretation of d-frames, see [24] and Chapter 6 of [22].

The aim of this thesis

This thesis is concerned with the question of finding a suitable notion of pointfree bisubspace. Along the way, the question will also naturally arise of what is the most suitable notion of pointfree bispace, if it is the study of pointfree bisubspaces that we are interested in. A few sections of the thesis will be akin to digressions, and even skipping them would not interrupt the flow of the narrative. More importantly, there is a clear and defined main message that this thesis aims to convey. The main message of this thesis can be condensed, sacrificing some precision, into the following.

For any pointfree space L, there is another pointfree space $\mathsf{A}(L)$ which is isomorphic to the ordered collection of all pointfree subspaces of L. The pointfree space $\mathsf{A}(L)$ has a natural bitopological structure. The frame $\mathsf{A}(L)$ is generated, in the subbasis sense, by the subframe consisting of the closed pointfree subspaces and the subframe generated by the open ones.

There is no agreed upon notion of what a pointfree bispace ought to be. In this thesis we will explore two pointfree translations of this notion: d-frames and biframes. What follows applies to both these structures. A pointfree bispace $\mathcal{L}$ is a pair of frames L^+ and L^-, plus some extra information relating the two frames. Pointfree bisubspaces of $\mathcal{L}$ are congruences on L^+ and L^-, plus some extra information. There are certain pointfree bispaces of $\mathcal{L}$, which we call *positive*, which are completely determined by the congruence they determine on L^+. We call *negative* the pointfree bisubspaces of $\mathcal{L}$ which are completely determined by the congruence they induce on L^-. In this thesis, we argue towards the following point. For a pointfree bispace $\mathcal{L}$ there is a pointfree bispace $\mathsf{A}(\mathcal{L})$ which – in a sense which is more subtle than in the frame case – represents all the pointfree bisubspaces of $\mathcal{L}$. The pointfree patch of $\mathsf{A}(\mathcal{L})$ is a pointfree space which is bitopological in more than one natural way. This gives raise to the pointfree bispaces $\mathsf{A}_{cf}(\mathcal{L})$ and $\mathsf{A}_{\pm}(\mathcal{L})$. The first represents the collection of all pointfree bisubspaces of $\mathcal{L}$; the positive topology is given by the patch-closed pointfree bisubspaces, and the negative one by the patch-fitted ones. The second pointfree bispace still represents the collection of all pointfree bisubspaces of $\mathcal{L}$; the positive topology is given by the *positive* pointfree bisubspaces, and the negative one by the *negative* ones. These two constructions uncover the bitopological nature of the collection of pointfree bisubspaces of a pointfree bispace.

Background

Frames and their duality

The results that we present in this section are well-known; we refer the reader to [26] and to [37]. Here we present the main results on the topic without proofs. A *frame* is a complete lattice[1] L in which we have the following distributivity law for $x, y_i \in L$.

$$\bigvee_i (y_i \wedge x) = (\bigvee_i y_i) \wedge x.$$

The category **Frm** is the category whose objects are frames, and whose morphisms are maps $f : L \to M$ between the underlying sets of the two frames such that they preserve finite meets and all joins. Every frame is a Heyting algebra, with the Heyting operator on a frame L calculated as

$$a \to b = \bigvee \{x \in L : x \wedge a \leq b\}$$

for every $a, b \in L$. Conversely, every complete Heyting algebra is a frame. The categories **Frm** and **Heyt** of complete Heyting algebras are not equivalent: in the category **Heyt** the Heyting implication needs to be preserved by the morphisms, and not all frame maps preserve it.

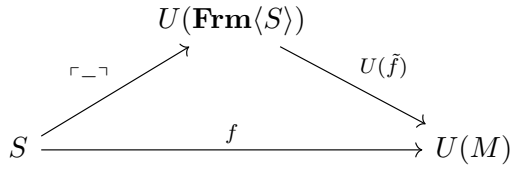
Example 0.0.1. The following is an example of a frame morphism which does not preserve the Heyting arrow. Let $C_2 = \{0 < 1\}$ be the two-element chain, and let $C_3 = \{0 < a < 1\}$ be the three-element chain. Define a map $f : C_3 \to C_2$ as $f(0) = 0$, $f(a) = 0$, and $f(1) = 1$. We have that $f(\neg a) = f(0) = 0$ but $\neg f(a) = \neg 0 = 1$.

The forgetful functor $U : \mathbf{Frm} \to \mathbf{Set}$ has a left adjoint. This means that one can then speak of the free frame on a set of generators. For a set S we denote as

$$\mathbf{Frm}\langle S \rangle$$

[1] When we say that a lattice is complete, we also implicitly assume it has the empty meet and the empty join, and it is therefore bounded.

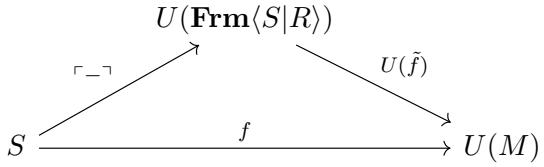
the free frame with generators from S. We denote as $\ulcorner - \urcorner : S \to U(\mathbf{Frm}\langle S\rangle)$ the canonical map of generators. Explicitly, the universal property of this frame states that for every function $S \to U(M)$ for a frame M there is a unique frame map $\tilde{f} : \mathbf{Frm}\langle S\rangle \to M$ such that the following commutes in **Set**.



We can also construct free frames subject to certain relations. For any set S and any relation $R \subseteq S \times S$, there is a frame

$$\mathbf{Frm}\langle S|R\rangle$$

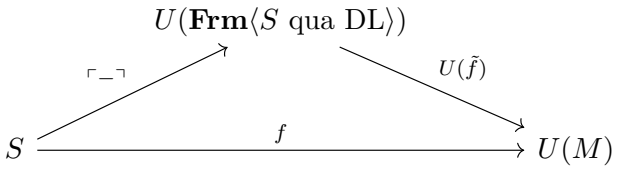
with a canonical map $\ulcorner - \urcorner : S \to U(\mathbf{Frm}\langle S|R\rangle)$ such that we have the following universal property. Whenever we have a map $f : S \to U(M)$ for some frame M such that it respects R in the sense that whenever $(s_1, s_2) \in R$ then also $f(s_1) \leq f(s_2)$, then there is a unique frame map $\tilde{f} : \mathbf{Frm}\langle S|R\rangle$ such that the following commutes.



In the special case where S is equipped with some order or algebraic structure, say it is a distributive lattice, we may ask that the free frame generated from S also preserves the distributive lattice structure. There exists a frame

$$\mathbf{Frm}\langle S \text{ qua DL}\rangle$$

with a map $\ulcorner - \urcorner : S \to U(\mathbf{Frm}\langle S \text{ qua DL}\rangle)$ such that we have the following universal property. Whenever we have a map $f : S \to U(M)$ for some frame M such that the lattice operations of S are preserved, there is a unique frame map $\tilde{f} : \mathbf{Frm}\langle S \text{ qua DL}\rangle \to M$ such that the following commutes.



Similarly, if S is a meet semilattice is called $\mathbf{Frm}\langle S \text{ qua } \wedge\text{-SL}\rangle$. We also use a similar notation for join semilattices.

The spatial-sober adjunction

Let us now recall the dual adjunction between the category of frames and that of topological spaces from which the view of frames as abstract spaces stems from. We limit ourselves to providing the definitions of the two functors involved, as well as that of the unit and the counit of the adjunction.

- Let us define the functor $\Omega : \mathbf{Top} \to \mathbf{Frm}^{op}$. On objects, we assign to each space X the frame $\Omega(X)$, i.e., the collection of opens of X ordered under set inclusion. On morphisms, for each continuous map $f : X \to Y$ we define the morphism $\Omega(f) : \Omega(Y) \to \Omega(X)$ as the preimage map f^{-1}.

- Let us define the functor $\mathsf{pt} : \mathbf{Frm}^{op} \to \mathbf{Top}$. For a frame L we define the space $\mathsf{pt}(L)$ as follows. The underlying set of points is $\mathbf{Frm}(L, 2)$. An open is defined to be any subset of $\mathbf{Frm}(L \to 2)$ of the form

  $$\{f \in \mathbf{Frm}(L, 2) : f(a) = 1\}$$

  for some $a \in L$. For a frame map $f; L \to M$ we define $\mathsf{pt}(f) : \mathsf{pt}(M) \to \mathsf{pt}(M)$ to be the precomposition map $- \circ f$.

Remark 0.0.2. Let us recall the motivation behind choosing maps to 2 to be a suitable representation of the points of a frame L, seen as an abstract space. For every space X, every point $x \in X$ determines a neighborhood map $N_x : \Omega(X) \to 2$ defined as $N_x(U) = 1$ if and only if $x \in U$. This will always be a frame map.

Now for the unit and the counit of the adjunction.

- For a space X we define the map $\psi_X : X \to \mathsf{pt}(\Omega(X))$ as the map $x \mapsto N_x$, where $N_x : \Omega(X) \to 2$ is the neighborhood map relative to x.

- For a frame L we define the map $\varphi_L : L \to \Omega(\mathsf{pt}(L))$ as $a \mapsto \{f \in \mathbf{Frm}(L, 2) : f(a) = 1\}$ for every element $a \in L$. This map is always a frame map, and it is surjective by definition of the topology on $\mathsf{pt}(L)$.

Theorem 0.0.3. *There is an adjunction $\Omega : \mathbf{Top} \leftrightarrows \mathbf{Frm}^{op} : \mathsf{pt}$, with $\Omega \dashv \mathsf{pt}$, with unit ψ and counit φ.*

We would like to restrict this adjunction to an equivalence of categories. We say that a space X is *sober* if every irreducible closed set of X is the closure of a unique point. We have the following.

Proposition 0.0.4. *The following are equivalent for a space X.*

1. *The space X is sober.*
2. *The space X is homeomorphic to $\mathsf{pt}(L)$ for some frame L.*
3. *The map $\psi_X : X \to \mathsf{pt}(\Omega(X))$ is a homeomorphism.*
4. *The map $\psi_X : X \to \mathsf{pt}(\Omega(X))$ is a bijection.*

For every space X, the map ψ_X is called the *sobrification map*, and the space $\mathsf{pt}(\Omega(X))$ is called the *sobrification* of X. We say that a frame L is *spatial* if whenever for $a, b \in L$ we have $a \not\leq b$ there is a frame map $f : L \to 2$ with $f(a) = 1$ and $f(b) = 0$.

Proposition 0.0.5. *The following are equivalent for a frame L.*

1. *The frame L is spatial.*
2. *The frame L is isomorphic to $\Omega(X)$ for some space X.*
3. *The map $\varphi_L : L \to \Omega(\mathsf{pt}(L))$ is an isomorphism.*
4. *The map $\varphi_L : L \to \Omega(\mathsf{pt}(L))$ is injective.*

For every frame L, the map φ_L is called the *spatialization map*, and the frame $\Omega(\mathsf{pt}(L))$ is called the *spatialization* of L. We know that every adjunction $(F, G, \eta, \varepsilon)$ with $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ restricts to the maximal equivalence of categories given by the functors F and G. The equivalence holds between the two full subcategories $\mathsf{Fix}_{GF}(\mathcal{C})$ and $\mathsf{Fix}_{FG}(\mathcal{D})$, where $\mathsf{Fix}_{GF}(\mathcal{C})$ is the full subcategory of $\mathcal{C}$ determined by the objects

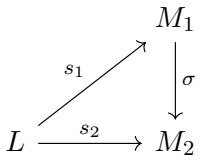
$$\{X \in \mathsf{Obj}(\mathcal{C}) : \eta_X \text{ is an isomorphism}\},$$

and the category $\mathsf{Fix}_{FG}(\mathcal{D})$ is defined similarly. In light of this fact, and of the two proposition we have just stated, we may deduce the following fact.

Theorem 0.0.6. *The functors pt and Ω determine a dual equivalence of categories between the full subcategory of* **Top** *whose objects are the sober spaces and the full subcategory of* **Frm** *whose objects are the spatial frames.*

Pointfree subspaces: sublocales, congruences, and nuclei

We move our attention to the pointfree notion of subspace. In this section we look at some known results, which can all be found in the classic references for pointfree topology, such as [26], [21], [37], [55]. We begin our explanation by asking what is a good pointfree representation of the notion of subspace. It is known that an arrow $i : Y \to X$ in **Top** is such that Y is isomorphic to the subspace $i[Y] \subseteq X$ if and only if the dualization $\Omega(i) : \Omega(X) \to \Omega(Y)$ is a frame surjection; and this is why we start from the consideration that frame surjections are good representations of subspaces. However, we would prefer to have a notion of pointfree subspace which identifies subspaces with isomorphic subspace inclusions. In other words, we want a notion of pointfree subspace which represents *equivalence classes of isomorphic frame surjections*, where we regard surjections $s_1 : L \to M_1$ and $s_2 : L \to M_2$ to be isomorphic if there is a frame isomorphism $\sigma : M_1 \to M_2$ such that the following diagram commutes in **Frm**.

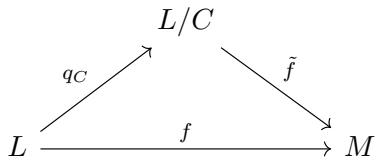


We will now see that in pointfree topology we have three equivalent ways of representing equivalence classes of isomorphic frame surjections. These are sublocales, congruences, and nuclei. Given a frame L, we say that $S \subseteq L$ is a *sublocale* if it is closed under arbitrary meets, and $s \in S$ implies $x \to s \in S$ for every $x \in L$. The collection of all sublocales of a frame L ordered under set inclusion is denoted as $\mathsf{S}(L)$. The second equivalent pointfree notion of subspace is that of frame congruence. We say that a relation $C \subseteq L \times L$ is a *frame congruence* if it is an equivalence relation and it respects the frame operations in the sense that both

- $(x_m, y_m) \in C$ implies that $(x_1 \wedge ... \wedge x_n, y_1 \wedge ... \wedge y_n)$ for finite families $x_m, y_m \in L$;
- $(x_i, y_i) \in C$ implies that $(\bigvee_i x_i, \bigvee_i y_i) \in C$ for any two families $x_i, y_i \in L$.

Given a frame L and a congruence C on it, for every element $x \in L$ we denote as $[x]_C$ the equivalence class of the element x under the equivalence relation C. We may always quotient a frame by a congruence: given a frame L and a congruence C on L, there is a frame L/C such that its set of elements is $\{[x]_C : x \in L\}$, and such that $[x]_C \leq [y]_C$ if there are $x', y' \in L$ such that $x' \leq y'$ and both $(x, x') \in C$ and $(y, y') \in C$. We denote this structure as L/C. The canonical map $q_C := [-]_C : L \to L/C$ is always a frame map, and we usually call it the *quotient map* of L relative to the congruence C. The quotient L/C has the universal property

that whenever $f : L \to M$ is a frame map such that $f(a) = f(b)$ whenever $(a, b) \in C$, there is a unique frame map $\tilde{f} : L/C \to M$ such that the following commutes.



As an instance of a more general fact of universal algebra, frame quotients are canonical representatives of equivalence classes of isomorphic frame surjections: two frame surjections $s_1 : L \to M_1$ and $s_2 : L \to M_2$ are isomorphic precisely when they induce the same congruence – that is – when $\{(x, y) : s_1(x) = s_1(y)\} = \{(x, y) : s_2(x) = s_2(y)\}$. The collection of all congruences on a frame L ordered under set inclusion is always a frame, which is called the *assembly* of the frame L. We denote the assembly of a frame L as $\mathsf{A}(L)$.

The third equivalent notion of pointfree subspace is that of nucleus. Given a frame L, we say that a map $\nu : L \to L$ is a nucleus on it if it is inflationary, idempotent, and it preserves finite meets. The collection of all nuclei on a frame L ordered pointwise is denoted as $\mathsf{N}(L)$.

These three notions (nuclei, sublocales, frame congruences) are interdefinable, in the sense that we have isomorphisms $\mathsf{S}(L)^{op} \cong \mathsf{A}(L) \cong \mathsf{N}(L)$. We define below the isomorphisms connecting these three structures.

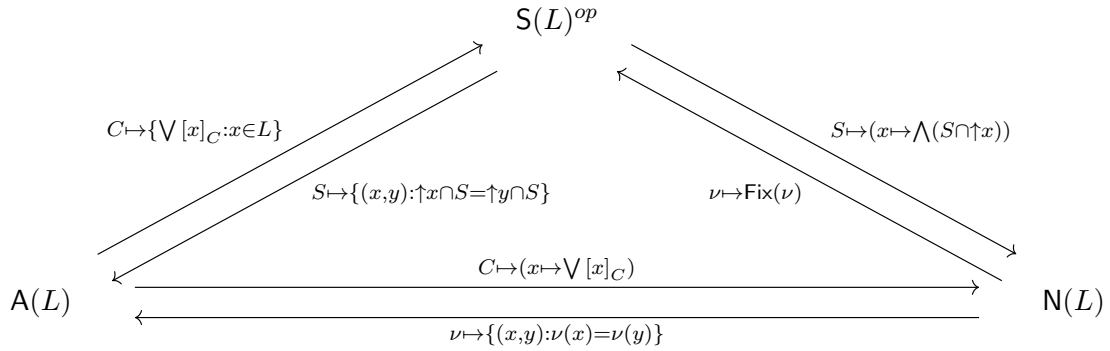
- We have mutually inverse isomorphisms $S_{(-)} : \mathsf{A}(L) \cong \mathsf{S}(L)^{op} : C_{(-)}$. For a congruence C the sublocale S_C is defined to be the collection of the top elements $\bigvee [x]_C$ of equivalence classes of C. For a sublocale S the congruence C_S is defined to be the relation $\{(x, y) \in L \times L : \uparrow x \cap S = \uparrow y \cap S\}$.

- We also have mutually inverse isomorphisms $\nu_{(-)} : \mathsf{S}(L)^{op} \cong \mathsf{N}(L) : S_{(-)}$. For a sublocale $S \subseteq L$, we define the nucleus ν_S to be the closure operator $x \mapsto \bigwedge(S \cap \uparrow x)$ having the elements of S as fixpoints. For a nucleus ν, we define the sublocale S_ν to be the ordered collection $\mathsf{Fix}(\nu) \subseteq L$ of its fixpoints.

- We also have mutually inverse isomorphisms $\nu(-) : \mathsf{A}(L) \cong \mathsf{N}(L) : C(-)$. For a congruence C on L the nucleus $\nu(C)$ is defined to be $x \mapsto \bigvee [x]_C$, and for a nucleus ν on L the congruence $C(\nu)$ is defined to be the congruence $\{(x, y) \in L \times L : \nu(x) = \nu(y)\}$.

In the following diagram we summarize the action of the various frame isomorphisms at

play.



In this thesis, we will switch between these three notions depending on which makes our calculations easier.

Let us now look at open and closed sublocales, pointfree versions of open and closed subspaces. Under the spatial-sober duality, whenever we have a subspace inclusion $i : Y \subseteq X$, its dualization $\Omega(i) : \Omega(X) \to \Omega(Y)$ is a frame surjection, and it is explicitly computed as $U \mapsto Y \cap U$. When Y happens to be an open $V \in \Omega(X)$, this map is $- \cap V : \Omega(X) \to \downarrow V$, where the downset $\downarrow V$ is computed in $\Omega(X)$. Two elements are identified by this map if and only if their intersection with V is the same. Abstracting this property of dualizations of open subspace inclusions, we call a congruence on a frame L an *open* congruence if it is of the form $\{(x, y) : x \wedge a = y \wedge a\}$ for some $a \in L$. We denote this congruence as $\Delta(a)$. For a frame L and for $a \in L$, the sublocale corresponding to the open congruence $\Delta(a)$ is the subset $\mathfrak{o}(a) := \{a \to x : x \in L\} \subseteq L$. The nucleus corresponding to an open sublocale $\mathfrak{o}(a) \subseteq L$ of a frame L is $x \mapsto (a \to x)$. We call *open* sublocales of a frame L those sublocales of L of the form $\mathfrak{o}(a)$ for some $a \in L$. Similarly, if we have a space X and an open $V \subseteq X$, the dualization of the subspace inclusion $V^c \subseteq X$ identifies two opens of X precisely when they have the same intersection with V^c, that is, when they have the same union with V. Motivated by this, for a frame L we call a congruence on L a *closed* congruence if it is of the form $\{(x, y) : x \vee a = y \vee a\}$ for some $a \in L$. For a frame L and an element $a \in L$, we denote the closed congruence $\{(x, y) : x \vee a = y \vee a\}$ as $\nabla(a)$. The sublocale corresponding to a closed congruence $\nabla(a)$ of a frame L is $\mathfrak{c}(a) := \{x \vee a : x \in L\} = \uparrow a \subseteq L$, and the corresponding nucleus is $x \mapsto x \vee a$. For a frame L, we call *closed* sublocales those sublocales of the form $\mathfrak{c}(a)$ for some $a \in L$.

Open and closed sublocales are important to us because they represent in pointfree language open and closed subspaces. In pointfree topology these are also important because their collection generates, in a certain sense, all of the pointfree subspaces of a pointfree space. We

shall see this shortly, but before let us see some basic results on the coframe $\mathsf{S}(L)$ of sublocales of a frame L, and let us see that the analogy with point-set open and closed subspaces can be pushed further.

Proposition 0.0.7. *For any frame L, the following facts hold.*

- *The lattice $\mathsf{S}(L)$ is a coframe, with top element L and bottom element $\{1\}$.*
- *Arbitrary meets in $\mathsf{S}(L)$ are set theoretical intersections.*
- *We have $\mathfrak{o}(1) = L$ and $\mathfrak{c}(1) = \{1\}$.*
- *We have $\mathfrak{o}(0) = \{1\}$ and $\mathfrak{c}(0) = L$.*
- *For any finite family $x_m \in L$ we have $\mathfrak{o}(x_1 \wedge ... \wedge x_n) = \mathfrak{o}(x_1) \cap ... \cap \mathfrak{o}(x_n)$. For any family $x_i \in L$ we have $\mathfrak{o}(\bigvee_i x_i) = \bigvee_i \mathfrak{o}(x_i)$.*
- *For any finite family $x_m \in L$ we have $\mathfrak{c}(x_1 \vee ... \vee x_n) = \mathfrak{c}(x_1) \vee ... \vee \mathfrak{c}(x_n)$. For any family $x_i \in L$ we have $\mathfrak{c}(\bigwedge_i x_i) = \bigcap_i \mathfrak{c}(x_i)$.*

A meet in $\mathsf{S}(L)$ of open sublocales is called a *fitted* sublocale. As we have seen in Proposition 0.0.7 above, open sublocales are behaved like open subspaces in that their collection is closed under arbitrary joins and finite meets in the lattice $\mathsf{S}(L)$ of pointfree subspaces of a pointfree space L. Similarly, the collection of closed sublocales is closed under arbitrary meets and finite joins. Let us now see counterexamples to see that the collection of opens sublocales of a frame L fails to be closed under arbitrary meets, and that the collection of closed ones fails to be closed under arbitrary joins.

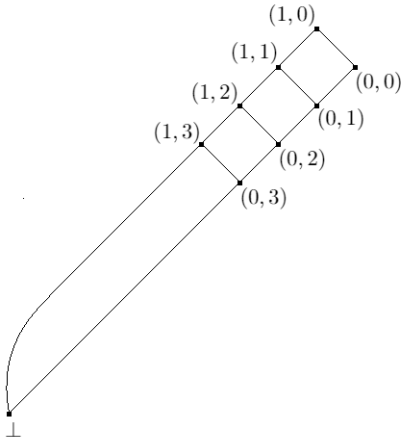
Examples 0.0.8. Let us see an example of a fitted sublocale which is not open, and of a join of closed sublocales which is not closed.

- Consider the frame L isomorphic to the opposite of the ordinal $(\omega+1)$. Suppose that we want to calculate the join of the congruences $\Delta(n)$, with $n \in \omega$. It can be checked that the equivalence relation depicted below is a frame congruence. It also can be checked that it is the smallest equivalence relation that contains all open congruences $\Delta(n)$, and so it is their join.

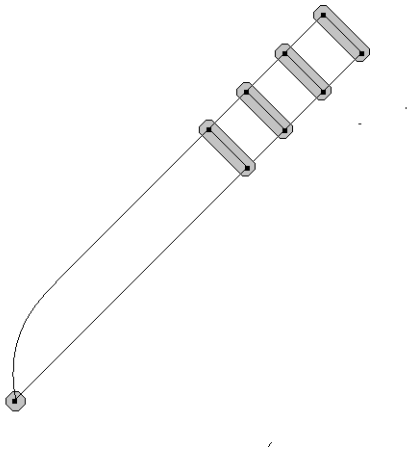


This congruence is different from the congruence $\Delta(\bigwedge_{n\in\omega} n) = \Delta(\omega+1)$, which is the trivial congruence identifying all elements of L.

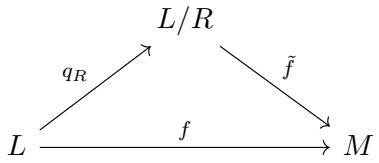
- Consider the following frame. This consists of two chains $\{... < (0,n) < ... < (0,1) < (0,0)\}$ and $\{... < (1,n) < ... < (1,1) < (1,0)\}$ both of cardinality ω together with a bottom element $\perp$, and such that $(0,n) < (1,n)$ for every $n \in \mathbb{N}$.



The intersection of closed congruences from all elements of the form $(1,n)$ forms a congruence with the equivalence classes depicted below.



For a frame L and a relation $R \subseteq L \times L$, the congruence $\bigcap\{C \in \mathsf{A}(L) : R \subseteq C\}$ is the smallest congruence on L containing R. We say that the smallest congruence on L containing R is the congruence *induced* by R on L. For an arbitrary relation R on a frame L, we refer to the equivalence classes of the congruence induced by R simply as $[x]_R$. Similarly, we call L/R the quotient of the frame L by the congruence induced by R. The quotient L/R has the universal property that whenever $f : L \to M$ is a frame map such that $f(a) \leq f(b)$ whenever $(a, b) \in R$, there is a unique frame map $\tilde{f} : L/R \to M$ such that the following commutes.



Here, $q_R : L \to L/R$ is the obvious quotient map. For a frame L and two elements $a, b \in L$, the relation $\{(a, b)\}$ induces $\nabla(a) \cap \Delta(b)$. We will refer to congruences of the form $\nabla(a) \cap \Delta(b)$ as *basic* congruences. This piece of terminology is justified by the following result.

Proposition 0.0.9. *For a frame L, and a congruence C on L we have*

$$C = \bigvee\{\nabla(a) \cap \Delta(b) : (a, b) \in C\} = \bigvee\{\nabla(a) \cap \Delta(b) : [a]_C \leq [b]_C\},$$

where the inequality $[a]_C \leq [b]_C$ above is understood to hold in the quotient frame L/C.

Translated in terms of sublocales, this result implies that for a frame L every sublocale is of the form $\bigcap_i \mathfrak{o}(a_i) \vee \mathfrak{c}(b_i)$. It is in this sense that open and closed pointfree subspaces generate the entire collection of subspaces of a pointfree space. Pointfree subspaces and point-set ones

have very different behaviours. As a famous example of this, consider for instance the notion of density. For a space X a subspace $Y \subseteq X$ is dense if it intersects all nonempty opens of X; but this can easily be phrased pointfreely: Y is dense precisely when the dualization $\Omega(i) : \Omega(X) \to \Omega(Y)$ is such that $\Omega(i)(U) = \emptyset$ implies that $U = \emptyset$. We can rephrase this result by saying that the subspace $i : Y \subseteq X$ is dense if and only if the sublocale corresponding to it contains the bottom element of $\Omega(X)$. Abstracting this, for a frame L we call a sublocale $S \subseteq L$ a *dense* sublocale if it contains the bottom element 0. Recall that sublocales are closed under intersections. This means that for every frame L we can define the sublocale

$$\bigcap \{S \in \mathsf{S}(L) : 0 \in S\}.$$

By its definition, this is the smallest dense sublocale of the frame L. We obtain Isbell's famous Density Theorem.

Theorem 0.0.10. *Every frame L has the smallest dense sublocale.*

One of the reason we are particularly interested in this theorem, when doing pointfree topology, is that it exhibits a crucial difference in the behaviour of point-set and pointfree subspaces: in point-set topology, dense subspaces of a space are not closed under intersections. In particular, Isbell's Density Theorem is a clue pointing strongly at the intuition that there are more sublocales of $\Omega(X)$ than there are subspaces of X. Let us see a famous example of this.

Example 0.0.11. Consider the real line $\mathbb{R}$, equipped with the Euclidean topology. The frame $\Omega(\mathbb{R})$ has a smallest dense sublocale. This is nontrivial, in the sense that it contains more elements than $\{\mathbb{R}, \emptyset\}$. In the concrete space $\mathbb{R}$. the situation is different. We do not have a smallest dense subspace. The frame $\Omega(\mathbb{R})$ has more sublocales than the space $\mathbb{R}$ has subspaces: the smallest dense sublocale of $\mathbb{R}$ is not the frame of opens of any subspace of $\mathbb{R}$. We notice that a subspace inclusion $i : Y \hookrightarrow X$ is dense if and only if the sublocale corresponding to its dualization is dense. This means (although we omit the proof here) that there is no subspace of $\mathbb{R}$ which induces the smallest dense sublocale on $\Omega(\mathbb{R})$.

Special kinds of frames

For a frame L we say that the element a is *rather below* the element b, and we write $a \prec b$, if there is some $c \in L$ such that $a \wedge c = 0$ and $b \vee c = 1$. In the literature, sometimes this relation is called being *well inside*. We say that a frame L is *regular* if every element of L is the join of the elements rather below it. We say that a frame L is *normal* if whenever $a, b \in L$ are such

that $a \vee b = 1$ there are $u, v \in L$ such that $u \wedge v = 0$, and such that $a \vee u = 1 = b \vee v$. We say that a frame L is *zero dimensional* if every element of L is the join of the complemented elements below it.

Bispaces

A *bitopological space* is a triple $(X, \Omega^+(X), \Omega^-(X))$ where X is a set and $\Omega^+(X)$ and $\Omega^-(X)$ are topologies on X. The category **BiTop** of bitopological spaces is defined to be the category whose objects are bitopological spaces, and whose morphisms are *bicontinuous maps*: maps $f : X \to Y$ between the underlying sets of two bitopological spaces X and Y such that they are continuous with respect to both topologies. A bicontinuous map which is a homeomorphism with respect to both topologies is called a *bihomeomorphism*. Just as we denote the frame of opens of a topological space as $\Omega(X)$, we will denote the frame of positive opens of a bispace X as $\Omega^+(X)$, and its frame of negative opens as $\Omega^-(X)$. We also indicate that an open is positive with a "+" exponent, and use a similar convention for negative opens. We will also abuse notation slightly, and for a bispace X we denote as X its underlying set, too, unless doing this would cause ambiguity. Every bitopological space X determines its *patch*, the topological space $\mathsf{patch}(X)$ defined as $(X, \Omega^+(X) \vee \Omega^-(X))$, where the join $\vee$ is taken in the lattice of all possible topologies on X. We denote the topology on the underlying set of X generated by $\Omega^+(X) \cup \Omega^-(X)$ as $\Omega(X)$. We observe that we have the following, because for a map to be continuous it suffices for its preimage map to map subbasic opens to subbasic opens.

Lemma 0.0.12. *Bicontinuous maps are continuous with respect to the patch topology.*

By assigning to each bispace its patch we can construct a functor $\mathsf{patch} : \mathbf{BiTop} \to \mathbf{Top}$, which acts on morphisms simply by mapping a function to itself. The patch functor has a left adjoint: this is the functor $(-, \Omega(-), \Omega(-)) : \mathbf{Top} \to \mathbf{BiTop}$, assigning to each topological space X the bispace $(X, \Omega(X), \Omega(X))$; and mapping each function to itself.

Various bitopological analogues of topological properties have been introduced in the literature. We stress that in general there is no unanimous answer as to which is the correct bitopological notion of a certain topological property. For the following three separation axioms, we follow the definitions of [47].

- A bitopological space X is said to be *pairwise* T_0 if whenever we have $x, y \in X$ with $x \neq y$ there is some $U \in \Omega^+(X) \cup \Omega^-(X)$ containing one of x and y and omitting the other.

- A bitopological space X is said to be *pairwise* T_1 if whenever we have $x, y \in X$ with $x \neq y$ there is some $U \in \Omega^+(X) \cup \Omega^-(X)$ such that it contains x and omits y.
- A bitopological space X is said to be *pairwise* T_2, or *pairwise Hausdorff*, if whenever we have $x, y \in X$ with $x \neq y$ there are disjoint opens $U^+ \in \Omega^+(X)$ and $U^- \in \Omega^-(X)$ such that $x \in U^+$ and $y \in U^-$, or such that $y \in U^+$ and $x \in U^-$.

For bitopological compactness and regularity, we use the definition used in [46]. This is the same definition used in [29], [22], [12], [8], [48], too; all sources that we will greatly use for this thesis. We say that a bispace X is *bicompact*, or simply *compact*, if every cover of elements in $\Omega^+(X) \cup \Omega^-(X)$ has a finite subcover. It is clear that for a bispace X compactness of $\mathsf{patch}(X)$ implies bicompactness of X. The converse holds too, provided that we use Alexander's Subbasis Lemma. In this thesis, we use the Axiom of Choice freely, and so in our view bicompactness of a bispace X and compactness of its patch $\mathsf{patch}(X)$ are equivalent conditions. For a bispace X we write $U^+ \preceq V^+$ for positive opens U_1^+ and U_2^+ if there is a negative open U^- such that $U_1^+ \cap U^- = \emptyset$ and $U_2^+ \cup U^- = X$. We define similarly the relation $\preceq$ on the negative opens of X. We say that a bispace is *biregular*, or simply *regular*, if for every positive open U^+ we have $U^+ = \bigcup\{V^+ \in \Omega^+(X) : V^+ \preceq U^+\}$, and the same holds for the negative opens. For the following notion, too, there does not seem to be much disagreement in the literature that we have used: it is the one used in [12], [22], and [29], [8], [48]. A bispace X is *zero dimensional* if every positive open is a union $\bigcup_i U_i^+$ of positive opens U_i^+ such that each of them is closed in the negative topology, and every negative open satisfies the analogous property. For bitopological compactness and regularity, we use the definition used in [46], [49]. As in [29] and [22] we say that a bispace X is *binormal*, or simply *normal*, if whenever U^+ is a positive open of X and U^- is a negative open such that $U^{+c} \cap U^{-c} = \emptyset$, there is a positive open V^+ and a negative open V^- such that $U^{-c} \subseteq V^+$ and $U^{+c} \subseteq V^-$.

Biframes and their duality

We now look at the most literal translation of the notion of bispace into pointfree language. We will introduce an adjunction similar to the one for frames and spaces, relating the category of bitopological spaces with a possible notion of a pointfree version of it. The theory that we describe here is introduced in [8], and further developed in [48]. A *biframe* is a triple (L^+, L^-, L) where L^+, L^-, L are frames, together with frame injections $e_L^+ : L^+ \to L$ and $e_L^- : L^- \to L$ such that all elements of L are of the form $\bigvee_i e_L^+(x_i^+) \wedge e_L^-(x_i^-)$ for families $x_i^+ \in L^+$ and $x_i^- \in L^-$. In general, we denote a biframe just as a triple (L^+, L^-, L), leaving

it implicit that we call the two subframe inclusions e_L^+ and e_L^- as above. Additionally, when we a biframe with a triple (L^+, L^-, L) in which $L^+, L^- \subseteq L$ we will leave it implicit that we take the frame injections are the two subset inclusions. Additionally, when we have a biframe with triple (L^+, L^-, L) where L is a free construction with generators from L^+ and L^- we will leave it implicit that we will regard the syntactic map $\ulcorner - \urcorner : L^+ \to L$, and the analogous one on the negative component, to be the two frame injections. To simplify further, we will denote a biframe (L^+, L^-, L) as $\mathcal{L}$, a biframe (M^+, M^-, M) as $\mathcal{M}$, and so on. For a biframe $\mathcal{L}$, we call L its *main component*. We note that for any biframe L there is always a congruence C on $L^+ \oplus L^-$ such that we have an isomorphism $i : (L^+ \oplus L^-)/C \cong L$ which is well-behaved with respect to the subframe inclusion, in the sense that the following diagram and the analogous one for the negative component commute.

$$\begin{array}{ccc} L^+ & & \\ {\scriptstyle q_c \circ \ulcorner - \urcorner}\downarrow & \searrow^{e_L^+} & \\ (L^+ \oplus L^-)/C & \xrightarrow{\quad i \quad} & L \end{array}$$

Here, $q_C : L^+ \oplus L^- \to (L^+ \oplus L^-)/C$ is the quotient map, and $\ulcorner - \urcorner : L^+ \to L^+ \oplus L^-$ the canonical injection of L^+ into the coproduct. For a biframe $\mathcal{L}$, in particular, the quotient of $L^+ \oplus L^-$ corresponding to L is the quotient by the relation

$$R_L := \{(\bigvee_i \ulcorner x_i^+ \urcorner \wedge \ulcorner x_i^- \urcorner, \bigvee_j \ulcorner y_j^+ \urcorner \wedge \ulcorner y_j^- \urcorner) : \bigvee_i e_L^+(x_i^+) \wedge e_L^-(x_i^-) \leq \bigvee_j e_L^+(y_j^+) \wedge e_L^-(y_j^-) \text{ in } L\}.$$

A morphism between biframes $\mathcal{L}$ and $\mathcal{M}$ is a pair of frame maps (f^+, f^-) with $f^+ : L^+ \to M^+$ and $f^- : L^- \to M^-$ such that there is a frame map $f : L \to M$ making the following diagram and the analogous one on the negative component both commute.

$$\begin{array}{ccc} L^+ & \xrightarrow{f^+} & M^+ \\ \downarrow{\scriptstyle e_L^+} & & \downarrow{\scriptstyle e_M^+} \\ L & \xrightarrow{\quad f \quad} & M \end{array}$$

Sometimes, as an abbreviation, we will refer as f to a biframe map $(f^+, f^-) : \mathcal{L} \to \mathcal{M}$. It will always be clear from the context whether we are speaking about the biframe map itself or the frame map $f : L \to M$ witness of (f^+, f^-) being a biframe map. Let us see an alternative way of viewing biframe morphisms. For biframes $\mathcal{L}$ and $\mathcal{M}$, let C_L be the congruence on $L^+ \oplus L^-$ such that $(L^+ \oplus L^-)/C_L$ is canonically isomorphic to L. Let C_M be

the analogous congruence for $\mathcal{M}$. Combining this property of the congruences C_L and C_M, and using the defining property of biframe morphisms, we obtain that a suitable pair of frame maps (f^+, f^-) counts as a biframe morphism $(f^+, f^-) : \mathcal{L} \to \mathcal{M}$ if and only if there is a frame map $f : (L^+ \oplus L^-)/C_L \to (M^+ \oplus M^-)/C_M$ making the following diagram commute.

$$\begin{array}{ccc} L^+ \oplus L^- & \xrightarrow{f^+ \oplus f^-} & M^+ \oplus M^- \\ \downarrow{\scriptstyle q_{C_L}} & & \downarrow{\scriptstyle q_{C_M}} \\ (L^+ \oplus L^-)/C_L & \xrightarrow{f} & (M^+ \oplus M^-)/C_M \end{array}$$

We proceed by giving the definition of the two adjoint functors we are interested in, as well as the definitions of the unit and the counit of the adjunction.

- For every bispace X, the biframe $\mathsf{b}\Omega(X)$ is defined to be the triple

  $$(\Omega^+(X), \Omega^-(X), \Omega(X)),$$

  with set inclusions as the two frame injections. For a bicontinuous map $f : X \to Y$, we define the map $\mathsf{b}\Omega(f)$ as the pairing of preimage maps $(f^{-1}, f^{-1}) : \mathsf{b}\Omega(Y) \to \mathsf{b}\Omega(X)$. This defines a functor $\mathsf{b}\Omega : \mathbf{BiTop} \to \mathbf{BiFrm}^{op}$.

- For every biframe $\mathcal{L}$, we define the bispace $\mathsf{bpt}(\mathcal{L})$ as follows. Its underlying set of points is $\mathbf{BiFrm}(\mathcal{L}, \mathbf{2})$, where $\mathbf{2}$ denotes the biframe $(2, 2, 2)$. A positive open of $\mathsf{bpt}(\mathcal{L})$ is any subset of $\mathbf{BiFrm}(\mathcal{L}, \mathbf{2})$ of the form $\{(f^+, f^-) \in \mathbf{BiFrm}(\mathcal{L}, \mathbf{2}) : f^+(a^+) = 1\}$ for some $a^+ \in L^+$; negative opens are defined analogously. For every biframe map $f : \mathcal{L} \to \mathcal{M}$ we define the bicontinuous map $\mathsf{bpt}(f) : \mathsf{bpt}(\mathcal{M}) \to \mathsf{bpt}(\mathcal{L})$ as the pairing of precompositions $(- \circ f^+, - \circ f^-)$. This defines a functor $\mathsf{bpt} : \mathbf{BiFrm}^{op} \to \mathbf{BiTop}$, right adjoint to $\mathsf{b}\Omega$.

Now, let us look at the unit and the counit.

- For every bispace X, we define the map $\psi_X : X \to \mathsf{bpt}(\mathsf{b}\Omega(X))$ as $x \mapsto (N_x^+, N_x^-)$ for each $x \in X$. Here, $N_x^+ : \Omega^+(X) \to 2$ is defined to be $n_x^+(U^+) = 1$ of and only if $x \in U^+$ for every positive open $U^+ \subseteq X$. We call this map the *positive neighborhood map* of the bispace X. We call *negative neighborhood map* the frame map $N_x^- : \Omega^-(X) \to 2$, defined analogously as N_x^+. The map $\psi_X : X \to \mathsf{bpt}(\mathsf{b}\Omega(X))$ is always bicontinuous. We call this the *sobrification map* of the bispace X. This determines a natural transformation $\psi : 1_{\mathbf{BiTop}} \to \mathsf{bpt} \circ \mathsf{b}\Omega$, and this is the unit of the adjunction.

- For every biframe $\mathcal{L}$ we define the map $\varphi_{\mathcal{L}} : \mathcal{L} \to \mathsf{b}\Omega(\mathsf{bpt}(\mathcal{L}))$ as the pair of maps $(\varphi^+_{\mathcal{L}}, \varphi^-_{\mathcal{L}})$ where $\varphi^+_{\mathcal{L}}(a^+) = \{(f^+, f^-) \in \mathbf{BiFrm}(\mathcal{L}, \mathbf{2}) : f^+(a^+) = 1\}$ for every $a^+ \in L^+$. The negative component of $\varphi_{\mathcal{L}}$ is defined similarly. We call $\varphi_{\mathcal{L}}$ the *spatialization* of $\mathcal{L}$. This determines a natural transformation $\varphi : \mathsf{b}\Omega \circ \mathsf{bpt} \to 1_{\mathbf{BiFrm}^{op}}$, and this is the counit of the adjunction.

We call a bispace *sober* if its patch is a sober space.

Proposition 0.0.13. *The following are equivalent for a bispace X.*

- *The bispace X is sober.*
- *The map ψ_X is a bihomeomorphism.*
- *The bispace X is bihomeomorphic to $\mathsf{bpt}(\mathcal{L})$ for some biframe $\mathcal{L}$.*
- *Both components of the map ψ_X are bijections.*

We call a biframe *spatial* if its third component is a spatial frame.

Proposition 0.0.14. *The following are equivalent for a biframe $\mathcal{L}$.*

- *The biframe $\mathcal{L}$ is spatial.*
- *The map $\varphi_{\mathcal{L}}$ is an isomorphism of biframes.*
- *The biframe $\mathcal{L}$ is isomorphic to $\mathsf{b}\Omega(X)$ for some bispace X.*
- *Both components of the map $\varphi_{\mathcal{L}}$ are injections.*

Corollary 0.0.15. *The adjunction $\mathsf{b}\Omega : \mathbf{BiTop} \leftrightarrows \mathbf{BiFrm}^{op} : \mathsf{bpt}$ restricts to a dual equivalence of the full subcategory of* **BiTop** *with the sober bispaces as objects and the full subcategory of* **BiFrm** *with the spatial biframes as objects.*

The following definitions are those used in [48]. For a biframe $\mathcal{L}$ and for $a^+, b^+ \in L^+$ we write $a^+ \prec b^+$, and say that a^+ is *rather below* the element b^+ if there is some $c^- \in L^-$ such that $e^+_L(a^+) \wedge e^-_L(c^-) = 0$ and $e^+_L(b^+) \vee e^-_L(c^-) = 1$. We define similarly the analogous relation $\prec$ on elements of L^-. A biframe $\mathcal{L}$ is called *regular* if for every $a^+ \in L^+$ we have that $a^+ = \bigvee\{b^+ \in L^+ : b^+ \prec a^+\}$, and the analogous property holds for the elements of L^-. A biframe $\mathcal{L}$ is called *normal* if whenever there are $a^+ \in L^+$ and $a^- \in L^-$ with $e^+_L(a^+) \vee e^-_L(a^-) = 1$ in L, there also exist $b^+ \in L^+$ and $b^- \in L^-$ such that $e^+_L(b^+) \wedge e^-_L(b^-) = 0$ and $e^+_L(a^+) \vee e^-_L(b^-) = 1 = e^+_L(b^+) \vee e^-_L(a^-)$. For an element $a^+ \in L^+$, we say that $a^- \in L^-$ is

the *bicomplement* of a^+ if the elements $e_L^+(a^+)$ and $e_L^-(a^-)$ are complements of each other in L. If the bicomplement of an element exists, then it is of course unique. We define similarly bicomplements of elements of L^-. We say that an element of $L^+ \cup L^-$ is *bicomplemented* if it has a bicomplement. A biframe $\mathcal{L}$ is called *zero dimensional* if every element of L^+ is the join of the bicomplemented elements below it, and if the analogous property holds for elements of L^-. A biframe $\mathcal{L}$ is called *compact* if every finite cover of L of elements of the form $e_L^+(a^+)$ or $e_L^-(a^-)$ has a finite subcover, that is, if whenever $1 \leq \bigvee_i e_L^+(a_i^+) \vee e_L^-(a_i^-)$ there is a finite subset $F \subseteq I$ with $1 \leq \bigvee_{f \in F} \bigvee_{f \in F} e_L^+(a_f^+) \vee e_L^-(a_f^-)$. As in the point-set case, when deducing global compactness from bitopological compactness, we will use some form of Alexander's Lemma. The result below uses Zorn's Lemma; the proof is essentially a pointfree translation of the classical point set argument.

Proposition 0.0.16. *If (L^+, L^-, L) is a biframe, then L is compact if and only if $1 \leq \bigvee_i x_i^+ \vee x_i^-$ implies $1 \leq \bigvee_{f \in F} x_f^+ \vee x_f^-$ for some finite $F \subseteq I$, for all families with $x_i^+ \in L_i^+$ and $x_i^- \in L^-$.*

Proof. Only one direction of the desired bi-implication is not trivial. Suppose, then, that L is such that $1 \leq \bigvee_i x_i^+ \vee x_i^-$ implies $1 \leq \bigvee_{f \in F} x_f^+ \vee x_f^-$. Consider the poset $\mathcal{S}$, whose elements are all the families $F \subseteq L$ such that $1 \leq \bigvee F$ but $1 \not\leq f_1 \vee ... \vee f_n$ for all finite families $\{f_1, ..., f_n\} \subseteq F$, and whose order is set inclusion. Towards contradiction, we assume that $\mathcal{S}$ is nonempty. The first step is to show that all ascending chains of $\mathcal{S}$ have an upper bound, in order to be able to use Zorn's Lemma. Assume, then, that we have a chain $F_i \in \mathcal{S}$. We claim that the upper bound of this is $\bigcup_i F_i$. We certainly have $1 \leq \bigvee \bigcup_i F_i$. Suppose towards contradiction that we have any finite family $\{f_1, ..., f_n\} \subseteq \bigcup_i F_i$ with $1 \leq f_1 \vee ... \vee f_n$. Say that $f_m \in F_{i_m}$ for all $m \leq n$. Since $\{F_{i_m} : m \leq n\}$ is a total order, it has a maximum, call this maximum F_M. Then we have $f_1, ..., f_n \in F_M$, contradicting that $F_M \in \mathcal{S}$. We then must have that no such finite family $\{f_1, ..., f_n\} \subseteq \bigcup_i F_i$ exists, and so $\bigcup_i F_i \in \mathcal{S}$, which means that $\bigcup_i F_i$ is indeed an upper bound for the chain F_i. By Zorn's Lemma, then, since the poset $\mathcal{S}$ is nonempty, there must be a maximal element $F \in \mathcal{S}$. Maximality here amounts to the following condition:

Whenever $x \notin F$, there is a finite family $\{f_1, ..., f_n\} \subseteq F$ with $1 \leq x \vee f_1 \vee ... \vee f_n$.

Let $\{x_i : i \in I\} \cup \{x^- j : j \in J\}$ be the collection of subbasic elements appearing in F. We cannot have $1 \leq \bigvee_i x_i^+ \vee \bigvee_j x_j^+$ or we would contradict that L is compact with respect to subbasic covers. Then, as $1 \leq \bigvee F$, there must be some $f \in F$ with $f \not\leq \bigvee_i x_i^+ \vee \bigvee_j x_j^+$ and, since $L^+ \cup L^-$ is a subbasis for L, this implies that there is $f^+ \wedge f^- \leq f$ such that $f^+ \wedge f^- \not\leq$

$\bigvee_i x_i^+ \vee \bigvee_j x_j^+$. In particular, we have both $f^+ \not\leq \bigvee_i x_i^+ \vee \bigvee_j x_j^+$ and $f^- \not\leq \bigvee_i x_i^+ \vee \bigvee_j x_j^+$, which means, by definition of the family $\{x_i : i \in I\} \cup \{x^- j : j \in J\}$, that both $f^+ \notin F$ and $f^- \notin F$. By maximality of F, then, there must be two finite families $F^+, F^- \subseteq F$ such that

$$\begin{cases} 1 \leq \bigvee F^+ \vee f^+, \\ 1 \leq \bigvee F^- \vee f^-. \end{cases}$$

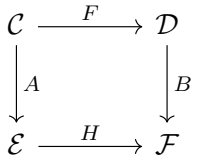
This implies that the family $F^+ \cup F^- \subseteq F$ is a finite family such that $1 \leq \bigvee(F^+ \cup F^-) \vee (f^+ \wedge f^-)$. Recall that we had assumed that $f^+ \wedge f^- \leq f$ for $f \in F$. We then obtain $1 \leq \bigvee(f^+ \cup F^-) \vee f$, contradicting our assumption on F. We then must have that the poset $\mathcal{S}$ is empty, that is, that L is compact. □

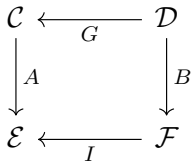
Connecting the point-set and the pointfree patch

For a biframe $\mathcal{L}$, let us denote as $\pi_3(\mathcal{L})$ its main component. We recall that a biframe map $(f^+, f^-) : \mathcal{L} \to \mathcal{M}$, by its defining property, is such that there is a unique frame map $f : L \to M$ making the following, and its analogue for the negative component, commute in **Frm**.

$$\begin{array}{ccc} L^+ & \xrightarrow{f^+} & M^+ \\ \downarrow{\scriptstyle e_L^+} & & \downarrow{\scriptstyle e_M^+} \\ L & \xrightarrow{f} & M \end{array}$$

For a biframe morphism $f : \mathcal{L} \to \mathcal{M}$, then, let us define the map $\pi_3(f) : \pi_3(\mathcal{L}) \to \pi_3(\mathcal{M})$ as $f : L \to M$. This determines a functor $\pi_3 : \mathbf{BiFrm} \to \mathbf{Frm}$, which is — we now claim — the pointfree analogue of the $\mathsf{patch} : \mathbf{BiTop} \to \mathbf{Top}$ for bispaces and spaces. We now work towards establishing what is means exactly for the functor π_3 to be the pointfree version of patch. We take this to mean that the pairing of the two functors connects the spatial-sober adjunction for biframes and that for frames in a particularly natural way. More specifically, we will see that the pair of functors constitutes, in a certain sense and up to isomorphism, a morphism between the two adjunctions. In [32] the concept of map between adjunctions is defined (Chapter IV, Section 7). For adjunctions $(F, G, \eta, \varepsilon)$ with $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ and (H, I, θ, ζ) with $H : \mathcal{E} \leftrightarrows \mathcal{F} : I$ we say that the pair of functors $A : \mathcal{C} \to \mathcal{E}$ and $B : \mathcal{D} \to \mathcal{F}$ is a map between the adjunctions $(F, G, \eta, \varepsilon)$ and (H, I, θ, ζ) if and only if the two squares below commute and if for every object $X \in \mathsf{Obj}(\mathcal{C})$ we have that $A(\eta_X) = \theta_{A(X)}$, and for every object $Y \in \mathsf{Obj}(\mathcal{D})$ we have $B(\varepsilon_Y) = \zeta_{B(Y)}$.

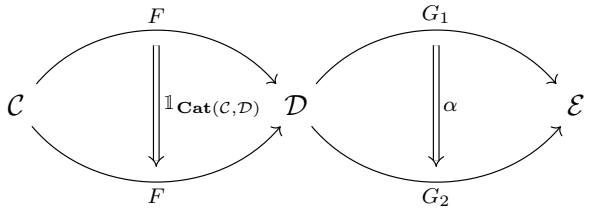




We wish to loosen this definition, to obtain a notion of what it means for a pair of functors to be a morphism between adjunctions up to natural isomorphism. Before we do so, we need to talk more generally about natural transformations and natural isomorphisms.

Lemma 0.0.17. *Suppose that $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are categories. Suppose that we have a functor $F : \mathcal{C} \to \mathcal{D}$ and two naturally isomorphic functors $G_1 : \mathcal{D} \to \mathcal{E}$ and $G_2 : \mathcal{D} \to \mathcal{E}$. Then, the functors $G_1 \circ F$ and $G_2 \circ F$ are naturally isomorphic, too. Similarly, if we have a functor $H : \mathcal{E} \to \mathcal{F}$, the functors $H \circ G_1$ and $H \circ G_2$ are naturally isomorphic, too.*

Proof. We depict the situation in the following diagram. Here, α is a natural isomorphism, and the natural transformation $\mathbb{1}_{\mathbf{Cat}(\mathcal{C},\mathcal{D})}$ is the assignment $\mathsf{Obj}(\mathcal{C}) \to \mathsf{Mor}(\mathcal{D})$ which for every object $C \in \mathsf{Obj}(\mathcal{C})$ gives the identity on $F(C)$.



We know that we can compose horizontally the two natural transformations (an operation on natural transformations which is sometimes called the "Godement product" (see [2])). In this case this composition yields a natural transformation $\alpha \circ \mathbb{1}_{\mathbf{Cat}(\mathcal{C},\mathcal{D})} : \mathsf{Obj}(\mathcal{C}) \to \mathsf{Mor}(\mathcal{E})$ defined as $X \mapsto \alpha_{F(X)}$ for every object $X \in \mathsf{Obj}(X)$. Let us that this is a natural transformation. Suppose that $f : X \to Y$ is an arrow in $\mathcal{C}$. The naturality square amounts to

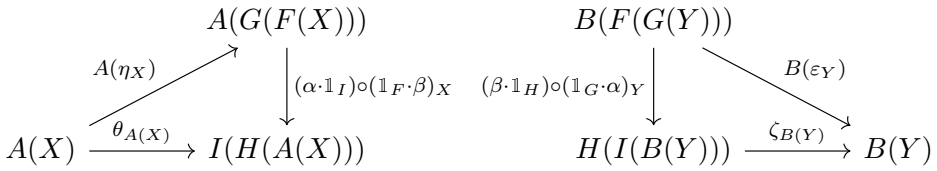
$$\begin{array}{ccc} G_1(F(X)) & \xrightarrow{\alpha_{F(X)}} & G_2(F(X)) \\ {\scriptstyle G_1(F(f))}\downarrow & & \downarrow{\scriptstyle G_2(F(f))} \\ G_1(F(Y)) & \xrightarrow{\alpha_{F(Y)}} & G_2(F(Y)) \end{array}$$

and this commutes by naturality of α. Since α_C is an isomorphism for every $C \in \mathsf{Obj}(\mathcal{C})$, we know that $\alpha \circ \mathbb{1}_{\mathbf{Cat}(\mathcal{C},\mathcal{D})}$, too, is an isomorphism when evaluated at each object of $\mathcal{C}$, and

so it is a natural isomorphism. The proof for postcomposition with a functor $H : \mathcal{E} \to \mathcal{F}$ is analogous. □

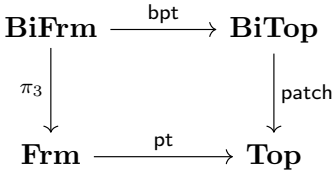
For adjunctions $(F, G, \eta, \varepsilon)$ with $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ and (H, I, θ, ζ) with $H : \mathcal{E} \leftrightarrows \mathcal{F} : I$ we say that the pair of functors $A : \mathcal{C} \to \mathcal{E}$ and $B : \mathcal{D} \to \mathcal{F}$ is a map between the adjunctions $(F, G, \eta, \varepsilon)$ and (H, I, θ, ζ) up to natural isomorphism if

1. the two squares above commute up to natural isomorphism (say the two natural isomorphisms are $\alpha : B \circ F \to H \circ A$ and $\beta : A \circ G \to I \circ B$, respectively); and
2. the two natural isomorphisms above determine in the way described in Lemma 0.0.17 natural isomorphism $(\alpha \cdot \mathbb{1}_I) \circ (\mathbb{1}_F \cdot \beta) : A \circ G \circ F \to I \circ H \circ A$ and $(\beta \cdot \mathbb{1}_H) \circ (\mathbb{1}_G \cdot \alpha) : B \circ F \circ G \to H \circ I \circ B$ such that the following two triangles commute for every object $X \in \mathsf{Obj}(\mathcal{C})$ and every object $Y \in \mathsf{Obj}(\mathcal{D})$. Here, $\cdot$ denotes the Godement product and $\mathbb{1}_X$ the identity natural transformation on a functor X.



We now claim that the pair of functors (π_3, patch) are a map, up to natural isomorphism, between the spatial-sober adjunction for biframes and that for frames.

Lemma 0.0.18. *The following square commutes up to natural isomorphism.*



Proof. Let us prove that for every biframe $\mathcal{L}$ we have a homeomorphism $\mathsf{pt}(\pi_3((\mathcal{L})) \cong \mathsf{patch}(\mathsf{bpt}(\mathcal{L}))$. Define the map

$$\alpha_{\mathcal{L}} : \mathsf{patch}(\mathsf{bpt}(\mathcal{L})) \to \mathsf{pt}(\pi_3((\mathcal{L}))$$
$$(f^+, f^-) \mapsto \pi_3((f^+, f^-)).$$

The assignment is injective, as if f_1^+, f_2^+ differ on one element $a^+ \in L^+$ then we have that $f_1, f_2 : L \to 2$ differ on the element $e_L^+(a^+)$, by the definition of these two frame maps. The assignment is also surjective: whenever we have a frame map $f : L \to 2$, this is $\pi_3((f^+, f^-))$ for map $f^+ : L^+ \to 2$ defined as $a^+ \mapsto f(e_L^+(a^+))$, and the map f^- analogously defined. This holds because the following and the analogous diagram for the negative component commute, by definition of f^+ and f^-.

$$\begin{array}{ccc} L^+ & \xrightarrow{f^+} & 2 \\ \downarrow{\scriptstyle e_L^+} & & \downarrow{\scriptstyle \mathsf{id}_2} \\ L & \xrightarrow{f} & 2 \end{array}$$

To check that $\alpha_{\mathcal{L}}$ is a homeomorphism, we show that the forward images under this map of the subbasic opens of $\mathsf{patch}(\mathsf{bpt}(\mathcal{L}))$ coincide with the subbasic opens of $\mathsf{pt}(\pi_3(\mathcal{L}))$. There are two kinds of subbasic opens in $\mathsf{pt}(\pi_3(\mathcal{L}))$; those of the form

$$\varphi_{\pi_3(\mathcal{L})}(e_L^+(a^+)) = \{f \in \mathsf{pt}(\pi_3(\mathcal{L})) : f(e_L^+(a^+)) = 1\}$$

for some $a \in L^+$ and those of an analogous form for some element of L^-. On the other hand, the subbasic opens of $\mathsf{patch}(\mathsf{bpt}(\mathcal{L}))$ are those subsets of $\mathsf{bpt}(\mathcal{L})$ of the form

$$\varphi_{\mathcal{L}}^+(a^+) = \{(f^+, f^-) \in \mathsf{bpt}(\mathcal{L}) : f^+(a^+) = 1\}$$

for some $a^+ \in L^+$, together with those of an analogous form for some $a^- \in L^-$. By definition of the map $\alpha_{\mathcal{L}}$, we have that $(f^+, f^-) \in \mathsf{bpt}(\mathcal{L})$ is such that $f^+(a^+) = 1$ if and only if $\alpha_{\mathcal{L}}((f^+, f^-)) = f$ is such that $f(e_L^+(a^+)) = 1$. Additionally, $\alpha_{\mathcal{L}}$ is a surjection and so all elements of a basic open of the form $\varphi_{\pi_3(\mathcal{L})}(e_L^+(a^+))$ are of the form $\alpha_{\mathcal{L}}((f^+, f^-))$ for some $(f^+, f^-) \in \mathsf{bpt}(\mathcal{L})$. This means that the forward images under $\alpha_{\mathcal{L}}$ of subbasic opens of the first kind of the first space are subbasic opens of the first kind in the second space, and similarly for subbasic opens of the second kind. Let us now show that $\alpha : \mathsf{patch} \circ \mathsf{bpt} \to \mathsf{pt} \circ \pi_3$ is a natural transformation. Suppose that we have a map $f : \mathcal{L} \to \mathcal{M}$ of biframes. The commutativity square is as follows.

$$\begin{array}{ccc} \mathsf{patch}(\mathsf{bpt}(\mathcal{L})) & \xrightarrow{\alpha_{\mathcal{L}}} & \mathsf{pt}(\pi_3(\mathcal{L})) \\ \uparrow{\scriptstyle \mathsf{patch}(\mathsf{bpt}(f))} & & \uparrow{\scriptstyle \mathsf{pt}(\pi_3(f))} \\ \mathsf{patch}(\mathsf{bpt}(\mathcal{M})) & \xrightarrow{\alpha_{\mathcal{M}}} & \mathsf{pt}(\pi_3(\mathcal{M})) \end{array}$$

Commutativity of this diagram in **Top** amounts to commutativity of the following in **Set**.

We have used the definition of several of the functors above.

$$\begin{array}{ccc} \mathsf{bpt}(\mathcal{L}) & \xrightarrow{\alpha_{\mathcal{L}}} & \mathsf{pt}(\pi_3(\mathcal{L})) \\ {\scriptstyle (-\circ f^+, -\circ f^-)}\uparrow & & \uparrow{\scriptstyle -\circ \pi_3(f)} \\ \mathsf{bpt}(\mathcal{M}) & \xrightarrow{\alpha_{\mathcal{M}}} & \mathsf{pt}(\pi_3(\mathcal{M})) \end{array}$$

Suppose, now, that $(g^+, g^-) \in \mathsf{bpt}(\mathcal{M})$ and let $g : M \to 2$ be $\pi_3((g^+, g^-))$. We have that $\alpha_{\mathcal{M}}((g^+, g^-)) \circ f = g \circ f$. On the other hand, we also have that $\alpha_{\mathcal{L}}(g^+ \circ f^+, g^- \circ f^-) = g \circ f$, too. Then, the square commutes, as desired. □

Lemma 0.0.19. *The following square commutes.*

$$\begin{array}{ccc} \mathbf{BiFrm} & \xleftarrow[\mathsf{b}\Omega]{} & \mathbf{BiTop} \\ {\scriptstyle \pi_3}\downarrow & & \downarrow{\scriptstyle \mathsf{patch}} \\ \mathbf{Frm} & \xleftarrow[\Omega]{} & \mathbf{Top} \end{array}$$

Proof. This follows from the definition of the biframe of opens of a bispace. □

Now, let us check that the functor π_3 is compatible with the units of the two adjunctions, and that patch is compatible with the counits.

We are now ready to proceed.

Lemma 0.0.20. *We have a natural isomorphism $\mathbb{1}_\Omega \cdot \alpha : \Pi \circ \mathsf{b}\Omega \circ \mathsf{bpt} \to \Omega \circ \mathsf{pt} \circ \Pi$ such that for every biframe $\mathcal{L}$ the following triangle commutes in* **Frm**.

$$\begin{array}{ccc} & & \pi_3(\mathsf{b}\Omega(\mathsf{bpt}(\mathcal{L}))) \\ & {\scriptstyle \pi_3(\varphi_{\mathcal{L}})}\nearrow & \uparrow{\scriptstyle \mathbb{1}_\Omega \cdot \alpha_{\mathcal{L}}} \\ \pi_3(\mathcal{L}) & \xrightarrow{\varphi_{\pi_3(\mathcal{L})}} & \Omega(\mathsf{pt}(\pi_3(\mathcal{L}))) \end{array}$$

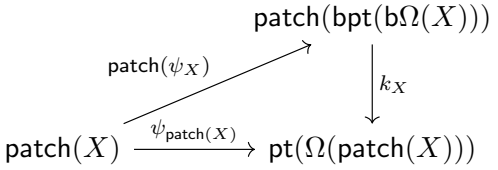
Proof. By Lemma 0.0.19, we have $\pi_3 \circ \mathsf{b}\Omega \circ \mathsf{bpt} = \Omega \circ \mathsf{patch} \circ \mathsf{bpt}$. By Lemma 0.0.17, we have a natural isomorphism $i := \mathbb{1}_\Omega \cdot \alpha : \Omega \circ \mathsf{patch} \circ \mathsf{bpt} \to \Omega \circ \mathsf{pt} \circ \pi_3$, and by the equality established before in this proof, we have a natural isomorphism $i : \pi_3 \circ \mathsf{b}\Omega \circ \mathsf{bpt} \to \Omega \circ \mathsf{pt} \circ \pi_3$. Let us check commutativity of the triangle in the statement. In this case, by definition of Godement product, we have that $i_{\mathcal{L}} = \Omega(\alpha_{\mathcal{L}}) = \alpha_{\mathcal{L}}^{-1}$ for every biframe $\mathcal{L}$. For a positive

generator $e_L^+(a^+) \in \pi_3(\mathcal{L})$ we have the following.

$$\begin{aligned}
&\alpha_{\mathcal{L}}^{-1}(\varphi_{\pi_3(\mathcal{L})}(e_L^+(a^+))) = \\
&= \alpha_{\mathcal{L}}^{-1}(\{f \in \mathsf{pt}(\pi_3(\mathcal{L})) : f(e_L^+(a^+)) = 1\}) = \\
&= \{(f^+, f^-) \in \mathsf{bpt}(\mathcal{L}) : f^+(a^+) = 1\} = \\
&= \{(f^+, f^-) \in \mathsf{bpt}(\mathcal{L}) : f(e_L^+(a^+)) = 1\} = \\
&= \pi_3(\varphi_{\mathcal{L}}(e_L^+(a^+)).
\end{aligned}$$

So, indeed, the triangle commutes, as desired. □

Lemma 0.0.21. *There is a natural isomorphism* $k : \mathsf{patch} \circ \mathsf{bpt} \circ \mathsf{b}\Omega \to \mathsf{pt} \circ \Omega \circ \mathsf{patch}$ *such that the following triangle commutes for every bispace* X*.*



Proof. By Lemma 0.0.19 we have that $\mathsf{pt} \circ \Omega \circ \mathsf{patch} = \mathsf{pt} \circ \pi_3 \circ \mathsf{b}\Omega$. By Lemma 0.0.17, we may compose the two natural transformations α and $\mathbb{1}_{\mathsf{b}\Omega}$ to obtain a natural isomorphism $k := \alpha \cdot \mathbb{1}_{\mathsf{b}\Omega} : \mathsf{patch} \circ \mathsf{bpt} \circ \mathsf{b}\Omega \to \mathsf{pt} \circ \pi_3 \circ \mathsf{b}\Omega$. By the equality established above, this is a natural isomorphism $\mathsf{patch} \circ \mathsf{bpt} \circ \mathsf{b}\Omega \to \mathsf{pt} \circ \Omega \circ \mathsf{patch}$. For every $x \in X$, we have the following.

$$\begin{aligned}
&\alpha_{\mathsf{b}\Omega(X)}(\mathsf{patch}(\psi_X)) = \\
&\alpha_{\mathsf{b}\Omega(X)}((N_x^+, N_x^-)) = \\
&= \pi_3((N_x^+, N_x^-)) = \\
&= \psi_{\mathsf{patch}(X)},
\end{aligned}$$

where for the last equality we have used the fact that the frame map $\pi_3((N_x^+, N_x^-)) : \Omega(X) \to 2$ is the unique frame map acting on positive generators as N_x^+ and on negative generators as n_x^-, and is therefore the classical topological neighborhood map $n_x : \Omega(X) \to 2$. Then, the triangle commutes, as desired. □

Proposition 0.0.22. *The pairing* (π_3, patch) *constitutes a map, up to isomorphism, between the adjunctions* $(\mathsf{b}\Omega, \mathsf{bpt}, \varphi, \psi)$ *and* $(\Omega, \mathsf{pt}, \varphi, \psi)$*.*

Proof. This follows by combining Lemmas 0.0.18, 0.0.19, 0.0.20, and 0.0.21. □

Chapter 1

D-frames

Introduction

In this chapter we introduce the basics of the theory of d-frames. Most of the results that we state and prove here are known, and we will often refer the reader to [29] and [22]. Although there are some new results and some new proofs in this chapter, we regard this as a background chapter, where we provide setting and technical tools for this thesis.

In Section 1.1 we start with an exposition of the dual adjunction between bispaces and d-frames. We then will analyze the d-frame analogues of the notions of sobriety and spatiality, with a special attention to how these compare with the frame theoretical notions. In particular, we will see that the notion of d-sobriety does not collapse to that of sobriety of its patch topology. We will also see that, just as in the frame case, a space can fail to be d-sober because of having too many points, too few, or both. We then introduce the d-frame analogue of the prime spectrum of a frame, and show in detail that the prime spectrum functor is isomorphic to the d-spectrum one.

In Section 1.2 we explore the d-frame duality in greater depth. First we will look at several concrete properties of d-frames: we introduce, following [22], several separation axioms for d-frames, which we will relate to separation axioms for bispaces. We then show which of these properties imply each other. In the second part of this section, we will instead look at the d-frame duality more abstractly, and we put it into context by looking at how it relates to the biframe and the frame duality. In particular, we will define a patch functor $\Pi : \mathbf{dFrm} \to \mathbf{Frm}$, and we will look at how this and the patch functor $\mathsf{patch} : \mathbf{BiTop} \to \mathbf{Top}$ connect the dual adjunction for d-frames with that of frames.

We devote all of Section 1.3 to quotients of d-frames. We will not prove the results, instead,

we will only state the results and point the reader to [25] for most of the proofs. Our main focus will be giving an intuitive explanation of what these results mean.

1.1 D-frames and their duality

As we have already mentioned in the introduction, the d-frame of a bitopological space captures information on covering and disjointness relations between the two topologies. D-frames are quadruples (L^+, L^-, c_L, t_L), where L^+ and L^- are frames, and where $c_L, t_L \subseteq L^+ \times L^-$ satisfy certain axioms. We think of L^+ and L^- as being two abstract topologies, c_L as being an abstract disjointness relation, and t_L as being an abstract covering relation. For two sets L^+ and L^-, we say that a family $(x_i^+, x_i^-) \subseteq L^+ \times L^-$ is *homogeneous* if it is either of the form (x_i^+, x^-) or (x^+, x_i^-). If L^+ and L^- are posets, we say that a join (resp. a meet) in $L^+ \times L^-$ is homogeneous when it is a join (resp. a meet) of a homogeneous family. We are now ready to give the formal definition of a d-frame.

A *d-frame* is a quadruple (L^+, L^-, c_L, t_L) such that L^+ and L^- are frames, and such that c_L and t_L are subsets of $L^+ \times L^-$ satisfying the following axioms.

1. The subset c_L is a downset of $L^+ \times L^-$.
2. The subset c_L is closed under arbitrary homogeneous joins.
3. The subset t_L is an upset of $L^+ \times L^-$.
4. The subset t_L is closed under finite homogeneous meets.
5. Whenever we have $(a^+, a_1^-) \in c_L$ and $(a^+, a_2^-) \in t_L$, we must have $a_1^- \leq a_2^-$. Similar inequalities hold in the frame L^+.

We remark that the condition on the con subsets are weaker than those in the definition found in [22]. There, con subsets are required to be Scott-closed, too, and this does not follow from closure under arbitrary homogeneous joins.

Axiom (5) is usually called *balance*, and it is the pointfree version of the spatial fact that, whenever we have subsets $A, B_1, B_2 \subseteq X$ of some set X, if we have both $A \cap B_1 = \emptyset$ and $A \cup B_2 = X$, we must have that $B_1 \subseteq B_2$.

In general, when we refer to a d-frame called $\mathcal{L}$, we implicitly say that its four components are called (L^+, L^-, c_L, t_L). We refer to c_L as the con component - or the consistency component - of the d-frame $\mathcal{L}$ and to t_L as the tot component - or totality component - of $\mathcal{L}$. A morphism $f = (f^+, f^-) : \mathcal{L} \to \mathcal{M}$ in the category of d-frames consists of a pair of frame maps $f^+ : L^+ \to$

M^+ and $f^- : L^- \to M^-$ such that they respect the con and the tot subsets, in the sense that $(a^+, a^-) \in c_L$ implies that $(f^+(a^+), f^-(a^-)) \in c_M$, and similarly for the tot component. We call **dFrm** the category of d-frames with d-frame morphisms.

Proposition 1.1.1. *For any two frames L^+ and L^- with more than one element, consider the sets*

$$c_m := \{(x^+, 0^-) : x^+ \in L^+\} \cup \{(0^+, x^-) : x^- \in L^-\},$$
$$t_m := \{(x^+, 0^-) : x^+ \in L^+\} \cup \{(0^+, x^-) : x^- \in L^-\}.$$

These are the smallest consistency and totality subsets on $L^+ \times L^-$, respectively. Additionally, they are balanced.

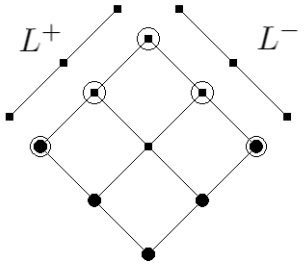
Proof. We first show that, whenever $c \subseteq L^+ \times L^-$ is a consistency subset, we must have $c_m \subseteq c$, and similarly for the totality component. Observe that the family $\{(x^+, x^-) : x^- \in \emptyset\}$ is homogeneous for each $x^+ \in L^+$, and since it is the empty set it is a subset of c. Then its join $(x^+, 0^-)$ must be in c, too. By a similar argument we must also have $(0^+, x^-) \in c$ for every $x^- \in L^-$. The argument for showing that $t_m \subseteq t$ is similar, and uses the fact that empty meets are top elements. By its definition, the set c_m is clearly a downset. By its definition, the set c_m is closed under all homogeneous joins. Similarly we check that t_m is a totality subset of $L^+ \times L^-$. Let us now check that these are balanced. Suppose that we have $(x^+, x_1^-) \in c_m$ and $(x^+, x_2^-) \in t_m$. There are two possibilities: either $x^+ = 0^+$ or $x_1^- = 0^-$. If $x^+ = 0^+$, by assumption we must have $x^+ \neq 1^+$, and so $x_2^- = 1^-$, and so $x_1^- \leq x_2^-$. If, instead, we have $x_1^- = 0^-$, we deduce immediately that $x_1^- \leq x_2^-$. □

When we speak of the *minimal* consistency and totality subsets on a binary product of frames, we will mean the ones defined in the proposition above. We denote these as c_m and t_m. For each d-frame $\mathcal{L}$, we call $\mathcal{L}_m$ the d-frame (L^+, L^-, c_m, t_m).

Let us explore a bit more the order relation between the various con subsets of a binary product of frames. For any two frames L^+ and L^-, we will call $\mathsf{con}(L^+ \times L^-)$ the ordered collection of all possible con subsets, and we call $\mathsf{tot}(L^+ \times L^-)$ the analogous poset for tot subsets. We then know that both $\mathsf{con}(L^+ \times L^-)$ and $\mathsf{tot}(L^+ \times L^-)$ have a bottom element, namely c_m and t_m, respectively. The result below tells us that, in addition, both these posets are complete lattices. In Chapter 2 we will show that, in fact, these are both frames.

Proposition 1.1.2. *For any two frames L^+ and L^-, the collection $\mathsf{con}(L^+ \times L^-)$ of consistency subsets of $L^+ \times L^-$ is closed under arbitrary intersections, and so is the collection $\mathsf{tot}(L^+ \times L^-)$ of totality subsets.*

Let us now spend some words on our graphical rendition of d-frames. When we depict a d-frame $\mathcal{L}$ we will show the L^+ component, the L^- component, and the product $L^+ \times L^-$. On the product, the nodes representing pairs in the con component c_L will be marked with black circles, and the nodes corresponding to pairs in the tot components will be marked with white circles. Thus, for example, the d-frame $(3, 3, c_m, t_m)$ will be represented as follows.



1.1.1 The dual adjunction

Now, let us describe the dual adjunction between bispaces and d-frames. All the results that we present here are proven in [22]. For a bispace X, define the d-frame $\mathsf{d}\Omega(X)$ as the quadruple

$$(\Omega^+(X), \Omega^-(X), \{(U^+, U^-) : U^+ \cap U^- = \emptyset\}, \{(U^+, U^-) : U^+ \cup U^- = X\}).$$

The pairs (U^+, U^-) are understood to range over $\Omega^+(X) \times \Omega^-(X)$. With routine set-theoretical calculations one can check that the quadruple $\mathsf{d}\Omega(X)$ is a d-frame for every bispace X. We extend this to a functor by defining the $\mathsf{d}\Omega$-image of a bicontinuous map $f : X \to Y$ as the pair of preimage maps $(f^{-1}, f^{-1}) : \mathsf{d}\Omega(Y) \to \mathsf{d}\Omega(X)$. Preimage maps preserve the frame operations, and in particular they also preserve the property of two opens being disjoint and covering. Therefore, indeed $\mathsf{d}\Omega(f)$ is a d-frame map for every bicontinuous map f. Preimages respect composition of maps as well as identities, and so the assignment is functorial.

Let us now define the d-frame analogue of a spectrum. As in the frame case, there are several ways of describing the spectrum of a d-frame. First, we will introduce the definition of the spectrum of a d-frame in terms of morphisms to a dualizing object. Let us call $\mathbf{2}$ the d-frame $(2, 2, c_m, t_m)$. For clarity, we will denote the elements of the first component of this as $0^+ < 1^+$, and those of the negative frame component as $0^- < 1^-$.

For any d-frame $\mathcal{L}$, let us define as $\mathsf{dpt}(\mathcal{L})$ the collection of d-frame morphisms $f : \mathcal{L} \to \mathbf{2}$. In terms of pairs of frame morphisms, the set $\mathsf{dpt}(\mathcal{L})$ can be described as the set of all of those pairs of frame maps (f^+, f^-), with $f^+ : L^+ \to 2$ and $f^- : L^- \to 2$, such that they respect the

con and the tot components. Now, let us equip this set with two topologies. Define a map $\phi^+_{\mathcal{L}} : L^+ \to \mathcal{P}(\mathsf{dpt}(\mathcal{L}))$ as

$$\phi^+_{\mathcal{L}}(a^+) = \{(f^+, f^-) \in \mathsf{dpt}(\mathcal{L}) : f^+(a^+) = 1^+\}.$$

Define similarly the map $\phi^-_{\mathcal{L}} : L^- \to \mathcal{P}(\mathsf{dpt}(\mathcal{L}))$. For a d-frame $\mathcal{L}$, define its *d-spectrum*, or simply its *spectrum*, as the bispace such that its underlying set of points is $\mathsf{dpt}(\mathcal{L})$, its positive opens are the sets of the form $\phi^+_{\mathcal{L}}(a^+)$ for some $a^+ \in L^+$, and its negative opens are the sets of the form $\phi^-_{\mathcal{L}}(a^-)$ for some $a^- \in L^-$. From now on, when there is no ambiguity, we will identify notationally a d-spectrum with its underlying set of points $\mathsf{dpt}(\mathcal{L})$, just like with general bispaces.

Remark 1.1.3. Just as in the frame case, we can give an argument by duality to further convince ourselves that d-frame morphisms $\mathcal{L} \to \mathbf{2}$ are the right notion of point. Points of a bispace X are the bicontinuous maps $\{*\} \to X$, where the singleton $\{*\}$ is equipped with the only possible bitopology on this set. The d-spectrum of $\mathbf{2}$ is homeomorphic to the bispace $\{*\}$ since there is only one possible d-frame morphism $\mathbf{2} \to \mathbf{2}$, namely, the identity. If we dualize a morphism $\mathcal{L} \to \mathbf{2}$, then, we obtain a bicontinuous map $\{*\} \to \mathsf{dpt}(\mathcal{L})$.

Let us extend dpt to a functor, by defining the dpt-image of a d-frame morphism $f : \mathcal{L} \to \mathcal{M}$ as the precomposition map $(- \circ f^+, - \circ f^-) : \mathsf{dpt}(\mathcal{M}) \to \mathsf{dpt}(\mathcal{L})$. The argument to show that precomposition is a bicontinuous mapping is analogous to the argument to show that $\mathsf{pt}(f)$ is continuous for any frame map f. The fact that precomposition respects identities and compositions is known from frame theory already, too. Before proceeding to define the unit and counit of the desired adjunction, let us pause to look more closely at how d-spectra are constructed. The first thing to notice is that for any d-frame $\mathcal{L}$ the space $\mathsf{dpt}(\mathcal{L})$ is a bisubspace of $\mathsf{dpt}(\mathcal{L}_m)$. The following lemma can be checked by looking at the definitions of the topologies on $\mathsf{dpt}(\mathcal{L})$ and $\mathsf{dpt}(\mathcal{L}_m)$.

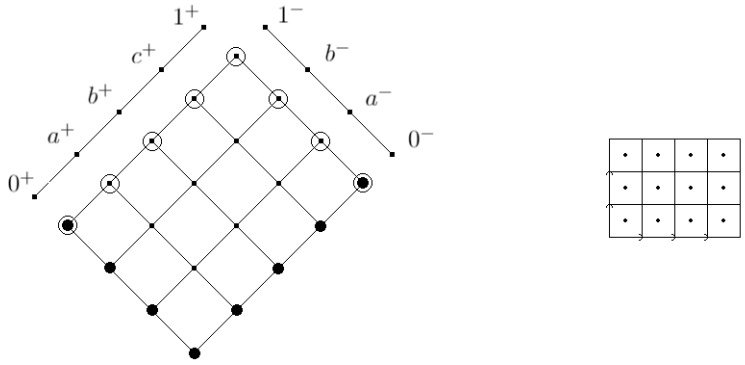
Lemma 1.1.4. *For every d-frame $\mathcal{L}$, the bispace $\mathsf{dpt}(\mathcal{L})$ is a bisubspace of $\mathsf{dpt}(\mathcal{L}_m)$. It is, in particular, the bisubspace of $\mathsf{dpt}(\mathcal{L}_m)$ obtained by taking away*

- *all points which are in $\phi^+_{\mathcal{L}_m}(a^+) \cap \phi^-_{\mathcal{L}_m}(a^-)$ for some $(a^+, a^-) \in c_L$,*
- *all points which are in $(\phi^+_{\mathcal{L}_m}(a^+) \cup \phi^-_{\mathcal{L}_m}(a^-))^c$ for some $(a^+, a^-) \in t_L$.*

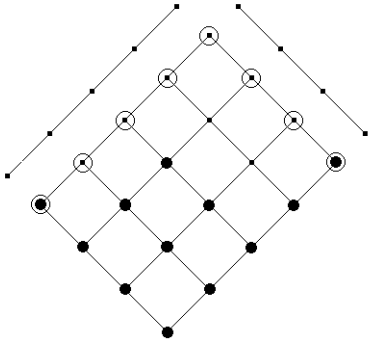
This lemma starts to guide our intuition towards the idea that adding elements to the con and the tot components of a d-frame means taking away points from its d-spectrum. Let us look at an example to see this happening concretely.

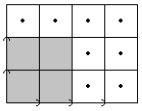
Example 1.1.5. We choose L^+ to be the five element chain, whose elements we denote as $0^+ < a^+ < b^+ < c^+ < 1^+$ and L^- to be the four element chain, which we denote as $0^- < a^- < b^- < 1^-$. Let us look at various possible d-frames with L^+ and L^- as their frame components.

- First, let us equip our two frames with the minimal con and tot subsets. By definition, the set of points of its spectrum is a subset of $\mathsf{pt}(L^+) \times \mathsf{pt}(L^-)$. We have depicted it as a rectangle, with the space $\mathsf{pt}(L^+)$ as its horizontal side and $\mathsf{pt}(L^-)$ as its vertical one. The positive opens are depicted as left sections of the rectangle, and the negative ones as lower sections. Given that con and tot are minimal, all of the points of $\mathsf{pt}(L^+) \times \mathsf{pt}(L^-)$ qualify as points of the d-spectrum $\mathsf{dpt}(\mathcal{L}_m)$. We can then think of $\mathsf{dpt}(\mathcal{L}_m)$ as a grid of points.



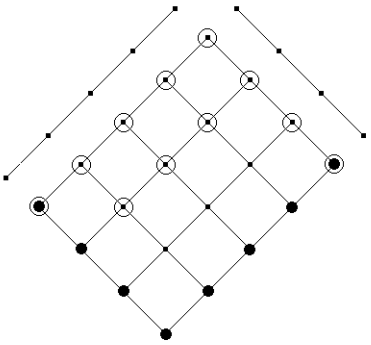
- Now, let us add the pair (b^+, b^-), and all the pairs below it, to the con relation. Lemma 1.1.4 is telling us that the d-spectrum of this d-frame can be seen as a the bisubspace of $\mathsf{dpt}(\mathcal{L}_m)$ that we obtain by taking away all of the points in the intersection $\phi^+_{\mathcal{L}_m}(b^+) \cap \phi^-_{\mathcal{L}_m}(b^-)$.

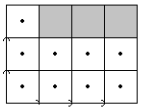




We notice that adding elements to con amounts to taking away lower left sections of our space $\mathsf{dpt}(\mathcal{L}_m)$. This can be made into a precise statement by appealing to the specialization order of the two topologies. However, we limit ourselves to pointing out this visual intuition.

- Let us look at what happens when we add elements to the tot components. Say that we add to this the pair (a^+, b^-), and all those above it. By Lemma 1.1.4, the d-spectrum of this d-frame amounts to taking away from $\mathsf{dpt}(\mathcal{L}_m)$ all of the points which are not in $\phi^+_{\mathcal{L}_m}(a^+) \cup \phi^-_{\mathcal{L}_m}(b^-)$.





Adding elements to tot, in general, reflects on its d-spectrum as taking away upper right sections of our rectangle.

Let us now conclude the definition of our adjunction. Just as a topological space X has associated with each point x a neighborhood map $N_x : \Omega(X) \to 2$ defined as $N_x(U) = 1$ if and only if $x \in U$, a bispace has a pair of neighborhood maps for each point. For a bispace X, and a point x, define the map $N_x^+ : \Omega^+(X) \to 2$ as $N_x^+(U^+) = 1$ if and only if $x \in U^+$, and define similarly the negative neighborhood map $N_x^- : \Omega^-(X) \to 2$. We are now ready to define the unit of adjunction. For any bispace X and any point $x \in X$, define

$$\psi_X : X \to \mathsf{dpt}(\mathsf{d}\Omega(X))$$
$$x \mapsto (N_x^+, N_x^-).$$

The argument that this is bicontinuous is analogous to the classical argument that sobrification of a space is a continuous map. And just as in the frame case, the map above need not be injective or surjective. More on this in the next subsection, where we will talk in greater depth about the d-frame versions of sobriety and spatiality. Let us now look at the counit of the adjunction. For any d-frame $\mathcal{L}$, define the map

$$(\phi_{\mathcal{L}}^+, \phi_{\mathcal{L}}^-) : \mathcal{L} \to \mathsf{d}\Omega(\mathsf{dpt}(\mathcal{L}))$$
$$a^+ \mapsto \phi_{\mathcal{L}}^+(a^+) = \{(f^+, f^-) \in \mathsf{dpt}(\mathcal{L}) : f^+(a^+) = 1^+\}.$$
$$a^- \mapsto \phi_{\mathcal{L}}^-(a^-) = \{(f^+, f^-) \in \mathsf{dpt}(\mathcal{L}) : f^-(a^-) = 1^-\}$$

That this is a pair of frame maps is already known from classical frame theory. To see that they respect the con and the tot component, notice that by definition of d-point, whenever we have $(a^+, a^-) \in c_L$ we cannot have that a d-point (f^+, f^-) maps both components to 1, or we would contradict that it is a d-frame map $(f^+, f^-) : \mathcal{L} \to \mathbf{2}$. So no d-point can be in $\varphi_L^+(a^+) \cap \varphi_L^-(a^-)$ Similarly, for $(a^+, a^-) \in t_L$, whenever we have a d-point (f^+, f^-) we cannot have that it maps both elements to 0, and therefore we must have $(f^+, f^-) \in \varphi_L^+(a^+) \cup \varphi_L^-(a^-)$.

Now, let us show that we can obtain a natural transformation $1_{\mathbf{BiTop}} \to \mathsf{dpt} \circ \mathsf{d}\Omega$. Here $1_{\mathbf{BiTop}}$ is the identity functor relative to the category $\mathbf{BiTop}$.

Lemma 1.1.6. *The assignment* $\mathsf{Obj}(\mathbf{Bitop}) \to \mathsf{Mor}(\mathbf{BiTop})$ *defined as for each bispace* X *as* $X \mapsto \psi_X$ *is a natural transformation* $\psi : 1_{\mathbf{BiTop}} \to \mathsf{dpt} \circ \mathsf{d}\Omega$.

Proof. Suppose that $f : X \to Y$ is a bicontinuous map between bispaces. Let us check that

the following naturality square commutes.

$$\begin{array}{ccc} X & \xrightarrow{\psi_X} & \mathsf{dpt}(\mathsf{d}\Omega(X)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathsf{dpt}(\mathsf{d}\Omega(f))} \\ Y & \xrightarrow{\psi_Y} & \mathsf{dpt}(\mathsf{d}\Omega(Y)) \end{array}$$

First, let us unravel the definitions of the functors dpt and $\mathsf{d}\Omega$ on morphisms to obtain $\mathsf{dpt}(\mathsf{d}\Omega(f)) = (- \circ f^{-1}, - \circ f^{-1})$. Now, let us use this and unravel the upper right half of the square. We have the following chain of equalities for each $x \in X$.

$$\mathsf{dpt}(\mathsf{d}\Omega(f))(\psi_X(x)) = \mathsf{dpt}(\mathsf{d}\Omega(f))(N_x^+, N_x^-) = (N_x^+ \circ f^{-1}, N_x^- \circ f^{-1}).$$

On the other hand, unraveling the bottom left half of the square yields $\psi_Y(f(x)) = (N^+_{f(x)}, N^-_{f(x)})$. Commutativity of the square, then, amounts to this being the same as the final line of the derivation. The fact that they are equal is already known from classical frame theory. □

Let us see that the same holds for the morphisms of the form $\phi_{\mathcal{L}}$, and that we can paste them together to obtain a natural transformation $\phi : \mathsf{d}\Omega \circ \mathsf{dpt} \to 1_{\mathbf{dFrm}^{op}}$. Here $1_{\mathbf{dFrm}^{op}}$ is the identity functor relative to the category $\mathbf{dFrm}^{op}$.

Lemma 1.1.7. *The assignment* $\mathsf{Obj}(\mathbf{dFrm}^{op}) \to \mathsf{Mor}(\mathbf{dFrm}^{op})$ *defined for each d-frame* $\mathcal{L}$ *as* $\mathcal{L} \mapsto (\phi^+_{\mathcal{L}}, \phi^-_{\mathcal{L}})$ *is a natural transformation* $\phi : \mathsf{d}\Omega \circ \mathsf{dpt} \to 1_{\mathbf{dFrm}^{op}}$.

Proof. Suppose that $\mathcal{L} \to \mathcal{M}$ is a d-frame morphism. To check that the naturality square in $\mathbf{dFrm}^{op}$, as usual, we reverse the morphisms and check that the following square commutes in **dFrm**.

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{(\phi^+_{\mathcal{L}}, \phi^-_{\mathcal{L}})} & \mathsf{d}\Omega(\mathsf{dpt}(\mathcal{L})) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathsf{d}\Omega(\mathsf{dpt}(f))} \\ \mathcal{M} & \xrightarrow{(\phi^+_{\mathcal{M}}, \phi^-_{\mathcal{M}})} & \mathsf{d}\Omega(\mathsf{dpt}(\mathcal{M})) \end{array}$$

Because dpt and $\mathsf{d}\Omega$ on morphisms are defined as in the classical frame theoretical case, commutativity of this square amounts to making the classical argument of naturality of φ, one for each of the frames L^+ and L^-. □

Finally, let us prove that all the pieces we have constructed come together to form an adjunction.

Theorem 1.1.8. *There is an adjunction* $\mathsf{d}\Omega : \mathbf{BiTop} \leftrightarrows \mathbf{dFrm}^{op} : \mathsf{dpt}$ *with* $\mathsf{d}\Omega \dashv \mathsf{dpt}$. *The unit of adjunction in* **BiTop** *is evaluated at a bispace* X *as the bicontinuous neighborhood map*

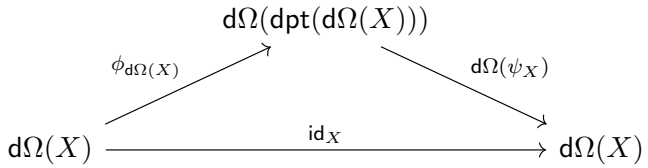
$$\begin{aligned}\psi_X : X &\to \mathsf{dpt}(\mathsf{d}\Omega(X))\\ x &\mapsto (N_x^+, N_x^-).\end{aligned}$$

We describe the counit of adjunction in $\mathbf{dFrm}^{op}$ *in terms of maps in* **dFrm**. *The counit is evaluated at a d-frame* $\mathcal{L}$ *as the d-frame map*

$$\begin{aligned}&(\phi_{\mathcal{L}}^+, \phi_{\mathcal{L}}^-) : \mathcal{L} \to \mathsf{d}\Omega(\mathsf{dpt}(\mathcal{L}))\\ &a^+ \mapsto \phi^+(a^+) = \{(f^+, f^-) \in \mathsf{dpt}(\mathcal{L}) : f^+(a^+) = 1^+\},\\ &a^- \mapsto \phi^-(a^-) = \{(f^+, f^-) \in \mathsf{dpt}(\mathcal{L}) : f^-(a^-) = 1^-\}.\end{aligned}$$

Proof. We show that this pair of functors is an adjunction by showing that the two unit-counit triangles commute.

- Suppose that X is a bitopological space. To check the commutativity of the unit-counit triangle in $\mathbf{dFrm}^{op}$, we reverse the arrows and check that the following triangle in **dFrm** commutes.

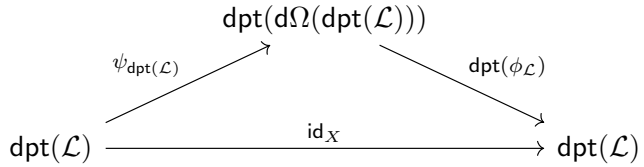


  For every positive open $U^+ \subseteq X$, we have the following chain of equalities.

$$\begin{aligned}&\mathsf{d}\Omega(\psi_X)(\phi^+_{\mathsf{d}\Omega(X)}(U^+)) =\\ &= \mathsf{d}\Omega(\psi_X)(\{f \in \mathsf{dpt}(\mathsf{d}\Omega(X)) : f^+(U^+) = 1^+\}) =\\ &= \psi_X^{-1}(\{f \in \mathsf{dpt}(\mathsf{d}\Omega(X)) : f^+(U^+) = 1^+\}) =\\ &= \{x \in X : N_x^+(U^+) = 1^+\} =\\ &= U^+.\end{aligned}$$

  We have an analogous chain of equalities for negative opens of X. Then, the triangle commutes.

- Suppose that $\mathcal{L}$ is a d-frame. We need to show that the following triangle in **BiTop** commutes.



For every $f \in \mathsf{dpt}(\mathcal{L})$, we have the following chain of equalities.

$$\begin{aligned}&\mathsf{dpt}(\phi_{\mathcal{L}})(\psi_{\mathsf{dpt}(\mathcal{L})})(f) = \\ &= \mathsf{dpt}(\phi_{\mathcal{L}})(N_f^+, N_f^-) = \\ &= (N_f^+ \circ \phi_{\mathcal{L}}^+, N_f^- \circ \phi_{\mathcal{L}}^-).\end{aligned}$$

Let us check that this is the same as $f = (f^+, f^-) : \mathcal{L} \to \mathbf{2}$. It suffices to check that the positive component of these two frame maps is the same since the argument for the negative one is similar. For every $a^+ \in L^+$, we have the following chain of equivalences.

$$\begin{aligned}&N_f^+(\phi_{\mathcal{L}}^+(a^+)) = 1^+ \text{ iff} \\ &f \in \phi_{\mathcal{L}}^+(a^+) \text{ iff} \\ &f^+(a^+) = 1^+.\end{aligned}$$

Then, indeed the map $f \mapsto (N_f^+ \circ \phi_{\mathcal{L}}^+, N_f^- \circ \phi_{\mathcal{L}}^-)$ is the identity on $\mathsf{dpt}(\mathcal{L})$. □

1.1.2 Spatial d-frames and d-sober bispaces

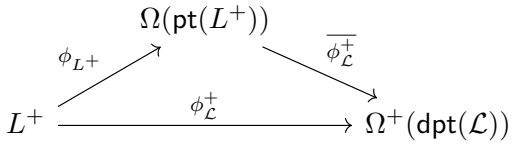
We will now see how the adjunction described in the previous subsection restricts to an equivalence of categories. Once again, all the results that we state and prove here are shown in Chapter 2 of [22]. We say that a d-frame $\mathcal{L}$ is *d-spatial*, or simply *spatial*, if the following conditions hold.

- L^+ and L^- are both spatial frames.
- If for all d-points $(f^+, f^-) : \mathcal{L} \to \mathbf{2}$ we have that either $f^+(a^+) = 0^+$ or $f^-(a^-) = 0^-$, then we must have $(a^+, a^-) \in c_L$.
- If for all d-points $(f^+, f^-) : \mathcal{L} \to \mathbf{2}$ we have that either $f^+(a^+) = 1^+$ or $f^-(a^-) = 1^-$, then we must have $(a^+, a^-) \in t_L$.

The last two conditions can be rephrased as follows.

- We have $(a^+, a^-) \in c_L$ if and only if there is no d-point $f : \mathcal{L} \to \mathbf{2}$ with $(f^+(a^+), f^-(a^-)) = (1^+, 1^-)$.
- We have $(a^+, a^-) \in t_L$ if and only if there is no d-point $f : \mathcal{L} \to \mathbf{2}$ with $(f^+(a^+), f^-(a^-)) = (0^+, 0^-)$.

Notice that for a general d-frame we already have the direction "$(a^+, a^-) \in c_L$ implies that there is no d-point $f : \mathcal{L} \to \mathbf{2}$ with $(f^+(a^+), f^-(a^-)) = (1^+, 1^-)$". Similarly for the second condition. From this, we see that spatiality of a d-frame amounts on the one hand to L^+ and L^- having enough points (for L^+ and L^- to be spatial), on the other hand to the con and the tot components being large enough. The d-spatialization map of a d-frame, then, spatializes L^+ and L^-, but when the two starting frames are already spatial it could also increase the con and the tot components. Let us explore a bit more the relation between the d-spatialization of a d-frame and the spatializations of its frame components. The d-spatialization map $(\phi^+_{\mathcal{L}}, \phi^-_{\mathcal{L}}) : \mathcal{L} \twoheadrightarrow \mathsf{d}\Omega(\mathsf{dpt}(\mathcal{L}))$ consists of a pair of frame surjections, both to spatial frames. Then, we must have that each of the two components factors as follows.



The map $\overline{\phi^+_{\mathcal{L}}} : \Omega(\mathsf{pt}(L^+)) \to \Omega^+(\mathsf{dpt}(\mathcal{L}))$ must be a frame surjection, and by commutativity of the diagram we know that it acts as $\phi_{L^+}(a^+) \mapsto \phi^+_{\mathcal{L}}(a^+)$. As the next example shows, this might fail to be injective, showing that in taking the d-spectrum of a d-frame we may lose information on the frame L^+.

Example 1.1.9. Consider the extreme case where L^- is any frame with no points. Then, the d-spectrum $\mathsf{dpt}(\mathcal{L})$ will have no points either, and so we will have that $\phi^+_{\mathcal{L}}(a^+) = \emptyset$ for each $a^+ \in L^+$. The map $\phi_{L^+}(a^+) \mapsto \phi^+_{\mathcal{L}}(a^+)$ here is not injective unless we have the trivial case where $\mathsf{pt}(L^+) = \emptyset$. Intuitively, this means that if the frame L^- has very few points, taking the d-spectrum $\mathsf{dpt}(\mathcal{L})$ might cause us to lose quite a lot of information about the frame L^+, too, even if L^+ happens to be spatial. Or, to phrase this in yet another way: if we have a point $f^+ : L^+ \to 2$ it is not guaranteed that there will be a point $f^- : L^- \to 2$ with $(f^+, f^-) \in \mathsf{dpt}(\mathcal{L})$, and so the point f^+ might, so to speak, disappear when taking the d-spectrum of $\mathcal{L}$.

Let us move to characterizations of d-spatiality reminiscent of the characterizations of spatiality of a frame.

Proposition 1.1.10. *For a d-frame $\mathcal{L}$, the following are equivalent.*

- *The d-frame $\mathcal{L}$ is spatial.*
- *The d-frame $\mathcal{L}$ is isomorphic to $\mathsf{d\Omega}(X)$ for some bispace X.*
- *The two maps $\phi^+ : L^+ \to \mathcal{P}(\mathsf{pt}(\mathcal{L}))$ and $\phi^- : L^- \to \mathcal{P}(\mathsf{pt}(\mathcal{L}))$ are injective.*

For each d-frame $\mathcal{L}$, we refer to the map $(\phi^+_{\mathcal{L}}, \phi^-_{\mathcal{L}}) : \mathcal{L} \to \mathsf{d\Omega}(\mathsf{dpt}(\mathcal{L}))$ as its *d-spatialization* map. We call the d-frame $\mathsf{d\Omega}(\mathsf{dpt}(\mathcal{L}))$ the *d-spatialization* of $\mathcal{L}$. Let us now focus on the d-frame version of sobriety. We say that a bispace X is *d-sober* if ψ_X is a bihomeomorphism. In Section 4 of [29] the following is proven.

Proposition 1.1.11. *For a bispace X, the following are equivalent.*

- *The bispace X is bihomeomorphic to $\mathsf{pt}(\mathcal{L})$ for some d-frame $\mathcal{L}$.*
- *The bispace X is d-sober.*
- *The map ψ_X is a bijection.*

For every bispace X, we refer to the map $\psi_X : X \to \mathsf{dpt}(\mathsf{d\Omega}(X))$ as its *d-sobrification* map, and we call the bispace $\mathsf{dpt}(\mathsf{d\Omega}(X))$ the *d-sobrification* of the bispace X. For a bispace X, as is the case with the sobrification map of a space, it is not in general true that ψ_X is injective or surjective. A bispace may fail to be sober because it has too many points, or too few, or both. Analogously to in the monotopological case, having too many points means not being pairwise T_0, as the next lemma shows.

Lemma 1.1.12. *A bispace X is pairwise T_0 if and only if the d-sobrification map $\psi_X : X \to \mathsf{dpt}(\mathsf{d\Omega}(X))$ is an injection.*

Proof. If a bispace X is pairwise T_0, then for any two distinct points x and y there must be some positive or some negative open separating them. In other words, we must have that either $N^+_x \neq N^+_y$ or $N^-_x \neq N^-_y$. Conversely, if a bispace is not pairwise T_0, and if $x, y \in X$ are distinct points appearing precisely in the same positive and negative opens, we must have $(N^+_x, N^-_x) = (N^+_y, N^-_y)$. □

Recall that the bispaces which are "bisober" with respect to the biframe duality are exactly the ones whose patch is sober. We will now see that, unlike that of biframes, the duality of d-frames determines a bitopological notion of sobriety which does not collapse to the monotopological one. What we do have is one direction of the bi-implication.

Lemma 1.1.13. *All d-sober spaces are have a sober patch.*

Proof. If we have a d-sober bispace X, the assignment $x \mapsto (N_x^+, N_x^-)$ is injective, i.e. the space is pairwise T_0, which by definition is seen to be a property formally stronger than the patch topology being T_0. On the other hand, the assignment $x \mapsto (N_x^+, N_x^-)$ is also surjective. Suppose that we have a point $f : \Omega(X) \to 2$ of the frame of the patch topology, let us consider the restrictions of f to the two subframes of positive and negative opens, let us call them $f^+ : \Omega^+(X) \to 2$ and $f^- : \Omega^-(X) \to 2$. These two are frame maps and with routine calculations we can check that their pairing is a d-frame map $(f^+, f^-) : \mathsf{d}\Omega(X) \to \mathbf{2}$. By our assumption, this must be equal to (N_x^+, N_x^-) for some $x \in X$. Now we claim that f is N_x, the monotopological neighborhood map of this point. This holds because for every patch-open $\bigcup_i U_i^+ \cap U_i^-$ we have

$$
\begin{aligned}
& f(\bigcup_i U_i^+ \cap U_i^-) = 1 \text{ iff} \\
& f(U_i^+ \cap U_i^-) = 1 \text{ for some } i \in I, \text{ iff} \\
& f^+(U_i^+) = 1 \text{ and } f^-(U_i^-) = 1 \text{ for some } i \in I, \text{ iff} \\
& x \in U_i^+ \cap U_i^- \text{ for some } i \in I, \text{ iff} \\
& x \in \bigcup_i U_i^+ \cap U_i^-.
\end{aligned}
$$

Then, X equipped with the patch topology is also such that the map $\psi_X : X \to \mathsf{pt}(\Omega(X))$ with $x \mapsto N_x$ is surjective. Then, it is sober. □

To see that the converse of this lemma is not true, consider the following simple counterexample, which we can find in Section 4 of [29].

Example 1.1.14. Consider the following bispace. The underlying set of points has two elements, and both topologies are the Sierpinski topology on the set, with one of the two points isolated in both topologies. The topological space that we obtain by taking the patch of this bispace is the Sierpinski space, and so it is sober. Here, the d-frame of opens of this space is $(3, 3, c_m, t_m)$, and so its d-spectrum has four points, since all of the elements of $\mathsf{pt}(3) \times \mathsf{pt}(3)$ qualify as d-points, as con and tot are minimal. Then, its d-spectrum cannot be homeomorphic to our space.

Remark 1.1.15. It is possible to see the relation between d-sobriety and patch-sobriety in a more abstract and intuitive way. The patch-sober spaces are exactly those that can be completely recovered by knowing all of the order theoretical information about the patch

topology. The d-sober spaces are exactly those that can be completely recovered knowing only some of this information, involving covering and disjointness. Phrased in this way, it is clear that the first property is stronger than the second.

However, phrasing exactly what we mean by "recovering a space from some information" is not immediate. We will do this in Section 1.2.2, here we will relate the duality of d-frames and that of biframes more in detail.

For more on d-sobriety and its relation with other bitopological properties, see Section 4 of [29]. Let us denote as **dSob** the full subcategory of **BiTop** whose objects are the d-sober spaces, and let us denote as **dSpa** the full subcategory of **dFrm** whose objects are the spatial d-frames. We have the following analogue of the spatial-sober duality for spaces and frames.

Theorem 1.1.16. *The maximal equivalence of categories that the adjunction* $\mathsf{d}\Omega \dashv \mathsf{dpt}$ *restricts to is the equivalence* $\mathbf{dSob} \cong \mathbf{dSpa}^{op}$.

1.2 Concrete and abstract aspects of the d-frame duality

1.2.1 Concrete aspects: special kinds of d-frames

As with biframes, there already are several d-frame separation properties in the literature. In the biframe case, many of these pointfree properties rely on the notion of bipseudocomplement. Let us then begin this subsection by introducing the d-frame analogue of this. We begin by noticing that for every d-frame $\mathcal{L}$, as the c_L subset is closed under homogeneous arbitrary joins, we have that for each $a^+ \in L^+$ the element $(a^+, \bigvee\{a^- \in L^- : (a^+, a^-) \in c_L\})$ is still in c_L. This justifies our definition: for every element a^+, we define its *d-pseudocomplement* to be the element $\bigvee\{a^- \in L^- : (a^+, a^-) \in c_L\}$. We denote this as $\sim a^+$. For an element $x^+ \in L^+$, we call $x^- \in L^-$ the *d-complement* of x^+ if both $(x^+, x^-) \in c_L$ and $(x^+, x^-) \in t_L$.

Observation 1.2.1. *If, in a d-frame* $\mathcal{L}$*, we have that* $(x^+, x^-) \in c_L \cap t_L$*, then we must have* $x^- \leq \sim x^+$ *by definition of d-pseudocomplement, but also* $\sim x^+ \leq x^-$*, by balance. Thus, the d-complement of an element, when it exists, is always its d-pseudocomplement. Because of this, we also can deduce that d-complements are unique when they exist.*

We define similarly the d-pseudocomplement $\sim a^-$ of an element $a^- \in L^-$, as well its d-complement. We may see d-pseudocomplementation as a pair of maps $\sim^+ : L^+ \leftrightarrows L^- : \sim^-$, although for brevity we will still use the notation "$\sim$" and omit the exponent even when treating d-pseudocomplementation as a map. These two maps form an antitone Galois

connection, since for each pair $(a^+, a^-) \in L^+ \times L^-$ we have that the following three conditions are equivalent:

$$\frac{\dfrac{a^+ \leq \sim a^-}{a^- \leq \sim a^+}}{(a^+, a^-) \in c_L}$$

From the fact that this is a Galois connection, we can deduce that the operation of d-pseudocomplementations enjoys many properties similar to those of pseudocomplementation.

Lemma 1.2.2. *For a d-frame $\mathcal{L}$, the following facts hold.*

1. *The map $\sim$ turns arbitrary joins into meets.*
2. *The map $\sim\sim$ is monotone and inflationary.*
3. *We have $\sim\sim\sim = \sim$.*
4. *The map $\sim\sim$ is idempotent.*
5. *If $x \in L^+ \cup L^-$ has a d-complement, then $x = \sim\sim x$.*

Proof. Let us prove each item in turn. We will prove the statements for the maps $\sim : L^+ \to L^-$ and $\sim\sim : L^+ \to L^+$, since the proofs for the maps $\sim : L^- \to L^+$ and $\sim\sim : L^- \to L^-$ are of course analogous.

1. If we see L^+ and L^- as categories, an antitone Galois connection is the same as a dual adjunction, and so d-pseudocomplementation maps must turn colimits into limits.
2. The fact that $\sim\sim$ is monotone follows directly from antitonicity of $\sim$. To see that it is inflationary, notice that we have that $(x^+, \sim x^+) \in c_L$ for every $x^+ \in L^+$, and this is equivalent to $x^+ \leq \sim\sim x^+$ by definition of d-pseudocomplement.
3. By item (2) we know that $\sim\sim$ is inflationary, so we only need to check the inequality $\sim\sim\sim x^+ \leq \sim x^+$. Let us start from the inequality $x^+ \leq \sim\sim\sim\sim x^+$, which we know to be true because $\sim\sim$ is inflationary. We notice that this is equivalent to the desired inequality $\sim\sim\sim x^+ \leq \sim x^+$, by the defining property of Galois connections.
4. This follows from item (3).
5. Suppose that $x^+ \in L^+$ is a d-complemented element. By (2), the map $\sim\sim$ is inflationary and so we only need to check $\sim\sim x^+ \leq x^+$. Since we have that $(\sim\sim x^+, \sim x^+) \in c_L$, and that by assumption $(x^+, \sim x^+) \in t_L$, by balance we must also have $\sim\sim x^+ \leq x^+$. □

Remark 1.2.3. There are also some important places in which the analogies with the frame theoretical pseudocomplement stop. Firstly, from items (2) and (4) we deduce that double d-pseudocomplementation is always a closure operator. But, unlike double pseudocomplementation, it will not in general be a nucleus. Secondly, consider item (5) of the lemma above. Pseudocomplements satisfy a similar property: if for a frame L we have $a \in L$ with $a \vee \neg a = 1$, then we must also have $a = \neg\neg a$. Then, if all elements are complemented, the assignment $\neg\neg$ is the identity. The converse holds, too: a frame L in which $a = \neg\neg a$ for all $a \in L$ must be a Boolean algebra. The analogue of this, however, is not true for d-frames. We may have d-frames where $\sim\sim$ is the identity on both components, but where we have several elements with no d-complement. For instance, consider the unit interval with the lower open intervals as the positive topology, and the lower intervals as the negative one. Here, we have that the pseudocomplement of a typical positive open $(0, r)$ is $(r, 1)$, and so the double pseudocomplement is $(0, r)$. However, the only positive opens which are in tot with their pseudocomplement are $\emptyset$ and $(0, 1)$.

Let us now introduce the classical separation axioms for d-frames. We say that an element a^+ is *well inside* another element b^+, and we write $a^+ \lhd b^+$, if $(b^+, \sim a^+) \in t_L$. We say that a d-frame is *d-regular*, or simply *regular*, if every element of L^+ and L^- is the join of those well inside it. This equality actually amounts to just an inequality, by item (1) of the following lemma.

Lemma 1.2.4. *For every d-frame $\mathcal{L}$ we have the following.*

1. *If $a^+ \lhd b^+$ then we must have $a^+ \leq b^+$ and so for each $a^+ \in L^+$ we have $\bigvee\{b^+ \in L^+ : b^+ \lhd a^+\} \leq a^+$.*

2. *The family of elements well inside a given element is directed.*

Proof. Let us prove each item in turn.

1. If we have that $a^+ \lhd b^+$, then, by definition, this means that $(b^+, \sim a^+) \in t_L$, and so, by balance, we deduce that $a^+ \leq b^+$.

2. If we have $b_1^+, b_2^+ \lhd a^+$, then, since t_L is closed under homogeneous finite meets, we must also have $(a^+, \sim b_1^+ \wedge \sim b_2^+) \in t_L$, and since $\sim b_1^+ \wedge \sim b_2^+ = \sim(b_1^+ \vee b_2^+)$, by item (1) of Lemma 1.2.2. we must also have that $b_1^+ \vee b_2^+ \lhd a^+$. □

A d-frame is *d-normal*, or simply *normal*, if whenever $(a^+, a^-) \in t_L$ there is $(b^+, b^-) \in c_L$ such that $(a^+, b^-) \in t_L$ and $(b^+, a^-) \in t_L$. We say that a d-frame $\mathcal{L}$ is *zero dimensional* if

every element of L^+ and every element of L^- is the join of the d-complemented elements below it.

Observation 1.2.5. *If $\mathcal{L}$ is a zero dimensional d-frame, then we have that each $a^+ \in L^+$ is the join of d-complemented elements below it, but if $b^+ \leq a^+$ is d-complemented we have that $(b^+, \sim b^+) \in t_L$ and so also $(a^+, \sim b^+) \in t_L$, which means that $b^+ \lhd a^+$. Then, a^+ is also the join of the elements well inside it. All zero dimensional d-frames are then regular.*

We say that a d-frame is *Boolean* if every element of $L^+ \cup L^-$ is d-complemented.

Observation 1.2.6. *If $\mathcal{L}$ is a Boolean d-frame, and if we have $(a^+, a^-) \in t_L$, then the pair $(a^+, \sim a^+) \in c_L$ is always such that $(a^+, \sim a^+) \in t_L$ and $(a^+, a^-) \in t_L$. Then, all Boolean d-frames are normal.*

To create a bit of context, we now work towards replicating for d-frames the poset of implications that we have seen in the background chapter. We say that a d-frame $\mathcal{L}$ is *subfit* if whenever $a^+ \not\leq b^+$ there is $x^- \in L^-$ such that $(a^+, x^-) \in t_L$ and $(b^+, x^-) \notin t_L$.

Lemma 1.2.7. *If a d-frame is regular, then it is subfit.*

Proof. Suppose that $\mathcal{L}$ is a regular d-frame, and suppose that we have $a^+ \not\leq b^+$. Then, there must be an element, say c^+, which is well inside a^+ but not well inside b^+. Its pseudocomplement, then, is such that $(a^+, \sim c^+) \in t_L$ and $(b^+, \sim c^+) \notin t_L$. The argument for negative elements $a^- \not\leq b^-$ is, of course, analogous. □

We say that a d-frame $\mathcal{L}$ is *compact* if the set $t_L \subseteq L^+ \times L^-$ is inaccessible by directed joins. This is simply a d-frame translation of the notion of compact bispace that we have seen in Section .

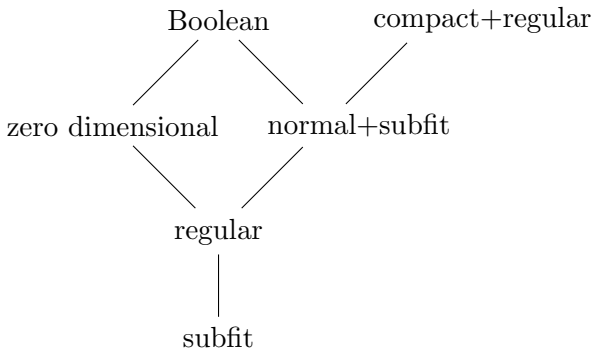
Lemma 1.2.8. *If a d-frame is compact and regular, then it is normal and subfit.*

Proof. Suppose that $\mathcal{L}$ is a d-frame which is compact and regular. The d-frame must be subfit by Lemma 1.2.7. For normality, suppose that we have a pair $(a^+, a^-) \in t_L$. Since each of a^+ and a^- is, by regularity, the join of elements well inside it, and since this join is directed by item (2) of Lemma 1.2.4, compactness tells us that we must have $(a_1^+, a^-) \in t_L$ for some $a_1^+ \lhd a^+$. Then, the pair $(a_1^+, \sim a_1^+) \in c_L$ is a witness of normality for the pair (a^+, a^-), in that it is such that both $(a_1^+, a^-) \in t_L$ and $(a^+, \sim a_1^+) \in t_L$. This means that the d-frame must be normal. □

Lemma 1.2.9. *If a d-frame is normal and subfit, then it is regular.*

Proof. Suppose that we have a normal subfit d-frame $\mathcal{L}$. By item (1) of Lemma 1.2.4, we only need to show the inequality $a^+ \leq \bigvee\{x^+ \in L^+ : x^+ \lhd a^+\} =: b^+$. By subfitness, it suffices to show that $(a^+, a^-) \in t_L$ implies that $(b^+, a^-) \in t_L$. Suppose, then, that $(a^+, a^-) \in t_L$. By normality, let us pick a pair $(x^+, x^-) \in c_L$ such that $(a^+, x^-) \in t_L$ and $(x^+, a^-) \in t_L$. Since t_L is an upset, the pair $(x^+, \sim x^+) \in c_L$, too, is a witness of normality, and since $(a^+, \sim x^+) \in t_L$ we have that, by definition of the well inside relation, $x^+ \lhd a^+$. Since $(x^+, a^-) \in t_L$, and since clearly $x^+ \leq b^+$, we must have $(b^+, a^-) \in t_L$. So indeed $a^+ \leq b^+$. Similarly one shows that elements of L^- are the join of the elements well inside it. □

Bringing together the implications that we have established in Lemmas 1.2.7, 1.2.8, and 1.2.9, as well Observations 1.2.5 and 1.2.6, and the ones which are immediate from the definitions, we obtain the following poset of implications for d-frames.



These axioms on d-frames connect with the analogous ones on bispaces as expected. We see that the following proposition holds.

Proposition 1.2.10. *For a bispace X we have the following.*

- *The space X is biregular if and only if $\mathsf{d}\Omega(X)$ is regular.*
- *The space X is zero dimensional if and only if $\mathsf{d}\Omega(X)$ is zero dimensional.*
- *The space X is binormal if and only if $\mathsf{d}\Omega(X)$ is normal.*
- *The space X is bicompact if and only if $\mathsf{d}\Omega(X)$ is compact.*

1.2.2 Abstract aspects: relations with frames and biframes

Given a d-frame $\mathcal{L}$, unlike the biframe case, we do not immediately get a pointfree version of the patch topology of $\mathcal{L}$ seen as an abstract bispace. Let us then see how, given a d-frame,

we can still give a pointfree description of the relative patch space. Recall that the coproduct of frames is a free construction, and so the generators $L^+ \cup L^-$ appear in it as syntactic expressions which, as usual, we denote as $\ulcorner x \urcorner$ for each $x \in L^+ \cup L^-$. We call the *patch* of a d-frame $\mathcal{L}$ the image of the quotienting of $L^+ \oplus L^-$ by the relation

$$R_{c_L t_L} := \{(\ulcorner a^+ \urcorner \wedge \ulcorner a^- \urcorner, 0) : (a^+, a^-) \in c_L\} \cup \{(1, \ulcorner a^+ \urcorner \vee \ulcorner a^- \urcorner) : (a^+, a^-) \in t_L\}.$$

We denote as $\Pi(\mathcal{L})$ the patch of a d-frame $\mathcal{L}$. Since the patch of a d-frame is still a free construction with generators $L^+ \cup L^-$, we call $\ulcorner - \urcorner : L^+ \to \Pi(\mathcal{L})$ the canonical map of generators, and we refer similarly to the same map on the negative component.

Remark 1.2.11. For a d-frame $\mathcal{L}$, the map $\ulcorner - \urcorner : L^+ \to \Pi(\mathcal{L})$ will not in general be an embedding. It is not so easy to find counterexamples to illustrate this, but we will encounter at least one of them, in Chapter 3. We will remind the reader of this remark when that happens.

We extend the definition of Π to d-frame morphisms as follows. Suppose that we have a d-frame map $f : \mathcal{L} \to \mathcal{M}$. Let us abbreviate the relation $R_{c_L t_L}$ as R_L and let us abbreviate similarly the relation $R_{c_M t_M}$. We define $\Pi(f)$ as the unique frame map making the following square commute.

$$\begin{array}{ccc} L^+ \oplus L^- & \xrightarrow{f^+ \oplus f^-} & M^+ \oplus M^- \\ \downarrow{\scriptstyle q_{R_L}} & & \downarrow{\scriptstyle q_{R_M}} \\ \Pi(\mathcal{L}) & \xrightarrow{\Pi(f)} & \Pi(\mathcal{M}) \end{array}$$

We do know that such a frame map exists, and that it is unique. Since f respects the con and the tot subsets, the map $q_{R_M} \circ (f^+ \oplus f^-)$ is such that

- $q_{R_M}(f^+(a^+)) \wedge q_{R_M}(f^-(a^-)) = 0$ whenever $(a^+, a^-) \in c_L$,
- $q_{R_M}(f^+(a^+)) \vee q_{R_M}(f^-(a^-)) = 1$ whenever $(a^+, a^-) \in t_L$.

This map must then factor through the quotient $q_{R_L} : L^+ \oplus L^- \to \Pi(\mathcal{L})$ uniquely, and so we indeed have a unique frame map that makes the square commute. Explicitly, the map $\Pi(f)$ simply acts on positive generators as $\ulcorner a^+ \urcorner \mapsto \ulcorner f^+(a^+) \urcorner$, and similarly on negative generators. We then know that the assignment Π is well defined on morphisms. The explicit definition of $\Pi(f)$ makes it clear that it respects identities and compositions, and so the assignment is functorial.

Remark 1.2.12. A difference between the patch functor Π and the patch functor π_3 for biframes that we should point out right away is that, while π_3 is a forgetful functor, the functor Π is more akin to a free construction.

Now, let us define the right adjoint to the functor Π. For a frame L let us first define

$$\mathsf{con}_{L_\wedge} = \{(a,b) \in L \times L : a \wedge b = 0\},$$
$$\mathsf{tot}_{L_\vee} = \{(a,b) \in L \times L : a \vee b = 1\}.$$

Then, define the d-frame $\delta(L)$ as $(L, L, \mathsf{con}_{L_\wedge}, \mathsf{tot}_{L_\vee})$. With routine calculations, using the distributivity laws of a frame, one can check that this quadruple is indeed a d-frame. To extend the action of δ on morphisms, define for a frame morphism $f : L \to M$ the d-frame map $\delta(f) : \delta(L) \to \delta(M)$ simply as the pairing (f, f). As a frame map is a lattice map, it preserves disjointness and covering, and so this assignment is well-defined on morphisms. Clearly it also respects identities and compositions, and so it is functorial. Let us now construct the unit and counit of this adjunction. For any frame L, explicitly, the frame $\Pi(\delta(L))$ is a quotient of $L \oplus L$. The pairing $\mathsf{id}_L \oplus \mathsf{id}_L : L \oplus L \to L$ factors through the quotient $q_{R_L} : L \oplus L \to \Pi(\delta(L))$, by definition of this quotient. We then define the map $\varepsilon_L : \Pi(\delta(L)) \to L$ the map $\overline{\mathsf{id}_L \oplus \mathsf{id}_L} : (L \oplus L)/R_L \to L$ coming from the universal property of this quotient. Explicitly, for every $a \in L$, this map acts as $\ulcorner a \urcorner \mapsto a$. Let us check that this defines a natural transformation.

Lemma 1.2.13. *The assignment* $\mathsf{Obj}(\mathbf{Frm}) \to \mathsf{Mor}(\mathbf{Frm})$ *assigning to each frame* L *the morphism* $\varepsilon_L : \Pi(\delta(L)) \to L$ *is a natural transformation* $\varepsilon : \Pi \circ \delta \to 1_{\mathbf{Frm}}$.

Proof. Suppose that $f : L \to M$ is a frame map. The naturality square is as follows.

$$\begin{array}{ccc} (L \oplus L)/R_L & \xrightarrow{\overline{\mathsf{id}_L \oplus \mathsf{id}_L}} & L \\ {\scriptstyle \Pi(\delta(f))}\downarrow & & \downarrow{\scriptstyle f} \\ (M \oplus M)/R_M & \xrightarrow{\overline{\mathsf{id}_M \oplus \mathsf{id}_M}} & M \end{array}$$

We observe that since $\delta(f)$ is just the pairing (f, f), by definition of π we have that $\Pi(\delta(f))$ acts on generators as $\ulcorner a \urcorner \mapsto \ulcorner f(a) \urcorner$. Recall also that by definition we have $\overline{\mathsf{id}_L \oplus \mathsf{id}_L} : \ulcorner a \urcorner \mapsto a$, and that the map at the bottom acts similarly. Commutativity of the square follows easily from these facts. □

Now, let us focus on the unit of adjunction. For each d-frame $\mathcal{L}$, first observe that, explicitly, the d-frame $\delta(\Pi(\mathcal{L}))$ is

$$(\Pi(\mathcal{L}), \Pi(\mathcal{L}), \mathsf{con}_{\Pi(\mathcal{L})_\wedge}, \mathsf{tot}_{\Pi(\mathcal{L})_\vee}).$$

Now define the d-frame map $\eta_{\mathcal{L}} : \mathcal{L} \to \delta(\Pi(\mathcal{L}))$ as $\eta^+_{\mathcal{L}}(a^+) = \ulcorner a^+ \urcorner$, and define η^-_L similarly. This is a d-frame morphism because, by definition of the patch of a d-frame, we have that is $(x^+, x^-) \in c_L$ then $\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner = 0$ in $\Pi(\mathcal{L})$ and so, by definition of the functor δ, we also have $(\ulcorner x^+ \urcorner, \ulcorner x^- \urcorner) \in \mathsf{con}_{\Pi(\mathcal{L})_\wedge}$. The argument to show that the map $\eta_{\mathcal{L}}$ respects the tot component is analogous.

Lemma 1.2.14. *The assignment* $\mathsf{Obj}(\mathbf{dFrm}) \to \mathsf{Mor}(\mathbf{dFrm})$ *assigning to each d-frame* $\mathcal{L}$ *the morphism* $\eta_{\mathcal{L}} : \mathcal{L} \to \delta(\Pi(\mathcal{L}))$ *is a natural transformation* $\eta : 1_{\mathbf{dFrm}} \to \delta \circ \Pi$.

Proof. Suppose that we have a map $f : \mathcal{L} \to \mathcal{M}$ of d-frames. Checking that the naturality square commutes amounts to showing that the following square in the category of frames commutes, as well as the analogous one on the negative component.

$$\begin{array}{ccc} L^+ & \xrightarrow{\eta^+_{\mathcal{L}}} & \Pi(\mathcal{L}) \\ {\scriptstyle f^+}\downarrow & & \downarrow{\scriptstyle \Pi(f)} \\ M^+ & \xrightarrow{\eta^+_{\mathcal{M}}} & \Pi(\mathcal{M}) \end{array}$$

We have the following chain of equalities for each $a^+ \in L^+$.

$$\Pi(f)(\eta^+_{\mathcal{L}})(a^+) = \Pi(f)(\ulcorner a^+ \urcorner) = \ulcorner f^+(a^+) \urcorner = \eta^+_{\mathcal{M}}(f^+(a^+)). \qquad \square$$

We are now ready to prove that we have the desired adjunction.

Proposition 1.2.15. *The functors* Π *and* δ *form an adjunction* $\Pi : \mathbf{Frm} \leftrightarrows \mathbf{dFrm} : \delta$ *with* $\Pi \dashv \delta$, *with unit* η *and counit* ε.

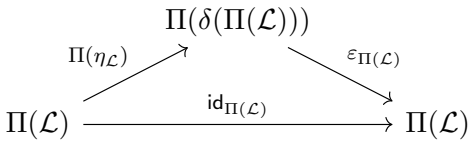
Proof. We prove that the data forms an adjunction by showing that the two unit-counit triangles commute.

- Suppose that L is a frame. The commutativity of the unit-counit triangle in **dFrm** amounts to the commutativity of the following triangle in **Frm**, as well as the analogous one on the negative component.

$$\begin{array}{ccccc} & & \Pi(\delta(L)) & & \\ & {\scriptstyle \eta^+_{\delta(L)}} \nearrow & & \searrow {\scriptstyle \varepsilon_L} & \\ L & & \xrightarrow{\quad \mathsf{id}_L \quad} & & L \end{array}$$

  We unravel the definitions of the morphisms and obtain $\varepsilon_L(\eta^+_{\delta(L)}(a)) = \varepsilon_L(\ulcorner a \urcorner) = a$.

- Suppose that $\mathcal{L}$ is a d-frame, and consider the following triangle in **Frm**.



  Unravelling the expression $\Pi(\delta(\Pi(\mathcal{L})))$ using the definitions of our two functors, we obtain that this is the frame quotient $(\Pi(\mathcal{L}) \oplus \Pi(\mathcal{L}))/R_{\mathsf{con}_{\Pi(\mathcal{L})_\wedge}\mathsf{tot}_{\Pi(\mathcal{L})_\vee}}$.We denote as $\ulcorner a^+ \urcorner$ the generator a^+ as appearing in $\Pi(\mathcal{L})$, and as $\ulcorner\ulcorner a^+ \urcorner\urcorner$ the same generator as appearing in $\Pi(\delta(\Pi(\mathcal{L})))$. For $a^+ \in L^+$ we have the following.

$$\begin{aligned}&\varepsilon_{\Pi(\mathcal{L})}(\Pi(\eta_{\mathcal{L}})(\ulcorner a^+ \urcorner)) = \\ &= \varepsilon_{\Pi(\mathcal{L})}(\ulcorner\ulcorner a^+ \urcorner\urcorner) = \\ &= \ulcorner a^+ \urcorner.\end{aligned}$$

□

If we modify the construction Π so that is preserves the information on how the two frames live in the patch, we can obtain a similar adjunction connecting the category of d-frames with that of biframes. For a d-frame $\mathcal{L}$ define the biframe $\Gamma(\mathcal{L})$ as the triple $(\ulcorner L^+ \urcorner, \ulcorner L^- \urcorner, \Pi(\mathcal{L}))$, where $\ulcorner L^+ \urcorner$ is the subframe $\{\ulcorner a^+ \urcorner : a^+ \in L^+\} \subseteq \Pi(\mathcal{L})$ and $\ulcorner L^- \urcorner$ is defined similarly. Recall that for a biframe $\mathcal{L}$ and a frame M, when we have a frame map $f : L \to M$ we denote as f^+ the restriction of f to $\ulcorner L^+ \urcorner$, and as f^- its restriction to $\ulcorner L^- \urcorner$. For a d-frame morphism $f : \mathcal{L} \to \mathcal{M}$, we define the biframe morphism $\Gamma(f) : \Gamma(\mathcal{L}) \to \Gamma(\mathcal{M})$ as the triple

$$(\Pi(f)^+, \Pi(f)^-, \Pi(f)) : (\ulcorner L^+ \urcorner, \ulcorner L^- \urcorner, \Pi(\mathcal{L})) \to (\ulcorner M^+ \urcorner, \ulcorner M^- \urcorner, \Pi(\mathcal{M})).$$

By definition of the frame map $\Pi(f)$, this is a well defined biframe morphism. The argument for functoriality of this is identical to that for the functor Π.

To construct the right adjoint to the functor Γ, assigning to each d-frame the free biframe constructed from it, let us define a forgetful functor from biframes to d-frames. For a biframe $\mathcal{L}$, let us define the d-frame $\Delta(\mathcal{L})$ as $(L^+, L^-, \mathsf{con}_{L_\wedge}, \mathsf{tot}_{L_\vee})$. With routine calculations we can check that this is well defined. For a biframe morphism $f : \mathcal{L} \to \mathcal{M}$ define the d-frame morphism $\Delta(f) : \Delta(\mathcal{L}) \to \Delta(\mathcal{M})$ as the pairing (f^+, f^-). This is always a d-frame morphism since the frame map $f : L \to M$ must preserve all the lattice operations, and so it respects the relations of disjointness and covering, too. The fact that Δ respects identity and composition is clear, and so it is a functor. In subsection 5.2 of [29] the following is proven.

Proposition 1.2.16. *The two functors* Δ : **BiFrm** $\leftrightarrows$ **dFrm** : Γ *constitute an adjunction* $\Delta \dashv \Gamma$.

We do not prove the adjunction in detail, because the arguments are very similar to those we used to prove the adjunction $\Pi \dashv \delta$. Let us see at least what the unit and the counit of this adjunction are.

- For the unit of adjunction, observe that for every d-frame $\mathcal{L}$ we have that the d-frame $\Delta(\Gamma(\mathcal{L}))$ is the d-frame

$$(\ulcorner L^+ \urcorner, \ulcorner L^- \urcorner, \mathsf{con}_{\Pi(\mathcal{L})_\wedge}, \mathsf{tot}_{\Pi(\mathcal{L})_\vee}).$$

  Whenever $(a^+, a^-) \in c_L$ we must have that $\ulcorner a^+ \urcorner \wedge \ulcorner a^- \urcorner = 0$ in the patch $\Pi(\mathcal{L})$, and for all pairs $(a^+, a^-) \in t_L$ we must have $\ulcorner a^+ \urcorner \vee \ulcorner a^- \urcorner = 1$. Then there must be a d-frame map $\eta_{\mathcal{L}} : \mathcal{L} \to \Delta(\Gamma(\mathcal{L}))$ sending each generator a^+ to its syntactic expression $\ulcorner a^+ \urcorner$, and similarly for negative generators. This is the unit of adjunction evaluated at $\mathcal{L}$.

- For every biframe $\mathcal{L}$, we observe that the biframe $\Gamma(\Delta(\mathcal{L}))$ is the biframe

$$(\ulcorner L^+ \urcorner, \ulcorner L^- \urcorner, (L^+ \oplus L^-)/R_{\mathsf{con}_{L_\wedge} \mathsf{tot}_{L_\vee}}).$$

  If we pair the two embeddings of L^+ and L^- we obtain a map $e : L^+ \oplus L^- \to L$, which factors through the quotient above since obviously all pairs in $\mathsf{con}_{L_\wedge}$ are disjoint in L and all pairs in $\mathsf{tot}_{L_\vee}$ are covering. By its definition, this map restricts as desired to the frames $\ulcorner L^+ \urcorner$ and $\ulcorner L^- \urcorner$ and so it determines a biframe map $\varepsilon_{\mathcal{L}} : \Gamma(\Delta(\mathcal{L})) \to \mathcal{L}$ which is the counit of the adjunction evaluated at $\mathcal{L}$.

The adjunction $\Gamma \dashv \Delta$ finally gives us a way of comparing the duality of d-frames with that of biframes. Let us see what happens when we compose adjunction $\Delta \dashv \Gamma$ with the spectral adjunction for bispaces and biframes.

$$\mathbf{BiTop} \underset{\mathsf{bpt}}{\overset{\mathsf{b}\Omega}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{BiFrm}^{op} \underset{\Gamma}{\overset{\Delta}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{dFrm}^{op}$$

Proposition 1.2.17. *We have that* $\Delta \circ \mathsf{b}\Omega = \mathsf{d}\Omega$ *and* $\mathsf{bpt} \circ \Gamma = \mathsf{dpt}$

Proof. The two frame components of the d-frames $\mathsf{d}\Omega(X)$ and $\Delta(\mathsf{b}\Omega(X))$ are clearly the same. The pairs in $\mathsf{con}_{\mathsf{b}\Omega(X)_\wedge}$ are the disjoint pairs, by definition of $\mathsf{b}\Omega(X)$, and similarly one sees that the totality components of $\Delta(\mathsf{b}\Omega(X))$ and $\mathsf{d}\Omega(X)$ coincide. For a morphism $f : X \to Y$ we have that $\Delta(\mathsf{b}\Omega(f)) = \Delta((f^{-1}, f^{-1}, f^{-1})) = (f^{-1}, f^{-1}) = \mathsf{d}\Omega(f)$. For a biframe $\mathcal{L}$ and a pair $f^+ : L^+ \to 2$ and $f^- : L^- \to 2$, below, we denote as $f : L \to 2$ the unique frame map

that coincides with f^+ when restricted to L^+, and with f^- when restricted to L^-. For any d-frame $\mathcal{L}$, we have the following.

$$
\begin{aligned}
&\mathsf{bpt}(\Gamma(\mathcal{L})) = \\
&= \{(f^+, f^-) \in \mathsf{pt}(L^+) \times \mathsf{pt}(L^-) : (f^+, f^-, f) : \Gamma(\mathcal{L}) \to \mathbf{2} \text{ is a biframe map}\} = \\
&= \{(f^+, f^-) \in \mathsf{pt}(L^+) \times \mathsf{pt}(L^-) : f^+(a^+) \wedge f^-(a^-) = 0 \text{ when } (a^+, a^-) \in c_L \\
&\text{and } f^+(a^+) \vee f^-(a^-) = 1 \text{ when } (a^+, a^-) \in t_L\} = \\
&\{(f^+, f^-) \in \mathsf{pt}(L^+) \times \mathsf{pt}(L^-) : f^+(a^+) = 0 \text{ or } f^-(a^-) = 0 \text{ when } (a^+, a^-) \in c_L, \\
&\text{and } f^+(a^+) = 1 \text{ or } f^-(a^-) = 1 \text{ when } (a^+, a^-) \in t_L\} = \\
&= \mathsf{dpt}(\mathcal{L}).
\end{aligned}
$$

Showing that the positive and the negative topologies of the two bispaces $\mathsf{bpt}(\Gamma(\mathcal{L}))$ and $\mathsf{dpt}(\mathcal{L})$ are the same is easy; one does it by showing that this bijection is a bihomeomorphism. For a d-frame morphism $f : \mathcal{L} \to \mathcal{M}$, we have that $\mathsf{bpt}(\Gamma(f)) = \mathsf{bpt}((f^+, f^-, f)) = (- \circ f, - \circ f) = \mathsf{dpt}(f)$. □

This tells us something about the two dualities that perhaps is not clear just from their definitions: in a certain sense, the biframe duality subsumes the d-frame duality. The fixpoints of the adjunction $\mathsf{b}\Omega \dashv \mathsf{bpt}$, then, are a subset of the fixpoints of the adjunction $\mathsf{d}\Omega \vdash \mathsf{dpt}$. Concretely, this means that the d-sober spaces are a subset of the patch-sober spaces. This is the precise phrasing of the intuition that we gave in Remark 1.1.15.

When we say that "we can recover a space from some information" we mean that the space is the fixpoint of a composition of two adjoints $F \dashv G$. We can see F as keeping only some information, and G as the "systematic recovering" of the spaces. Certain spaces are fixpoints of $G \circ F$, and these are exactly those that can be completely recovered via G. If we compose the adjunction $F \dashv G$ with another adjunction $F' \dashv G'$, the adjoint $F' \circ F$ keeps even less information than F. Then, for a space, being completely recoverable via $G' \circ G$ is a stronger property.

Let us now compare the point-set patch functor to its d-frame analogue Π.

We wish to check whether the pairing of the patch functors (Π, patch) constitutes a morphism between the spatial-sober adjunction of d-frames and that of frames, as we did for biframes. Once again, we loosen the definition to also include the situation where the relevant squares commute up to natural isomorphism, and where for the unit-counit compatibility we have suitable isomorphisms instead of equalities. Let us first check the first part of the definition of morphism between adjunctions.

Proposition 1.2.18. *The following square commutes up to natural isomorphism.*

$$\begin{array}{ccc} \mathbf{dFrm}^{op} & \xrightarrow{\mathsf{dpt}} & \mathbf{BiTop} \\ \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \mathsf{patch}} \\ \mathbf{Frm}^{op} & \xrightarrow{\mathsf{pt}} & \mathbf{Top} \end{array}$$

Proof. In light of Proposition 1.2.17, it suffices to show that the external rectangle of the following diagram commutes up natural isomorphism.

$$\begin{array}{ccccc} \mathbf{dFrm}^{op} & \xrightarrow{\Gamma} & \mathbf{BiFrm} & \xrightarrow{\mathsf{bpt}} & \mathbf{BiTop} \\ \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \pi_3} & & \downarrow{\scriptstyle \mathsf{patch}} \\ \mathbf{Frm}^{op} & \xrightarrow{1_{\mathbf{Frm}}} & \mathbf{Frm} & \xrightarrow{\mathsf{pt}} & \mathbf{Top} \end{array}$$

The third component of the biframe $\Gamma(\mathcal{L})$ for a d-frame $\mathcal{L}$ is defined to be $\Pi(\mathcal{L})$, and a similar result holds for morphisms. Then, the left square of this rectangle commutes. Furthermore, by Proposition 0.0.22, we know that there is a natural isomorphism $\mathsf{patch} \circ \mathsf{bpt} \cong \mathsf{pt} \circ \pi_3$. By Lemma 0.0.17, can deduce that there is also a natural isomorphism $\mathsf{patch} \circ \mathsf{bpt} \circ \Gamma \cong \mathsf{pt} \circ \pi_3 \circ \Gamma$. By commutativity of the left square, we can substitute equal functors into this expressions to obtain that there is a natural isomorphism $\mathsf{patch} \circ \mathsf{bpt} \circ \Gamma \cong \mathsf{pt} \circ 1_{\mathbf{Frm}} \circ \Pi$. □

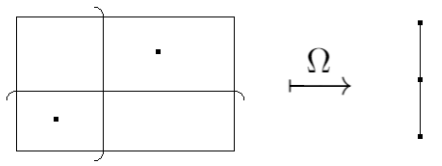
However, contrary to the biframe duality, the patch functors do not connect the two left adjoints so naturally. The following square *does not commute up to natural isomorphism*, and the most that we can say is that there is a frame surjection $\Pi(\mathsf{d}\Omega(X)) \twoheadrightarrow \Omega(\mathsf{patch}(X))$ for every bispace X.

$$\begin{array}{ccc} \mathbf{dFrm}^{op} & \xleftarrow[\mathsf{d}\Omega]{} & \mathbf{BiTop} \\ \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \mathsf{patch}} \\ \mathbf{Frm}^{op} & \xleftarrow[\Omega]{} & \mathbf{Top} \end{array}$$

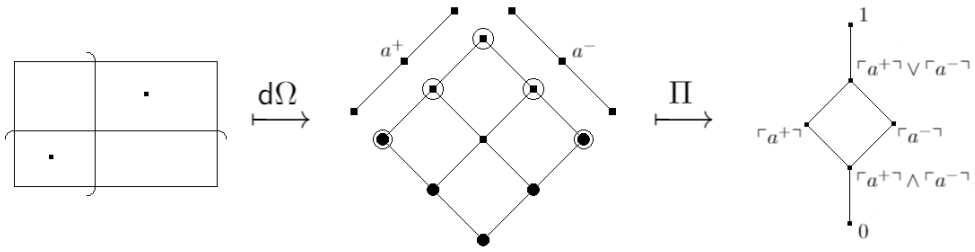
Then, the functors Π and patch do not constitute a map between the two adjunctions. We cannot therefore say that they establish a very meaningful connection between the d-frame and the frame duality.

Example 1.2.19. The following is a very simple example showing that the square above is not commutative. Let us start from the bispace with two points, such that one of the points

is open in both topologies, and the other in none. The frame of opens of its patch, which is simply homeomorphic to the Sierpinski space, is just the three element chain.



On the other hand, this is what we get when we take first the d-frame of opens, and then the pointfree patch of this. This the con and tot components are minimal, the patch of the d-frame of opens is the coprodduct $\Omega^+(X) \oplus \Omega^-(X)$, that is, the coproduct of two three element chains.



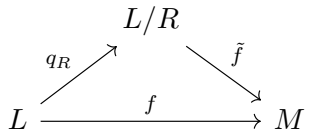
What has happened is that the d-frame of opens of a bispace only preserves the information on disjointness and covering, and when we construct the free frame from this data alone, we get a structure which is "more free" than the actual frame of opens of the patch.

1.3 Quotients of d-frames

D-frame quotients will be a very important tool in this thesis: many of the constructions which we will use to express the main message of the thesis will be built using d-quotients.

We may always quotient a frame by a relation. For any frame L and any relation $R \subseteq L \times L$, we can always find the smallest congruence $C(R)$ containing R, and obtain a frame L/R together with a frame surjection $q_R : L \twoheadrightarrow L/R$ sending each element of L to its $C(R)$-equivalence class. The map q_R satisfies the following universal property. Whenever there is a frame map $f : L \to M$ which respects the relation R, in the sense that $f(x) \leq f(y)$ whenever

xRy, there is a unique frame map $\tilde{f} : L/R \to M$ such that the following triangle commutes.



In [25] it is shown that an analogous result holds for d-frames. Let us explain, then, what is the d-frame version of a quotient.

D-frames have a double nature: they are structures which are both algebraic and relational. More precisely, they are algebraic (akin e.g. to rings and groups) in that we have the algebraic operations $\wedge$ and $\bigvee$ in the two frame components, and they are relational (akin e.g. to graphs) in that the con and the tot components are relations between the concrete elements of a d-frame. The algebraic and relational structure interact with one another, because of the axiom of balance. Quotienting, both for algebraic and for relational structures, means having an equivalence relation on the underlying set of elements, and constructing a new structure whose elements are the equivalence classes of the old one.

For algebraic structures, a quotient map may be seen as a canonical representative of an equivalence class of isomorphic surjections. Let us see this with frames. We say that a frame surjection $i : L \twoheadrightarrow M$ is isomorphic to a surjection $j : L \twoheadrightarrow N$ if there is an isomorphism $\alpha : M \to N$ such that $\alpha \circ i = j$. The quotient map representing the equivalence class of frame surjections isomorphic to $i : L \twoheadrightarrow M$ is that relative to L/R_i, where the relation R_i is $\{(x, y) \in L : i(x) = i(y)\}$. For relational structures, quotients do not have this role. Suppose that we have graphs $\mathcal{G}$ and $\mathcal{H}$, and suppose that we have a bijection $b : V(\mathcal{G}) \to V(\mathcal{H})$ between the sets of vertices of two graphs; suppose also that we have that vertices v_1 and v_2 in $\mathcal{G}$ which are not adjacent, but such that $f(v_1)$ and $f(v_2)$ are adjacent in $\mathcal{H}$. There cannot possibly be any quotient of $\mathcal{G}$ whose surjection $q_R : V(\mathcal{G}) \twoheadrightarrow V(\mathcal{G}/R)$ is isomorphic to the surjection $V(\mathcal{G}) \twoheadrightarrow V(\mathcal{H})$. Then, for graphs we cannot hold the view of quotients as canonical representatives of equivalence classes of isomorphic surjections.

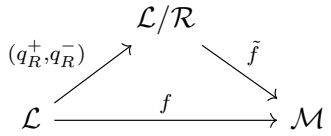
However, for d-frames we do want our notion of quotient to have this property. The reason is that we want quotients to be pointfree versions of bisubspace inclusions, and when we dualize bisubspace inclusions we get d-frame surjections. We take d-frame surjection to be pointfree versions of bisubspace inclusions; but we also want to take care that – just like in the frame case – isomorphic d-frame surjections represent the same abstract bisubspace of a d-frame. This is why we want d-frame quotients to be canonical representatives of equivalence classes of isomorphic d-frame surjections. D-frame surjections are determined not only by where the elements of the two frames are sent to, but also by which elements in the codomain

are in the con and the tot subsets. This is why when we are dealing with a d-frame we do not speak of quotienting by relations or pairs of relations, but by *relation structures*.

A *relation structure* on a d-frame $\mathcal{L}$ is a quadruple (R^+, R^-, c, t), such that R^+ and R^- are relations on L^+ and L^-, respectively, and c and t are subsets of $L^+ \times L^-$. The fact that d-frame quotients exist means the following. For any relation structure $\mathcal{R} = (R^+, R^-, c_R, t_R)$ on a d-frame $\mathcal{L}$, there is a d-frame $\mathcal{L}/\mathcal{R}$ and a canonical map $(q_R^+, q_R^-) : \mathcal{L} \twoheadrightarrow \mathcal{L}/\mathcal{R}$ consisting of two frame surjections, with the following universal property. Whenever $f : \mathcal{L} \to \mathcal{M}$ is a d-frame map such that

- $f^+(a^+) \leq f^+(b^+)$ whenever $a^+ R^+ b^+$,
- $f^-(a^-) \leq f^-(b^-)$ whenever $a^- R^- b^-$,
- $(f^+(a^+), f^-(a^-)) \in c_M$ whenever $(a^+, a^-) \in c_R$,
- $(f^+(a^+), f^-(a^-)) \in t_M$ whenever $(a^+, a^-) \in t_R$,

there is a unique d-frame map $\tilde{f} : \mathcal{L}/\mathcal{R} \to \mathcal{M}$ making the following diagram commute.



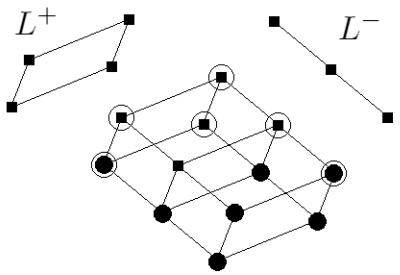
Crucially, here (q_R^+, q_R^-) will not be in general the pairing of the frame quotients $q_{R^+} : L^+ \twoheadrightarrow L^+/R^+$ and the analogous one on the negative component. Let us pause and talk about this aspect some more. For any d-frame $\mathcal{L}$ and any relation structure (R^+, R^-, c_R, t_R) on $\mathcal{L}$, consider the quadruple $(q_{R^+}[L^+], q_{R^-}[L^-], q_R[c_L], q_R[t_L])$, where $q_R[c_L]$ is an abbreviation for $\{(q_R^+(x^+), q_R^-(x^-)) \in q_{R^+}[L^+] \times q_{R^-}[L^-] : (x^+, x^-) \in c_L\}$, and $q_R[t_L]$ is defined similarly. We call this the *direct image quadruple* of the relation structure $\mathcal{R}$. In [25], the following is proven.

Proposition 1.3.1. *For any d-frame $\mathcal{L}$ and any relation structure (R^+, R^-, c_R, t_R) on $\mathcal{L}$, the direct image quadruple $(q_{R^+}[L^+], q_{R^-}[L^-], q_R[c_L], q_R[t_L])$ satisfies the following.*

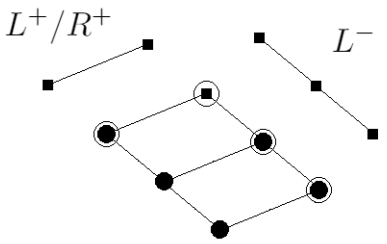
- *The subset $q_R[c_L]$ is a downset of $q_{R^+}[L^+] \times q_{R^-}[L^-]$ which is closed under finite homogeneous joins.*
- *The subset $q_R[t_L]$ is an upset of $q_{R^+}[L^+] \times q_{R^-}[L^-]$ which is closed under finite homogeneous meets.*

Direct image quadruples, however, in general fail to satisfy both closure under infinite homogeneous joins of $q_R[c_L]$ and balance, and so they are not d-frames. Perhaps it is instructive to look at two concrete examples of this. In particular, we can learn from these examples that the issues of infinite homogeneous joins and that of balance are two separate ones: there are direct images that respect the former property but not the latter, and others that respect the latter and not the former.

- *Balance.* Let $\mathcal{L}$ be the following d-frame.



As we can see from the picture, the totality component is minimal while we added only one extra pair to the consistency component. Let us call b^+ the element of L^+ represented by the rightmost "middle node" in the picture. Suppose that we want to quotient it by the relation structure $(R^+, \mathsf{id}^-, c_m, t_m)$ with $R^+ = \Delta(b^+)$. Taking the direct image quadruple $(L^+/R^+, L^-, q_R[c_L], q_R[t_L])$ relative to this relation structure gives

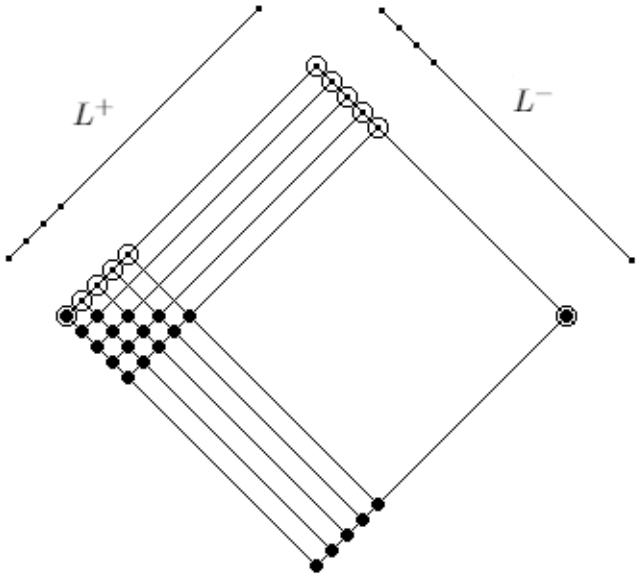


Here there are no infinite homogeneous joins to worry about. On the other hand, this direct image quadruple did not preserve balance of the starting d-frame. Denoting the elements of L^- as $0^- < a^- < 1^-$, from the picture above we see that

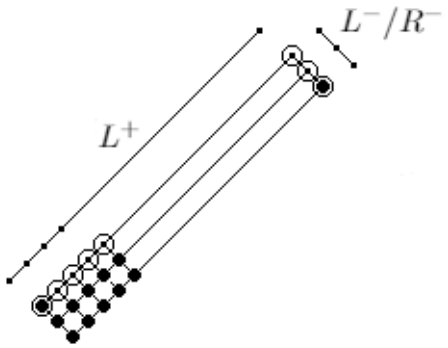
$$\begin{cases} (q_R^+(1^+), a^-) \in q_R[c_L] \\ (q_R^+(1^+), 0^-) \in t_R[t_L], \end{cases}$$

but $a^- \not\leq 0^-$. Then, the structure is not balanced.

- *Infinite homogeneous joins.* Now, consider the following d-frame. The L^+ component is isomorphic to the ordinal $\omega + 1$, while the L^- component is anti-isomorphic to it. The con component contains all pairs below the main diagonal of the square $L^+ \times L^-$, and the tot component is minimal. The d-frame is depicted below.



Now suppose that we want to quotient it by the relation structure $(\mathsf{id}^+, R^-, c_m, t_m)$, where R^- is the equivalence relation on L^- such that it identifies all elements different from 0^- and 1^-. Taking the direct image quadruple relative to the quotient $(q_{\mathsf{id}^+}, q_{R^-})$ yields the following structure.



From the picture, we can check that the direct image quadruple is a balanced structure. However, we have that the con component contains a homogeneous family of pairs, but not their join.

1.3.1 The lattice of d-quotients of a d-frame

But then, the reader who is new to this topic will be wondering how we can describe the quotient of a d-frame, if its image is not the direct image quadruple of a relation structure. To answer this, we need to add one more concept to the picture. In [25] the authors identify, among relation structures, the ones such that their two relation components are congruences, and such that their direct image quadruple is a d-frame. These are called the *reasonable congruence structures*. Their importance is captured in the following result, appearing in [25].

Proposition 1.3.2. *There is a bijection between reasonable congruence structures on a d-frame $\mathcal{L}$ and d-frame quotients from $\mathcal{L}$. The bijection assigns to each reasonable congruence structure (R^+, R^-, c_R, t_R) the d-quotient $q_R : \mathcal{L} \twoheadrightarrow (L^+/R^+, L^-/R^- q_R[c_R], q_R[c_R])$.*

Now let us consider the natural order relation between quotients of a d-frame. For a d-frame $\mathcal{L}$ and two quotients $q_R : \mathcal{L} \twoheadrightarrow \mathcal{L}/\mathcal{R}$ and $q_S : \mathcal{L} \twoheadrightarrow \mathcal{L}/\mathcal{S}$ we say that $q_R \leq q_S$ if there is a surjection $q_{SR} : \mathcal{L}/\mathcal{S} \twoheadrightarrow \mathcal{L}/\mathcal{R}$ such that $q_R = q_{SR} \circ q_S$. D-frame quotients are always epimorphisms (see [25]), and so if such a surjection exists it is uniquely determined. Let us call $\mathsf{S}(\mathcal{L})$ the ordered collection of d-quotients of $\mathcal{L}$. In [25] it is shown that the bijection of Proposition 1.3.2 establishes an anti-isomorphism between $\mathsf{S}(\mathcal{L})$ and the ordered collection of reasonable congruence structures on $\mathcal{L}$. In [25] the authors also prove that reasonable congruence structures are closed under componentwise set theoretical intersections. We then have the following.

Theorem 1.3.3. *The ordered collection* $\mathsf{S}(\mathcal{L})$ *of d-quotients of a d-frame* $\mathcal{L}$ *is a complete lattice.*

Just like a relation on a frame determines the smallest congruence containing it, a relation structure on a d-frame determines the smallest reasonable congruence structure containing it. This is just the componentwise set theoretical intersection of all reasonable congruence structures that contain it componentwise. The result of quotienting a d-frame $\mathcal{L}$ by a relation structure $\mathcal{R}$ is the d-frame surjection

$$q_R : \mathcal{L} \twoheadrightarrow (q_{C(R)}[L^+], q_{C(R)}[L^-], q_{C(R)}[C(c_R)], q_{C(R)}[C(t_R)]),$$

where $C(\mathcal{R}) = (C(R^+), C(R^-), C(c_R), C(t_R))$ is the smallest reasonable congruence structure containing $\mathcal{R}$ componentwise. As in the frame theoretical case, however, the explicit form of $C(\mathcal{R})$ given $\mathcal{R}$ is quite unpredictable. To obtain the smallest reasonable congruence structure containing a relation structure componentwise, one has to follow an iterative process which possibly has transfinitely many steps. This is why, in general, it is not easy to describe a d-quotient explicitly. The situation is further explored in [22], where the author identifies some sufficient conditions for a reasonable congruence structure obtained from a relation structure on a d-frame to have a simple explicit description.

In this thesis, very often, all that we will need is to keep in mind the universal property of d-frame quotients. Very rarely will we be interested in the explicit form of a d-quotient and - in the rare cases where we will - it will be a "lucky case" where the direct image quadruple is already a d-frame. There is no need for us to go through the steps of the transfinite process described in [22]. Let us just keep at the back of our minds the idea that d-quotients elude a simple explicit description, and this is why this thesis will treat them, most of the time, as abstract objects.

We can now introduce some of the special kinds of quotients whose direct image quadruple is already a d-frame. First, we have the d-frames where con and tot are minimal.

Lemma 1.3.4. *For a d-frame* $\mathcal{L}_m$ *with minimal* con *and* tot *components, any pair of frame maps* $(f^+, f^-) : \mathcal{L}_m \to \mathcal{M}$ *is a d-frame map.*

Proof. We have that if $(a^+, a^-) \in c_m$ then we must have that $a^+ = 0^+$ or $a^- = 0^-$. Suppose without losing generality that the first holds. Then $f^+(a^+) = 0 \in M^+$ and so $(f^+(a^+), f^-(a^-)) \in c_M$ since $(f^+(a^+), f^-(a^-)) \in c_m$, and by Proposition 1.1.1 we must have $c_m \subseteq c_M$. A similar argument shows that the totality subset is respected. □

Proposition 1.3.5. *For a d-frame $\mathcal{L}_m$ with minimal* con *and* tot *components, for every relation structure on it of the form (R^+, R^-, c_m, t_m), the quotient $\mathcal{L}_m/\mathcal{R}$ is the pair of frame maps $(q_{R^+}, q_{R^-}) : \mathcal{L}_m \twoheadrightarrow (L^+/R^+, L^-/R^-, c_m, t_m)$.*

Proof. It is easy to check that, for a d-frame $\mathcal{L}_m$ and a relation structure (R^+, R^-, c_m, t_m) on it we have that $q_R[c_m]$ and $q_R[t_m]$ are the minimal consistency and totality subsets on $q_R^+[L^+] \times q_R^-[L^-]$. We also already know that these are always balanced. For every d-frame map $f : \mathcal{L}_m \to \mathcal{M}$ such that its frame components respect the R^+ and the R^- relations, we define a map $(q_{R^+}, q_{R^-}) : (L^+/R^+, L^-/R^-, c_m, t_m) \to \mathcal{M}$, and notice that it is a d-frame map since by Lemma 1.3.4 any pair of frame maps from a d-frame with con and tot minimal is a d-frame map. □

For the following proposition, we remind the reader (Proposition 1.1.2) that the collections of consistency and totality subsets on a binary product of frames are both closed under arbitrary intersections. Then, for any two frames L^+ and L^-, and any subset $S \subseteq L^+ \times L^-$, we may always find the smallest consistency component containing it, and the same holds for totality.

Proposition 1.3.6. *Suppose that $\mathcal{R} = (\mathsf{id}^+, \mathsf{id}^-, c_R, t_R)$ is a relation structure on a d-frame $\mathcal{L}_m$, and let C_R be the smallest consistency subset of $L^+ \times L^-$ containing c_R, and let T_R be the smallest totality component containing t_R. If the quadruple (L^+, L^-, C_R, T_R) is balanced, then the quotient $q_R : \mathcal{L}_m \twoheadrightarrow \mathcal{L}/\mathcal{R}$ is*

$$(q_{\mathsf{id}^+}, q_{\mathsf{id}^-}) : \mathcal{L} \twoheadrightarrow (L^+, L^-, C_R, T_R).$$

Many of the constructions that we will look at involve a quotient of this kind. Having just seen this kind of quotient gives us now an occasion to speak about a special property of d-frame quotients. Usually, in algebraic categories, monic quotients are trivial. In the category of frames, the monic morphisms are the injective frame maps, while quotients are always surjections. One can show that a frame map which is bijective is a frame isomorphism. In the category of d-frames, this does not hold. There are d-quotients which are monic, but are not d-frame isomorphisms. In [34] the following is shown.

Proposition 1.3.7. *A d-frame map $f : \mathcal{L} \to \mathcal{M}$ is monic in* **dFrm** *if and only if the two frame maps f^+ and f^- are injective.*

In the category of d-frames, then, we may have monic quotients which are not trivial. These are exactly the quotients that change the starting d-frame only by increasing the con

and the tot components. Every d-frame can be expressed as the image of a monic quotient in a canonical way. If, for a d-frame $\mathcal{L}$, we let $\mathcal{R}_{ct}$ be the relation structure $(\mathsf{id}^+, \mathsf{id}^-, c_L, t_L)$ on $\mathcal{L}_m$, we have that $\mathcal{L} = \mathcal{L}_m/\mathcal{R}_{ct}$. Let us now look at a property which is in a certain sense dual to monicity, for d-quotients. In [34], the following is proven.

Proposition 1.3.8. *The extremal epimorphisms in* **dFrm** *are the d-frame surjections of the form* $f : \mathcal{L} \twoheadrightarrow (f^+[L^+], f^-[L^-], \overline{f[c_L]}, f[t_L])$*, where* $\overline{f[c_L]}$ *is the smallest* con *subset containing* $f[c_L]$*.*

In terms of relation structures, we have the following result on extremality.

Proposition 1.3.9. *If we have a relation structure of the form* $\mathcal{R} = (R^+, R^-, c_m, t_m)$ *on a d-frame* $\mathcal{L}$*, the quotient* $q_R : \mathcal{L} \twoheadrightarrow \mathcal{L}/\mathcal{R}$ *is extremal.*

Proof. Suppose that $(C(R^+), C(R^-), C(c_m), C(t_m))$ is the smallest reasonable congruence structure containing $\mathcal{R}$ componentwise. Then we have that the quotient q_R is the direct image quadruple

$$q_R : \mathcal{L} \twoheadrightarrow (q_{C(R)}[L^+], q_{C(R)}[L^-], q_{C(R)}[C(c_m)], q_{C(R)}[C(t_m)]).$$

We know that $\overline{q_{C(R)}[c_L]} \subseteq q_{C(R)}[C(c_m)]$ and that $q_{C(R)}[t_L] \subseteq q_{C(R)}[C(t_m)]$, as the quotient $q_{C(R)}$ must preserve the con and tot components. The pair $(q_{C(R)}[C(c_m)], q_{C(R)}[C(t_m)])$ is balanced, so the pair $(\overline{q_{C(R)}[c_L]}, q_{C(R)}[t_L])$ must be balanced too. The quadruple

$$\mathcal{L}/\mathcal{R}' := (q_{C(R)}[L^+], q_{C(R)}[L^-], \overline{q_{C(R)}[c_L]}, q_{C(R)}[t_L])$$

is then a d-frame, and additionally the d-frame quotient $q_{C(R)} : \mathcal{L} \to \mathcal{L}/\mathcal{R}'$ is extremal by definition. □

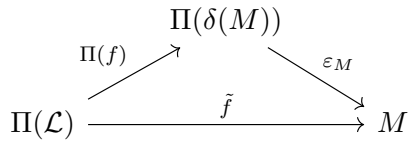
1.3.2 Patches of d-frame quotients

In this subsection we work towards proving a lemma which brings together the notion of d-frame quotient with that of frame quotient. It is a lemma that we will use in some crucial parts of our work.

Lemma 1.3.10. *Under the canonical natural bijection* $\mathbf{dFrm}(\mathcal{L}, \delta(M)) \cong \mathbf{Frm}(\Pi(\mathcal{L}), M)$ *given by the adjunction* $\Pi \dashv \delta$*, that is, under the bijection* $f \mapsto \varepsilon_M \circ \Pi(f)$*, we have that a morphism* $f : \mathcal{L} \to \delta(M)$ *respects the relation structure* $\mathcal{R}$ *if and only if its bijective image respects the relation*

$$\Pi(\mathcal{R}) := \{(\ulcorner x^+ \urcorner, \ulcorner y^+ \urcorner) : (x^+, y^+) \in R^+\} \cup \{(\ulcorner x^- \urcorner, \ulcorner y^- \urcorner) : (x^-, y^-) \in R^-\} \cup$$
$$\cup \{(\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner, 0) : (x \in c_R^+, x \in c_R^-)\} \cup \{(1, \ulcorner x^+ \urcorner \vee \ulcorner x^- \urcorner) : (x^+, x^-) \in t_R\}.$$

Proof. For a d-frame morphism $f : \mathcal{L} \to \delta(M)$, let us denote as $\tilde{f}$ its bijective image mentioned in the statement, that is, the morphism $\varepsilon_M \circ \Pi(f)$, in other words, the unique frame map $\tilde{f} : \Pi(\mathcal{L}) \to M$ such that the following diagram commutes.



We recall that $\Pi(f) : \Pi(\mathcal{L}) \to \Pi(\delta(M))$ is defined on positive generators as $\ulcorner x^+ \urcorner \mapsto \ulcorner f^+(x^+) \urcorner$, and similarly for negative generators. The map ε_M acts on generators as $\ulcorner x^+ \urcorner \mapsto x^+$. The condition that $\Pi(f)$ respects the first disjunct of $\Pi(\mathcal{R})$ is equivalent to saying that we have $(x^+, y^+) \in R^+$ implies that $\tilde{f}(\ulcorner x^+ \urcorner) \leq \tilde{f}(\ulcorner y^+ \urcorner)$; by commutativity of the diagram, this is equivalent to saying that $(x^+, x^-) \in R^+$ implies that $\varepsilon_M(\Pi(f)(\ulcorner x^+ \urcorner)) \leq \varepsilon_M(\Pi(f)(\ulcorner x^+ \urcorner))$, that is, $f^+(x^+) \leq f^+(y^+)$ in M. So, $\Pi(f)$ respecting the first disjunct of $\Pi(\mathcal{R})$ is equivalent to f respecting the first component of $\mathcal{R}$. The result for the second disjunct and component are proved analogously. The map $\Pi(f)$ respecting the third disjunct is equivalent to saying that whenever $(x^+, x^-) \in c_R$, we have $\tilde{f}(\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner) = \varepsilon_M(\Pi(f)(\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner)) = 0$. This is equivalent to saying that $(x^+, x^-) \in c_R$ implies that $f^+(x^+) \wedge f^-(x^-) = 0$ in M. Then, the same can be said about the third disjunct and the third component. The argument for the fourth ones is analogous. □

Lemma 1.3.11. *If L^+ and L^- are frames, and $\mathcal{R}$ is a relation structure on $\mathcal{L}$, we have that $\Pi(\mathcal{L}/\mathcal{R})$ is isomorphic to the patch $\Pi(\mathcal{L})$ quotiented by*

$$\Pi(\mathcal{R}) = \{(\ulcorner x^+ \urcorner, \ulcorner y^+ \urcorner) : (x^+, y^+) \in R^+\} \cup \{(\ulcorner x^- \urcorner, \ulcorner y^- \urcorner) : (x^-, y^- \in R^-)\} \cup$$
$$\cup \{(\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner, 0) : (x^+, x^-) \in c_R\} \cup \{(1, \ulcorner x^+ \urcorner \vee \ulcorner x^- \urcorner) : (x^+, x^-) \in t_R\}.$$

Proof. We will show that the patch $\Pi(\mathcal{L}/\mathcal{R})$ and the quotient described in the claim are isomorphic to the same quotient of $L^+ \oplus L^-$ by showing that they satisfy the same universal property in **Frm**. In order to do this, we reason in terms of hom-sets. By definition of d-frame and frame quotients, and by the fact that we have an adjunction $\Pi \dashv \delta$, we may obtain the following chain of bijections. We omit the proof that these are natural. The third bijection

follows from Lemma 1.3.10 above.

$$\begin{aligned}
&\mathbf{Frm}(\Pi(\mathcal{L}/\mathcal{R}), M) \cong \\
&\cong \mathbf{dFrm}(\mathcal{L}/\mathcal{R}, \delta(M)) \cong \\
&\cong \{f \in \mathbf{dFrm}(\mathcal{L}, \delta(\mathcal{M})) : f \text{ respects } \mathcal{R}\} \cong \\
&\cong \{\tilde{f} \in \mathbf{Frm}(\Pi(\mathcal{L}), M) : \tilde{f} \text{ respects } \Pi(\mathcal{R})\} \cong \\
&\cong \mathbf{Frm}(\Pi(\mathcal{L})/\Pi(\mathcal{R}), M).
\end{aligned}$$

□

1.3.3 D-quotients as pointfree bisubspaces

One of the main concepts we wish to study in this thesis is that of bitopological subspace inclusion, and its pointfree counterpart. Recall that, in the duality of frames and spaces, we have that a map $i : Y \to X$ between T_0 spaces is a subspace inclusion if and only if $\Omega(i) : \Omega(X) \to \Omega(Y)$ is a frame surjection. This is one piece of motivation behind the fact that, in pointfree topology, we regard d-quotients as good pointfree representations of subspaces. We refer to d-frame maps both of whose frame components are surjections simply as *d-frame surjections*. The result above has a d-frame analogue, and in fact one can show using just frame theoretical techniques that the following holds.

Proposition 1.3.12. *A bicontinuous map $i : Y \to X$ is a bisubspace inclusion if and only if $\mathsf{d}\Omega(i) : \mathsf{d}\Omega(X) \to \mathsf{d}\Omega(Y)$ is a d-frame surjection.*

This thesis is centered around the study of d-quotients. We will regard d-quotients as pointfree representations of bisubspaces, but the reader should be warned that this is not the only possible approach. In the category of frames, the frame surjections also coincide with the extremal epimorphisms. We can take equivalence classes of isomorphic extremal epimorphisms in **Frm** as being a more categorical definition of pointfree subspaces. In the category of d-frames it is no longer true that the surjections coincide with the extremal epimorphisms, and so we are left with a choice as to what counts as a pointfree bisubspace.

Example 1.3.13. Let us look at Example 1.1.5, where we showed how updating the con and the tot components of a d-frame translates as taking away points from its d-spectrum. It can be shown that all three d-frames in this example are spatial. Then, the bisubspace inclusion of the second space into the first, when dualized, amounts to a d-frame surjection $\mathcal{L} \twoheadrightarrow (L^+, L^-, c, t_m)$. That of the third space into the first, instead, is of the form $\mathcal{L} \twoheadrightarrow (L^+, L^-, c_m, t)$. These are clearly not extremal, but they are indeed dualizations of a bisubspace inclusion. Then, if we want all dualizations of bisubspace inclusions to count as pointfree bisubspaces, we should not restrict ourselves to the extremal morphisms.

It is really not clear whether the question of what is the "correct" notion of pointfree bisubspace is a well-posed one. Our choice of regarding all d-quotients as pointfree bisubspaces stems from the fact that here we are mostly interested in how the $\mathsf{d}\Omega \dashv \mathsf{dpt}$ adjunction concretely relates bisubspaces and d-quotients. Very often we will be looking at questions such as what is the dualization of a d-frame quotient, or that of a bisubspace inclusion.

Chapter 2

D-frames: frame theoretical aspects

In this chapter, we will analyze consistency and totality through a lattice theoretical lens. The aim of this chapter is to give the reader a more complete understanding of the basic notions in the theory of d-frames, by expressing them in the less recent and more well-understood language of frames. When we apply the patch construction to a d-frame, we are essentially translating from one language to the other. Under this translation, con subsets become closed sublocales, and tot subsets become fitted sublocales. One of the principal aims of this chapter is to analyze in detail how precise this correspondence between con and tot subsets and sublocales is. We will see, in particular, that the ordered collection $\mathsf{con}(L^+ \times L^-)$ of subsets of $L^+ \times L^-$ satisfying the consistency axioms is in bijective correspondence with that of closed sublocales of $L^+ \oplus L^-$. On the other hand, different elements of the collection $\mathsf{tot}(L^+ \times L^-)$ of all tot subsets may determine the same fitted sublocale of $L^+ \oplus L^-$. Proving these facts is the main goal of Sections 1 and 2.

The assignment of a fitted sublocale to a tot subset being not injective, ultimately, is a manifestation of the fact that different filters of a frame may induce the same fitted sublocale. More precisely, for every frame L we have an adjunction $\mathit{fit} : \mathsf{Filt}(L) \leftrightarrows \mathsf{S}(L)^{op} : \mathit{top}$, with $\mathit{top} \dashv \mathit{fit}$, defined as $\mathit{fit} : F \mapsto \bigcap_{f \in F} \mathfrak{o}(f)$, and $\mathit{top} : S \mapsto \{x \in L : S \subseteq \mathfrak{o}(x)\}$. The composition $\mathit{top} \circ \mathit{fit}$ is not in general the identity. A significant part of Section 2 will be aimed at analyzing this adjunction. We will characterize those filters which are fixpoints of $\mathit{top} \circ \mathit{fit}$, and use this fact to provide our own proof of the well-known fact that this adjunction restricts to an equivalence between Scott-open filters and compact fitted sublocales (see [27]).

One may wonder what the equivalence between Scott-open filters and compact fitted sublocales means, for tot subsets and fitted sublocales. Indeed, there is a notion of *compact* totality subset. And, indeed, the poset of compact totality subsets is in bijective correspondence with

the compact fitted sublocales of the coproduct $L^+ \oplus L^-$, but only if we assume a pointfree version of Alexander's Subbase Lemma, which we formulate and prove.

2.1 Consistency and totality as encoded in the frame coproduct

In this section we will analyze, for two frames L^+ and L^-, the posets of all con subsets of $L^+ \times L^-$, together with that of all possible tot subsets. Let us denote as $\mathsf{con}(L^+ \times L^-)$ the ordered collection of subsets of $L^+ \times L^-$ satisfying the consistency axioms, and as $\mathsf{tot}(L^+ \times L^-)$ the ordered collection of those satisfying the totality axioms.

Many of the results in this section require a concrete description of the coproduct in the category of frames, so let us analyze its structure. Our analysis is based on the concrete construction of the coproduct of frames found in [38]. In the following, for a lattice L we denote as $\mathcal{D}(L)$ the lattice of downsets of L. We denote the lattice of its upsets as $\mathcal{U}(L)$.

Lemma 2.1.1. *For any two meet-semilattices L^+ and L^-, we have*

$$\mathbf{Frm}\langle L^+ \cup L^- \ \textit{qua} \ \ \textit{meet-semilattices}\rangle \cong \mathcal{D}(L^+ \times L^-).$$

The canonical map from generators is an embedding $L^+ \hookrightarrow \mathcal{D}(L \times L)$ and is defined as $a^+ \mapsto {\downarrow}(a^+, 1^-)$.

Proof. The embedding $a^+ \mapsto {\downarrow}(a^+, 1^-)$ preserves finite meets, in particular we observe that it preserves the empty meet, since it maps a top element to a top element. For two finite meet preserving maps $f^+ : L^+ \to M$ and $f^- : L^- \to M$ to a frame M we define the map $\tilde{f} : \mathcal{D}(L^+ \times L^-) \to M$ as $\bigcup_i {\downarrow}(a_i^+, a_i^-) \mapsto \bigvee_i f^+(a_i^+) \wedge f^-(a_i^-)$. Simple calculations show that this is a frame map. The map restricts as desired for generators as $\tilde{f}({\downarrow}(a^+, 1^-)) = f^+(a^+) \wedge f^-(1^-)$. □

Lemma 2.1.2. *For two frames L^+ and L^-, the collection $\mathsf{con}(L^+ \times L^-) \subseteq \mathcal{D}(L^+ \times L^-)$ is a sublocale.*

Proof. Closure under meets is clear, it remains to show that $\mathsf{con}(L^+ \times L^-) \subseteq \mathcal{D}(L^+ \times L^-)$ is stable under $D \to -$. Suppose that D is an arbitrary downset and that c is a downset closed under homogeneous joins. Then, suppose that $(a_i^+, a^-) \in D \to c$. This condition is equivalent to having $(a_i^+ \wedge d^+, a^- \wedge d^-) \in c$ for all $(d^+, d^-) \in D$. By the closure property of c, this also implies that $(\bigvee_i a_i^+ \wedge d^+, a^- \wedge d^-) \in c$, which means that $(\bigvee_i a_i^+, a^-) \in D \to c$, and so indeed $D \to c$ is closed under homogeneous joins. □

Proposition 2.1.3. *For two frames L^+ and L^-, we have $L^+ \oplus L^- \cong \mathsf{con}(L^+ \times L^-)$. The canonical map from the generators is an embedding $L^+ \hookrightarrow \mathsf{con}(L^+ \times L^-)$ and is defined as $a^+ \mapsto c_m \cup {\downarrow}(a^+, 1^-)$.*

Proof. By Lemma 2.1.1, the frame $L^+ \oplus L^-$ is the quotient of $\mathcal{D}(L^+ \times L^-)$ making the embeddings $a^+ \mapsto {\downarrow}(a^+, 1^-)$ into join preserving maps. In particular this quotient has to identify each downset $\bigcup_i {\downarrow}(a_i^+, a^-)$ with the downset ${\downarrow}(\bigvee_i a_i^+, a^-)$. Then, the nucleus corresponding to this quotient must be the smallest containing the closure operator $hJ : \mathcal{D}(L^+ \times L^-) \to \mathcal{D}(L^+ \times L^-)$ which maps each downset at to its closure under homogeneous joins. Since the fixpoints of hJ form a sublocale, by Lemma 2.1.2, the closure operator hJ must in fact be a nucleus, and so the frame of its fixpoints $\mathsf{con}(L^+ \times L^-)$ must be isomorphic to $L^+ \oplus L^-$. The canonical map of generators, then, acts as $a^+ \mapsto hJ({\downarrow}(a^+, 1^-))$. We observe that $hJ({\downarrow}(a^+, 1^-)) = c_m \cup {\downarrow}(a^+, 1^-)$, and that the map $L^+ \to L^+ \oplus L^-$ defined as $a^+ \mapsto c_m \cup {\downarrow}(a^+, 1^-)$ is injective. □

Since the coproduct is a free construction, its elements can be seen as syntactic expressions of the form $\bigvee_i \ulcorner x_i^+ \urcorner \wedge \ulcorner x_i^- \urcorner$, we will often adopt this convention. The following are standard results about the coproduct of two frames, which appear in [37], and can be checked using its concrete description in terms of downsets.

Proposition 2.1.4. *For any two frames L^+ and L^-, the following facts hold in their coproduct $L^+ \oplus L^-$.*

- $\ulcorner x^+ \urcorner \leq \ulcorner x^- \urcorner$ *if and only if either* $x^+ = 0^+$ *or* $x^- = 1^-$.
- *If* $x^- \neq 0^-$ *then* $\ulcorner x^+ \wedge x^- \urcorner \leq \ulcorner y^+ \urcorner$ *if and only if* $x^+ \leq y^+$.
- *If* $y^- \neq 1^-$ *then* $\ulcorner x^+ \urcorner \leq \ulcorner y^+ \vee y^- \urcorner$ *if and only if* $x^+ \leq y^+$.
- *If* $x^\pm \neq 0^\pm$ *and* $y^\pm \neq 1^\pm$ *then* $\ulcorner x^+ \wedge x^- \urcorner \leq \ulcorner y^+ \vee y^- \urcorner$ *if and only if either* $x^+ \leq y^+$ *or* $x^- \leq y^-$.
- $\ulcorner \bigwedge_i a_i^+ \urcorner = \bigwedge_i \ulcorner a_i^+ \urcorner$.
- $(\bigwedge_i \ulcorner a_i^+ \urcorner) \vee \ulcorner a^- \urcorner = \bigwedge_i (\ulcorner a^+ \urcorner \wedge \ulcorner a^- \urcorner)$.

We will use the facts listed above without mention. Let us move onto finding a poset isomorphic to $\mathsf{tot}(L^+ \times L^-)$, reasoning in terms of free constructions. We begin by noting that, analogously to con subsets, tot subsets of $L^+ \times L^-$ too are a sublocale of $\mathcal{U}(L^+ \times L^-)$.

Lemma 2.1.5. *The collection $\mathsf{tot}(L^+ \times L^-) \subseteq \mathcal{U}(L^+ \times L^-)$ is a sublocale.*

Proof. This argument is the finitary version of the argument for 2.1.2, dualized. □

Lemma 2.1.6. *For two distributive lattices L^+ and L^-, we have an isomorphism of frames*

$$\mathsf{tot}(L^+ \times L^-) \cong \mathbf{Frm}\langle L^{+op} \cup L^{-op} \text{ qua } DL\rangle.$$

The canonical map from the generators is an embedding $L^{+op} \hookrightarrow \mathsf{tot}(L^+ \times L^-)$ and is defined as $a^+ \mapsto t_m \cup \uparrow(a^+, 0^-)$.

Proof. By Lemma 2.1.1, the frame $\mathbf{Frm}\langle L^{+op}, L^{-op} \text{ qua DL}\rangle$ is the quotient of $\mathcal{U}(L^+ \times L^-)$ making the canonical embeddings of generators $a^+ \mapsto \uparrow(a^+, 0^-)$ into finite join preserving maps. By an argument completely analogous to that in the proof of Proposition 2.1.3, the sublocale corresponding to this quotient consists of all the upsets closed under finite homogeneous meets, that is $\mathsf{tot}(L^+ \times L^-)$. Analogously as in 2.1.3, one shows that the generators are embedded into $\mathsf{tot}(L^+ \times L^-)$ as $a^+ \mapsto t_m \cup \uparrow(a^+, 0^-)$. □

Proposition 2.1.7. *For any two frames L^+ and L^-, we have an isomorphism of frames*

$$\mathsf{tot}(L^+ \times L^-) \cong \mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-).$$

Concretely, this isomorphism is defined as $t \mapsto \bigvee\{\ulcorner\uparrow x^+\urcorner \wedge \ulcorner\uparrow x^-\urcorner : (x^+, x^-) \in t\}$.

Proof. We begin by observing that, by the universal properties of the coproduct and of the ideal completion, the frame $\mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-)$ is isomorphic to $\mathbf{Frm}\langle L^{+op}, l^{-op} \text{ qua DL}\rangle$; and the generators are embedded in it as elements of the form $\ulcorner\uparrow x^+\urcorner$ and $\ulcorner\uparrow x^-\urcorner$. Then this structure, by Lemma 2.1.6, must also be isomorphic to $\mathsf{tot}(L^+ \times L^-)$. Let us describe the isomorphism linking these two structures explicitly. The isomorphism maps generators to generators, and must be a frame map. Then it must act on generators as $t_m \cup \uparrow(a^+, 0^-) \mapsto \ulcorner\uparrow a^+\urcorner$, and it maps tot subsets of the from $t_m \cup \uparrow(a^+, a^-)$ to $\ulcorner\uparrow a^+\urcorner \wedge \ulcorner\uparrow a^-\urcorner$. Finally, it must map an arbitrary tot subset $t = t_m \cup \bigcup\{\uparrow(x^+, x^-) : (x^+, x^-) \in t\}$ to the element $\bigvee\{\ulcorner\uparrow x^+\urcorner \wedge \ulcorner\uparrow x^-\urcorner : (x^+, x^-) \in t\}$. □

For two frames L^+ and L^-, define an element of $L^+ \oplus L^-$ to be *finitary* if it is a syntactic expression of the form $\bigvee_{m\leq n} \ulcorner x_m^+\urcorner \wedge \ulcorner x_m^-\urcorner$. We denote the collection of finitary elements, with their inherited order, as $\mathsf{fin}(L^+ \oplus L^-)$. All elements of a frame coproduct are directed joins of finitary elements. Define a filter of $L^+ \oplus L^-$ to be *finitary* if it is $\bigcup_i \uparrow x_i$ with each x_i a finitary element. We denote as $\mathsf{filt}(L^+ \oplus L^-)$ the collection of finitary filters of $L^+ \oplus L^-$, ordered by set inclusion. Let us note that the collection of finitary filter is a subframe of that of all the filters.

Lemma 2.1.8. *For any two frames L^+ and L^-, there is an isomorphism $\mathsf{fin}(L^+ \oplus L^-) \cong L^+ \oplus^D L^-$, the coproduct of L^+ and L^- in the category of distributive lattices.*

Proof. It is clear that the embeddings of generators into this structure preserve finite meets and joins, since these are computed in $\mathsf{fin}(L^+ \oplus L^-)$ as in $L^+ \oplus L^-$. For any two lattice maps $f^+ : L^+ \to D$ and $f^- : L^- \to D$ to a distributive lattice D, define the map $\mathsf{fin}(L^+ \oplus L^-) \to D$ as $\bigvee_{m \leq n} \ulcorner a^+_m \urcorner \wedge \ulcorner a^-_m \urcorner \mapsto \bigvee_{m \leq n} f^+(a^+_m) \wedge f^-(a^-_m)$. A few calculations, by using the distributivity laws inherited from the frame $L^+ \oplus L^-$, show that this assignments preserves lattice operations. Finally, it is also clear that this map restricts to generators as desired. □

Reasoning in terms of finitary elements enables us to find a concrete description of the coproduct $\mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-)$, and ultimately also an additional, more concrete, characterization of the poset $\mathsf{tot}(L^+ \times L^-)$.

Lemma 2.1.9. *We have isomorphisms of frames $\mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-) \cong \mathsf{Filt}(\mathsf{fin}(L^+ \oplus L^-)) \cong \mathsf{filt}(L^+ \oplus L^-)$.*

Proof. The functor $\mathsf{Idl} : \mathbf{Distr} \to \mathbf{Frm}$ is a left adjoint, and so it preserves colimits. Then $\mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-) \cong \mathsf{Filt}(L^+ \oplus^D L^-)$. Explicitly, this isomorphism is the frame map determined by the assignment $\ulcorner \uparrow x^+ \urcorner \mapsto \uparrow \ulcorner x^+ \urcorner$ and the analogous one on the negative component. There also is an isomorphism $\mathsf{Filt}(\mathsf{fin}(L^+ \oplus L^-)) \cong \mathsf{filt}(L^+ \oplus L^-)$, mapping $F \mapsto \uparrow \{\ulcorner f \urcorner : f \in F\}$. Its inverse assigns to a finitary filter the collection of finitary elements contained in it. □

Proposition 2.1.10. *For any two frames L^+ and L^-, we have an isomorphism $\mathsf{tot}(L^+ \times L^-) \cong \mathsf{filt}(L^+ \oplus L^-)$, mapping each $t \in \mathsf{tot}(L^+ \times L^-)$ to the filter generated by $\{\ulcorner x^+ \urcorner \vee \ulcorner x^- \urcorner : (x^+, x^-) \in t\}$.*

Proof. To see this, it suffices to compose the isomorphism in Proposition 2.1.7 with the isomorphisms of Lemma 2.1.9 above. □

2.2 Consistency and totality as closed and fitted sublocales

When constructing the patch of a d-frame $\mathcal{L}$, its con subset determines a closed sublocale of $L^+ \oplus L^-$, while its tot subset determines a fitted one. In this section, for two frames L^+ and L^- we will construct an assignment from arbitrary con subsets and arbitrary tot subsets of $L^+ \times L^-$ to sublocales of $L^+ \oplus L^-$ and analyze how strict this correspondence is.

For any frame L let us denote as $\mathsf{Cl}(L)$ the collection of all closed sublocales of L ordered under sublocale inclusion, and let us denote as $\mathsf{Fitt}(L)$ the ordered collection of fitted sublocales. Recall that both $\mathsf{CL}(L)$ and $\mathsf{Fitt}(L)$ are subcoframes of the coframe $\mathsf{S}(L)$ of sublocales.

Proposition 2.2.1. *The assignment* $c \mapsto \mathfrak{c}(\bigvee\{\ulcorner x^+\urcorner \wedge \ulcorner x^-\urcorner : (x^+, x^-) \in c\})$ *is an isomorphism* $\mathsf{con}(L^+ \times L^-) \cong \mathsf{Cl}(L^+ \oplus L^-)^{op}$.

Proof. In light of Proposition 2.1.3, we may identify each con subset with the element of the coproduct it determines. In particular, elements of the form $c_m \cup {\downarrow}(a^+, 1^-)$ correspond to generators $\ulcorner a^+\urcorner$, and each element of the form $c_m \cup {\downarrow}(a^+, a^-)$ corresponds to $\ulcorner a^+\urcorner \wedge \ulcorner a^-\urcorner$. A generic con subset $c = \bigcup\{{\downarrow}(x^+, x^-) : (x^+, x^-) \in c\}$ then corresponds to $\bigvee\{\ulcorner x^+\urcorner \wedge \ulcorner x^-\urcorner : (x^+, x^-) \in c\}$. The statement of the lemma then amounts to the assignment $x \mapsto \mathfrak{c}(x)$ from $L^+ \oplus L^-$ to its closed sublocales being an anti-isomorphism. This is a fact that holds for every frame. □

Define a fitted sublocale of a frame coproduct $L^+ \oplus L^-$ to be *finitary* if it is of the form $\bigcap_i \mathfrak{o}(x_i^+ \vee x_i^-)$. We denote the ordered collection of finitary fitted sublocales of L as $\mathsf{fitt}(L)$. A fitted sublocale of $L^+ \oplus L^-$ is finitary if and only if it is $\bigcap_{f \in F} \mathfrak{o}(f)$ for some finitary filter F. Finitary fitted sublocales form a subcoframe of $\mathsf{Fitt}(L^+ \oplus L^-)$.

Lemma 2.2.2. *For any frame* L*, the assignment* $fit : \mathsf{Filt}(L) \to \mathsf{Fitt}(L)^{op}$ *with* $F \mapsto \bigcap_{f \in F} \mathfrak{o}(f)$ *is a frame surjection.*

Proof. Let us observe that $fit(F \cap G) = \bigcap\{\mathfrak{o}(f \vee g) : f \in F, g \in G\}$ for any two filters $F, G \subseteq L$, using the coframe distributivity law of $\mathsf{S}(L)$ this is seen to be $fit(F) \vee fit(G)$. For preservation of arbitrary joins, we notice that, $fit(\bigvee_i F_i) = \bigcap\{\mathfrak{o}(x) : x \in \bigcup_i F_i\}$ and this is $\bigcap_i fit(F_i)$. □

Proposition 2.2.3. *The assignment* $t \mapsto \bigcap\{\mathfrak{o}(x^+ \vee x^-) : (x^+, x^-) \in t\}$ *is a frame surjection* $\mathsf{tot}(L^+ \times L^-) \twoheadrightarrow \mathsf{fitt}(L^+ \oplus L^-)^{op}$.

Proof. Recall that by Proposition 2.1.10 there is an order isomorphism between tot subsets of $L^+ \times L^-$ and finitary filters of $L^+ \oplus L^-$. For every tot subset t let $f(t)$ be the corresponding finitary filter $\uparrow\{\ulcorner x^+\urcorner \vee \ulcorner x^-\urcorner : (x^+, x^-) \in t\}$. The assignment described in the statement of the lemma then amounts to $t \mapsto \bigcap\{\mathfrak{o}x^+ \vee x^- : x^+ \vee x^- \in f(t)\} = \bigcap_{x \in f(t)} \mathfrak{o}(x)$. This assignment preserves arbitrary joins and finite meets, by Lemma 2.2.2. □

2.2.1 Filters and fitted sublocales

The fact that distinct finitary filters may determine the same fitted sublocale is a manifestation of the fact that we do not, in general, have an isomorphic correspondence between filters of a frame and fitted sublocales of its frame. Let us go on a small digression on the relation between filters and fitted sublocales of a frame.

Lemma 2.2.4. *There is an adjunction* $fit : \mathsf{Filt}(L) \leftrightarrows \mathsf{S}(L)^{op} : top$ *with* $fit \dashv top$*, with* $fit(F) = \bigcap_{f \in F} \mathfrak{o}(f)$ *and* $top(M) = \{x \in L : M \subseteq \mathfrak{o}(x)\}$*. The composition* $top \circ fit$ *is a nucleus on* $\mathsf{Filt}(L)$*.*

Proof. For the adjunction, we notice that $F \subseteq top(S)$ if and only if $S \subseteq \mathfrak{o}(f)$ whenever $f \in F$, and that this holds if and only if $S \subseteq \bigcap_{f \in F} \mathfrak{o}(f)$. To see that the closure operator $top \circ fit$ is a nucleus, we recall that the map *fit* preserves finite meets, and that *top* preserves finite meets of $\mathsf{S}(L)^{op}$ since it is a right adjoint. Then $top \circ fit$ preserves finite meets. □

The adjunction $top \dashv fit$ restricts to an equivalence between Scott-open filters and compact fitted sublocales. This well-known result, which is proven in [27], is usually regarded as a pointfree version of the Hofmann-Mislove Theorem. We will provide our own proof of this fact. A sublocale $S \subseteq L$ of a frame L is *compact* if whenever $S \subseteq \bigvee_i \mathfrak{o}(x_i)$, with $\{x_i : i \in I\}$ directed as a subposet of $\mathsf{S}(L)$, there is some $i \in I$ with $S \subseteq \mathfrak{o}(x_i)$. In our proof we will also use the concept of strongly exact meet, as defined in [6]. For a complete lattice L we say that a meet $\bigwedge_i x_i$ is *exact* if $\bigwedge_i x_i \vee y = \bigwedge_i (x_i \vee y)$ for all $y \in L$. Exact meets $\bigwedge_i x_i$ are characterized by the fact that $\bigvee_i \mathfrak{c}(x_i) = \mathfrak{c}(\bigwedge_i x_i)$. We also say that a meet $\bigwedge_i x_i \in L$ is *strongly exact* if and only if $x_i \to y = y$ for all $i \in I$ implies that $\bigwedge_i x_i \to y = y$, for all $y \in L$. Strongly exact meets $\bigwedge_i x_i$ are characterized by the fact that $\bigcap_i \mathfrak{o}(x_i) = \mathfrak{o}(\bigwedge_i x_i)$. Exact and strongly exact meets are explored and compared in detail in [6], where the characterizations that we have given are proved. The connection between strongly exact meets and the adjunction $fit \dashv top$ is explored in [35], where the following result is also proved. We call filters that are closed under strongly exact meets simply *strongly exact* filters.

Lemma 2.2.5. *For any frame* L*, a filter* $F \subseteq L$ *is a fixpoint of the nucleus* $top \circ fit$ *if and only if it is closed under strongly exact meets.*

Proof. First, let us note that a meet $\bigwedge_i x_i$ is strongly exact if and only if $\bigcap_i \mathfrak{o}(x_i) \subseteq \mathfrak{o}(\bigwedge_i x_i)$. That these two conditions are equivalent follows immediately from the fact that the membership condition $a \in \mathfrak{o}(b)$ amounts to $a = b \to a$. Suppose that F is a filter such that $F = top(fit(F))$. If $x_i \in F$ and the meet $\bigwedge_i x_i$ is strongly exact, then $fit(F) \subseteq \bigcap_i \mathfrak{o}(x_i) = \mathfrak{o}(\bigwedge_i x_i)$, and by assumption on F this implies $\bigwedge_i x_i \in F$. Conversely, suppose that F is a filter closed under strongly exact meets, and let $fit(F) \subseteq \mathfrak{o}(x)$, that is, $\bigcap_{f \in F} \mathfrak{o}(f \vee x) = \mathfrak{o}(x)$. Consider the collection $x \vee F := \{x \vee f : f \in F\}$. This is such that $x \leq \bigwedge(x \vee F)$ and this, combined with the previous result, gives that $\bigcap_{f \in F} \mathfrak{o}(f \vee x) = \mathfrak{o}(x) \subseteq \mathfrak{o}(\bigwedge(x \vee F))$. Then the meet $\bigwedge(x \vee F)$ must be strongly exact, and additionally we must have that this meet is x. By assumption on F, then, $x \in F$, and so $F = top(fit(F))$. □

Corollary 2.2.6. *If L is a frame and $F \subseteq L$ is a strongly exact filter, we have that $\bigcap_{f \in F} \mathfrak{o}(f) \subseteq \mathfrak{o}(x)$ implies $x \in F$ for any $x \in L$.*

Lemma 2.2.7. *All Scott-open filters of a frame L are closed under strongly exact meets.*

Proof. Note that $x \to y = y$ if and only if there is no element $z > y$ such that $z \wedge x \leq y$. Suppose that F is a Scott-open filter of a frame L, and that $x \notin F$. Our argument will show that x cannot be a strongly exact meet of elements in F. First, we claim that there must be some $y \in L$ such that $x \leq y$ with $y \notin F$ and such that $y < f$ implies that $f \in F$. If this were not the case, then we could find an infinite ascending chain $x \leq x_1 < ... < x_n < ...$, none of whose elements is in F. This would contradict Scott-openness of F. Then, let $y \in L$ be such an element, and in particular, let $\{y_k : k \in K\} = \uparrow y \cap F = \{z \in L : y < z\}$. Suppose that $x = \bigwedge_i x_i$ with $x_i \in F$, we will now show that the meet cannot be strongly exact. We must have $x_i \to y = y$ for all $i \in I$: this is because if there were some y_k with $y_k \wedge x_i \leq y$, we would obtain $y \in F$ which, by assumption on y, is false. On the other hand, since $x \leq y$, we have $x \to y = 1 \neq y$. Then, x can never be an exact meet of elements in F. □

Proposition 2.2.8. *The assignment*

$$\mathit{fit} : \mathsf{Filt}(L) \to \mathsf{Fitt}(L)$$

determines an anti-isomorphism between Scott-open filters and compact fitted sublocales, for any frame L.

Proof. By Lemma 2.2.7, the adjunction $\mathit{top} \dashv \mathit{fit}$ restricts to an isomorphism between Scott-open filters and the fitted sublocales they induce. It remains to show that these are the compact ones. It suffices to notice that a collection $\{x_i : i \in I\}$ being directed is equivalent to the collection of the open sublocales $\{\mathfrak{o}(x_i) : i \in I\}$ being directed in $\mathsf{S}(L)$. For a filter F closed under strongly exact meets, we additionally have that the condition $\mathit{fit}(F) \subseteq \mathfrak{o}(x)$ is equivalent to $x \in F$. Then, the filter F being Scott-open is equivalent to the sublocale $\mathit{fit}(F)$ being compact. □

2.2.2 Compact totality subsets

In light of these results on compactness, it is worth specializing our analysis to the case of compact tot subsets. Recall that for two frames L^+, L^-, a tot subset $t \subseteq L^+ \times L^-$ is *compact* if it is inaccessible by directed joins. For the next result, we will use a pointfree version of Alexander's Subbase Lemma, which we prove below. Our proof is a pointfree adaptation of

the classical proof for point-set spaces, which can be found, for instance, in [20]. The lemma, as its point-set version, uses Zorn's lemma.

Lemma 2.2.9. *Suppose that L is a frame generated by subframes L^+ and L^-. If a filter F is inaccessible by directed joins of families of the form $\{x_i^+ \vee x_i^- : i \in I\}$ with $x_i^+ \in L^+$ and $x_i^- \in L^-$, then the filter is Scott-open.*

Proof. Let $\mathcal{S}$ be the collection of all directed families $S \subseteq L$ such that $\bigvee S \in F$ but $s \notin F$ for each $s \in S$, ordered under set inclusion. For $S_i \in \mathcal{S}$ a chain, notice that $\bigcup_i {\downarrow} S_i$ is a directed union of ideals, and so it is itself an ideal, in particular it is directed. Additionally, $\bigvee \bigcup_i {\downarrow} S_i \in F$, and we cannot have $s \in F$ for any $s \in {\downarrow} S_i$. Then, $\bigcup_i {\downarrow} S_i \in \mathcal{S}$, and the poset $\mathcal{S}$ is such that every chain has an upper bound. By Zorn's Lemma, then, there is a family $\{x_i : i \in I\} \in \mathcal{S}$, which is maximal in the sense that whenever $y \neq 0$ with $y \neq x_i$, the join $y \vee \bigvee_i x_i$ is such that there is some $y \vee x_i \in F$. By maximality, this family is downward closed, and so it is closed under finite joins. Let $J \subseteq I$ be the collection of those indices j such that $x_j = x_j^+ \vee x_j^-$. The first claim we prove is that we have $\bigvee_{j \in J} x_j^+ \vee x_j^- \notin F$. The collection $x_j^+ \vee x_j^-$ is still directed, since it inherits from x_i the property of being closed under finite joins. Then we cannot have $\bigvee_j x_j^+ \vee x_j^- \in F$, or by assumption we would have $x_j^+ \vee x_j^- \in F$ for some $j \in J$, contradicting the assumption that $x_i \notin F$ for all $i \in I$. Then, there must be some $i \in I$ with $x_i \not\leq \bigvee_{j \in J} x_j^+ \vee x_j^-$. Since all elements of the coproduct are joins of finitary elements, there must also be a finitary element $f = (x_1^+ \vee x_1^-) \wedge ... \wedge (x_n^+ \vee x_n^-)$ with $f \leq x_i$ but $f \not\leq \bigvee_{j \in J} x_j^+ \vee x_j^-$. Then, for all $m \leq n$ and all $i \in I$, we must have $x_m^+ \vee x_m^- \neq x_i$. This implies, by maximality of the family x_i, that for each $m \leq n$ there must be some $i_m \in I$ such that $x_m^+ \vee x_m^- \vee x_{i_m} \in F$. Then, $f \vee \bigvee_{m \leq n} x_{i_m} \in F$, but since by assumption $f \leq x_i$, we must also have that $x_i \vee \bigvee_{m \leq n} x_{i_m} \in F$. Since the family x_i was assumed to be closed under finite joins, we must then have $x_i \vee \bigvee_{m \leq n} x_{i_m} = x_k$ for some $k \in I$. We have then reached the conclusion that $x_i \vee x_k \in F \cap \{x_i : i \in I\}$, contradicting our initial assumption. □

Lemma 2.2.10. *A totality $t \in \mathsf{tot}(L^+ \times L^-)$ is compact if and only if the corresponding finitary filter is Scott-open.*

Proof. By the pointfree version of Alexander's Lemma, it suffices to show that t is a compact tot subset of $L^+ \times L^-$ if and only if its corresponding filter $F(t)$ is inaccessible by joins of directed families of the form $\{\ulcorner x_i^+ \urcorner \vee \ulcorner x_i^- \urcorner : i \in I\}$. We observe that, by the isomorphism of Proposition 2.1.10, the conditions $(x^+, x^-) \in t$ and $\ulcorner x^+ \urcorner \vee \ulcorner x^- \urcorner \in F(t)$ are equivalent. We also claim that a family (x_i^+, x_i^-) is directed in $L^+ \times L^-$ if and only if the family $\ulcorner x_i^+ \urcorner \vee \ulcorner x_i^- \urcorner$ is directed in $L^+ \oplus L^-$. One direction of the implication is clear. For the other, suppose

that $\ulcorner x_i^+ \urcorner \vee \ulcorner x_i^- \urcorner$ is directed. Then, for $j, k \in I$ we must have some $l \in L$ with $\ulcorner x_j^+ \vee x_k^+ \urcorner \vee \ulcorner x_j^- \vee x_k^- \urcorner \leq \ulcorner x_l^+ \urcorner \vee \ulcorner x_l^- \urcorner$. By construction of the coproduct of frames, this implies that both $x_j^+ \vee x_k^+ \leq x_l^+$ and $x_j^- \vee x_k^- \leq x_l^-$, in other words the pair (x_l^+, x_l^-) is an upper bound for the pairs (x_j^+, x_j^-) and (x_k^+, x_k^-). Indeed, then, t being inaccessible by directed joins is equivalent to $F(t)$ being inaccessible by directed joins of families of the form $\ulcorner x_i^+ \urcorner \vee \ulcorner x_i^- \urcorner$, that is, to F being Scott-open. □

Proposition 2.2.11. *The assignment* $t \mapsto \bigcap\{\mathfrak{o}(x^+ \vee x^-) : (x^+, x^-) \in t\}$ *determines an isomorphism between compact* tot *subsets of* $L^+ \times L^-$ *and compact fitted sublocales of* $L^+ \oplus L^-$ *for any two frames* L^+ *and* L^-.

Proof. By Proposition 2.2.8, the adjunction $\mathit{fit} : \mathsf{Filt}(L^+ \oplus L^-) \leftrightarrows \mathsf{Fitt}(L^+ \oplus L^-) : \mathit{top}$ restricts to an isomorphism Scott-open filters and compact fitted sublocales. Let t be a totality subset of $L^+ \times L^-$, and let $F(t)$ be the corresponding finitary filter. We have observed in the proof of Proposition 2.2.3 that the sublocale $\bigcap\{\mathfrak{o}(\ulcorner x^+ \urcorner \vee \ulcorner x^- \urcorner) : (x^+, x^-) \in t\}$ coincides with $\mathit{fit}(F(t))$. Then, the map $\mathit{fit} \circ F$ determines an isomorphism between compact tot subsets and compact fitted sublocales of $L^+ \oplus L^-$. □

2.3 Consistency and totality as Galois connections

For any two posets P, Q, a *Galois connection* is a contravariant adjunction of posets $f : P \leftrightarrows Q : g$. In other words, it is a pair of antitone maps $f : P \leftrightarrows Q : g$ such that $q \leq f(p)$ if and only if $p \leq g(q)$ for all $p \in P$ and all $q \in Q$. We are particularly interested in Galois connections between frames. This section and the next, in their entirety, can be seen as an analysis of the interaction between the Galois connections in the following two examples.

Example 2.3.1. For any two frames L^+ and L^-, a con subset of $L^+ \times L^-$ determines pseudocomplementation maps $\sim^+ \colon L^+ \leftrightarrows L^- : \sim^-$. These always form a Galois connection (i.e. a contravariant adjunction of posets) between L^+ and L^-, since for a d-frame (L^+, L^-, c, t) clearly we have equivalences:

$$\frac{\dfrac{a^+ \leq \sim^- a^-}{a^- \leq \sim^+ a^+}}{(a^+, a^-) \in c}$$

Example 2.3.2. For any frame L, there is a Galois connection $\uparrow : L \leftrightarrows \mathsf{Filt}(L) : \bigwedge$. Indeed, we have an equivalence:

$$\frac{F \subseteq \uparrow x}{x \leq \bigwedge F}$$

For two frames L^+, L^-, let $\mathsf{Gal}(L^+, L^-)$ be the collection of Galois connections between L^+ and L^-, equipped with the pointwise order on the left components. It is known that this is a frame, and it has the universal property of $L^+ \oplus L^-$; this was proven by Wigner in [54]. There, the author identifies Galois connection between frames with join reversing maps between them. Since the coproduct of two frames is isomorphic to $\mathsf{con}(L^+ \times L^-)$, this means that there is an isomorphism $\mathsf{con}(L^+ \times L^-) \cong \mathsf{Gal}(L^+, L^-)$. Explicitly, this isomorphism is the assignment that we have seen in Example 2.3.1, i.e. the assignment to each con of its pseudocomplementation maps. Let us see an explicit proof that this is an isomorphism.

Lemma 2.3.3. *There is an isomorphism of frames $pc : \mathsf{con}(L^+ \times L^-) \to \mathsf{Gal}(L^+, L^-)$, assigning to each con subset its pseudocomplementation map.*

Proof. The inverse of the assignment acts as $(f, g) \mapsto \downarrow\{(x^+, f(x^+) : x^+ \in L^+)\} = con(f, g)$. Every con subset is completely determined by its pseudocomplementation maps, so the composition $con \circ pc$ is the identity. We claim $(y^+, y^-) \in con(f, g)$ if and only if $y^- \leq f(y^+)$. Using the fact that (f, g) is a contravariant adjunction, we obtain equivalences:

$$\frac{\dfrac{\dfrac{\dfrac{(y^+, y^-) \in con(f, g)}{(y^+, y^-) \leq (x^+, f(x^+)) \text{ for some } (x^+, x^-)}}{y^+ \leq x^+, x^+ \leq g(y^-) \text{ for some } (x^+, x^-)}}{y^+ \leq g(y^-)}}{y^- \leq f(y^+)}$$

This implies both that the collection is closed under arbitrary homogeneous joins, and that (f, g) are the pseudocomplementation maps of $con(f, g)$. Then, $pc \circ con$ is the identity. □

Let us also see explicitly the isomorphism between tot subsets of $L^+ \times L^-$ and Galois connections between $\mathsf{Filt}(L^+)$ and $\mathsf{Filt}(L^-)$.

Lemma 2.3.4. *There is an order-isomorphism $T_{(-)} : \mathsf{tot}(L^+ \times L^-) \to \mathsf{Gal}(\mathsf{Filt}(L^+), \mathsf{Filt}(L^-))$, defined on generators as $T_t(\uparrow x^+) = \{x^- \in L^- : (x^+, x^-) \in t\}$.*

Proof. We start by showing that the assignment we have defined is a Galois connection. We have equivalences

$$\dfrac{\dfrac{\dfrac{\dfrac{F^+ \subseteq T_t(F^-)}{F^+ \subseteq \bigcap_{f^- \in F^-} T_t(\uparrow f^-)}}{(f^+, f^-) \text{ for all } (f^+, f^-) \in F \times F}}{F^- \subseteq \bigcap_{f^+ \in F^+} T_t(\uparrow f^+)}}{F^- \subseteq T_t(F^+)}$$

The inverse *tot* of this assignment assigns to a Galois connection (S, T) the collection $\{(x^+, x^-) \in L^+ \times L^- : x^- \in S(\uparrow x^+)\}$. Every tot subset t is completely determined by the collection of filters $\{T_t(\uparrow x^+) : x^+ \in L^+\}$, and so the composition $tot \circ T_{(-)}$ is the identity. For a Galois connection (S, T) between $\mathsf{Filt}(L^+)$ and $\mathsf{Filt}(L^-)$, unravelling definitions we obtain $T_{tot(S,T)}(\uparrow x^+) = \{x^- \in L^- : x^- \in S(\uparrow x^+)\}$, and so the composition $T_{(-)} \circ tot$ is the identity. □

Both con subsets and tot subsets are frames of Galois connections, which are themselves linked by the Galois connection $(\uparrow, \bigwedge)$ as depicted in the diagram below. For any consistency with pseudocomplementation maps $(\sim^+, \sim^-)$, and for every totality t inducing the Galois connection (F_t^+, F_t^-), we may consider the following square.

$$\begin{array}{ccc} L^+ & \underset{\sim^-}{\overset{\sim^+}{\rightleftarrows}} & L^{-op} \\ {\scriptstyle\uparrow^+}\downarrow\uparrow{\scriptstyle\bigwedge^+} & & {\scriptstyle\bigwedge^-}\uparrow\downarrow{\scriptstyle\uparrow^-} \\ \mathsf{Filt}(L^+)^{op} & \underset{F_t^-}{\overset{F_t^+}{\rightleftarrows}} & \mathsf{Filt}(L^-) \end{array}$$

In looking at the square above, one may wonder whether there is a precise sense in which the two pairs of maps $(\uparrow, \bigwedge)$ induce a Galois connection between $\mathsf{Gal}(L^+, L^-)$ and $\mathsf{Gal}(\mathsf{Filt}(L^+), \mathsf{Filt}(L^-))$. This amounts to asking whether there is a natural way in which the Galois connections $(\uparrow, \bigwedge)$ induce a consistency subset of $(L^+ \oplus L^-) \times (\mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-))$. This is indeed the case, and we will explore this consistency subset, as well as a totality subset defined dually to this, in the following section.

2.4 The d-frame of all cons and tots

In this section it will be shown that there is a natural Galois connection between the two frames $L^+ \oplus L^-$ and $\mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-)$, induced by the Galois connection $(\uparrow, \bigwedge)$. There also exists a natural totality relation between these frames, and it is defined dually with respect to this consistency subset. Furthermore, as we will see, these two consistency and totality

subsets are balanced, meaning that for any two frames L^+ and L^- there is a d-frame of the form $(\mathsf{con}(L^+ \times L^-), \mathsf{tot}(L^+ \times L^-), c, t)$. Defining this d-frame explicitly is the aim of this section.

Lemma 2.4.1. *There is a consistency subset of* $(L^+ \oplus L^-) \times \mathsf{filt}(L^+ \oplus L^-)$ *defined as* $(x, F) \in \mathsf{Con}$ *if and only if* $x \leq \bigwedge F$. *There is also a totality subset, defined as* $(x, F) \in \mathsf{Tot}$ *if and only if* $x \in F$.

Proof. First, we remind the reader that finitary filters are a subframe of the collections of all filter, and so their finite meets are computed as set-theoretical meets, and their joins as the closure under finite meets of their set-theoretical union. The subset Con is a downset, since for $(y, G) \leq (x, F) \in \mathsf{Con}$ we have the chain of inequalities $y \leq x \leq \bigwedge F \leq \bigwedge G$. It is clearly closed under arbitrary homogeneous joins on the left. For homogeneous joins on the right, notice that in general $\bigwedge S = \bigwedge\{x_1 \wedge ... \wedge x_n : x_1, ..., x_n \in S\}$, so whenever $x \leq \bigwedge F_i$ for F_i a collection of filters, we have that $x \leq \bigwedge_i \bigwedge F_i = \bigwedge \bigcup_i F_i = \bigwedge(\bigvee_i F_i)$. The Tot subset is clearly an upset. Since filters are closed under finite meets, it is also closed under finite meets on the left. Closure under homogeneous finite meets on the right is immediate, since intersections of finitary filters are computed as set-theoretical intersections. □

Remark 2.4.2. Note that the consistency Con and the totality Tot are defined dually to one another, in the sense that $(x, F) \in \mathsf{Con}$ if and only if $F \subseteq {\uparrow}x$, and $(x, F) \in \mathsf{Tot}$ if and only if ${\uparrow}x \subseteq F$. In general, for a frame L and its filter completion $\mathsf{Filt}(L)$, there is a totality subset of $L \times \mathsf{Filt}(L)$ defined in terms of the Galois connection $(\uparrow, \bigwedge)$. This totality subset is, in fact, $\{(x, F) : {\uparrow}x \subseteq F\}$. It is not the case, however, that this procedure yields a totality subset of $L \times M$ when applied to any Galois connection $f : L \leftrightarrows M : g$. In particular, the set $\{(x, y) \in L \times M : f(x) \leq y\}$ is always an upset, but is only closed under finite homogeneous meets if f turns finite meets into joins.

Proposition 2.4.3. *The quadruple*

$$(L^+ \oplus L^-, \mathsf{filt}(L^+ \oplus L^-), \{(x, F) : x \leq \bigwedge F\}, \{(x, F) : x \in F\})$$

is a d-frame.

Proof. It remains to show that consistency and totality are balanced. For balance on the right notice that $x \leq \bigwedge F$ and $x \in G$ imply that the chain of inequalities $F \subseteq {\uparrow}x \subseteq G$ holds. For balance on the left, notice that $x \leq \bigwedge F$ and $y \in F$ imply that we have the chain of inequalities $x \leq \bigwedge F \leq y$. □

The d-frame $(L^+ \oplus L^-, \mathsf{filt}(L^+ \oplus L^-), \mathsf{Con}, \mathsf{Tot})$ is such that the elements in all pairs of the form $(\ulcorner x^+ \urcorner, \uparrow \ulcorner x^+ \urcorner)$ and of the form $(\ulcorner x^- \urcorner, \uparrow \ulcorner x^- \urcorner)$ are mutual d-complements. In fact, Con and Tot are the smallest possible consistency and totality on these two frames such that these are pairs of mutual d-complements. This enables us to describe the d-frame as a quotient.

Proposition 2.4.4. *Given two frames L^+ and L^-, let $\mathcal{C}$ be the collection $\{(\ulcorner x^+ \urcorner, \uparrow \ulcorner x^+ \urcorner) : \ulcorner x^+ \urcorner \in \ulcorner L^+ \urcorner\} \cup \{(\ulcorner x^- \urcorner, \uparrow \ulcorner x^- \urcorner) : \ulcorner x^- \urcorner \in \ulcorner L^- \urcorner\}$. The d-frame $(L^+ \oplus L^-, \mathsf{filt}(L^+ \oplus L^-), \mathsf{Con}, \mathsf{Tot})$ is the quotient of the d-frame $(L^+ \oplus L^-, \mathsf{filt}(L^+ \oplus L^-), c_m, t_m)$ by the relation structure*

$$(\mathsf{id}^+, \mathsf{id}^-, \mathcal{C}, \mathcal{C}).$$

Proof. Let $C(\mathcal{C})$ and $T(\mathcal{C})$ be the minimal consistency and totality subsets with making the elements in $\mathcal{C}$ into pairs of mutual d-complements. The claim we wish to prove amounts to the inequalities $\mathsf{Con} \subseteq C(\mathcal{C})$ and $\mathsf{Tot} \subseteq T(\mathcal{C})$. For the first inequality, suppose that $x \leq \bigwedge F$. Then, whenever $\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner \leq x$, we must have that $\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner \leq \bigwedge F$, that is $F \subseteq \uparrow \ulcorner x^+ \urcorner \vee \uparrow \ulcorner x^- \urcorner$. Additionally, $(\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner, \uparrow \ulcorner x^+ \urcorner \vee \uparrow \ulcorner x^+ \urcorner) \in C(\mathcal{C})$ and since this is a downset we must also have $(\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner, F) \in C(\mathcal{C})$, and since it is closed under homogeneous joins, indeed $(x, F) \in C(\mathcal{C})$. For the second inequality, suppose that $x \in F$. Since F is finitary, this implies that there is a finite collection $\ulcorner x_n^+ \urcorner \vee \ulcorner x_n^- \urcorner \in F$ such that their intersection is below x. Then $\uparrow(\ulcorner x_n^+ \urcorner \wedge \ulcorner x_n^- \urcorner) \subseteq F$, and since $(\ulcorner x_n^+ \urcorner \vee \ulcorner x_n^- \urcorner, \uparrow(\ulcorner x_n^+ \urcorner \wedge \ulcorner x_n^- \urcorner)) \in T(\mathcal{C})$ and $T(\mathcal{C})$ is an upset, we must have also $(\ulcorner x_n^+ \urcorner \vee \ulcorner x_n^- \urcorner, F) \in T(\mathcal{C})$. Since $T(\mathcal{C})$ is also closed under homogeneous finite meets, this implies $(x, F) \in T(\mathcal{C})$. □

From the fact that the isomorphism $\mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-) \cong \mathsf{filt}(L^+ \oplus L^-)$ acts on generators as $\uparrow \ulcorner x^+ \urcorner \mapsto \ulcorner \uparrow x^+ \urcorner$, we may obtain an alternative way of describing this d-frame.

Proposition 2.4.5. *The d-frame $(L^+ \oplus L^-, \mathsf{filt}(L^+ \oplus L^-), \mathsf{Con}, \mathsf{Tot})$ is isomorphic to the quotient of*

$$(L^+ \oplus L^-, \mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-), c_m, c_m)$$

by the relation structure

$$(\mathsf{id}^+, \mathsf{id}^-, \mathcal{C}', \mathcal{C}'),$$

where $\mathcal{C}' = \{(\ulcorner x^+ \urcorner, \ulcorner \uparrow x^+ \urcorner) : x^+ \in L^+\} \cup \{(\ulcorner x^- \urcorner, \ulcorner \uparrow x^- \urcorner) : x^- \in L^-\}$.

We have seen that the two frame components of this d-frame are isomorphic to $\mathsf{con}(L^+ \times L^-)$ and $\mathsf{tot}(L^+ \times L^-)$, respectively. Let us then write $(\mathsf{con}(L^+ \times L^-), \mathsf{tot}(L^+ \times L^-), \mathsf{Con}, \mathsf{Tot})$ for the d-frame isomorphic to this d-frame. It is possible to obtain an explicit description of these Con and Tot subsets, but for now this is not necessary for our work. What we do obtain is the following

Lemma 2.4.6. *The d-frame* $(\mathsf{con}(L^+ \times L^-), \mathsf{tot}(L^+ \times L^-), \mathsf{Con}, \mathsf{Tot})$ *is such that* Con *and* Tot *are the smallest consistency and totality making the pairs* $(c_m \cup {\downarrow}\{(x^+, x^-)\}, t_m \cup {\uparrow}\{(x^+, 0^-)\} \cup \{(0^+, 1^-)\})$ *into mutual d-complements.*

A generalized version of this d-frame will be one of the central constructions in this thesis.

Chapter 3

Quotients of d-frames concretely: spatializations and Booleanizations

Introduction

We recall that the category of d-frames has quotients, and that we quotient d-frames by quadruples called *relation structures*. The existence of d-quotients is proved in [25], and a more detailed and explicit description of d-quotients is given in [22]. There, the author describes the concrete form of a d-quotient as the result of a transfinite process. Then, some sufficient conditions are given for this process to halt after only one step. In general, however, it is quite difficult to describe d-quotients concretely, and we very quickly lose control of this transfinite process. The motivation behind the title of this chapter is that the most significant results that we present here all stem from finding concrete descriptions of certain especially well-behaved kinds of d-quotients, or of their spectra.

A significant part of this thesis is concerned with exploring the relation between the ordered collection of point-set bisubspaces of some bispace, and that of pointfree bisubspaces of the corresponding pointfree bispace. In particular, we will explore the question of how well the pointfree bisubspaces represent the bisubspaces, but also of how often the *spatial* pointfree bispaces represent them. This question will also be explored in the setting of d-frames, and this is why we are particularly interested in analyzing the ordered collection of the spatial d-quotients of a d-frame. It is known (see [25]) that the ordered collection $\mathsf{S}(\mathcal{L})$ of d-quotients of a d-frame $\mathcal{L}$ is a complete lattice. We will see that the lattice of spatial d-quotients $\mathsf{sp}[\mathsf{S}(\mathcal{L})]$ is generated, in a "co-subbase" sense, by two pairs of collections. These are both subcoframes of $\mathsf{sp}[\mathsf{S}(\mathcal{L})]$. These two pairs of collections are (1) the spatializations of d-quotients coming

from increasing con and from tot; and (2) the spatializations of d-quotients coming from congruences on L^+ and on L^-. In this sense, then, the collection of all spatial d-quotients of a d-frame is bitopological in nature, in two ways.

In the second section *Booleanizations* will be explored in the bitopological setting. We introduce a Booleanization construction for d-frames, and analyze its properties. We will see that, in the case of *de Morgan* d-frames, there is a concrete description of the Booleanization of a d-frame, and that this coincides with its smallest dense d-quotient.

A note on terminology: from this chapter onward, we sometimes will refer to increases in the con or the tot components as "updates".

3.1 The lattice of spatial d-quotients and its generators

We begin by analyzing the order structure of the collection of spatial d-quotients of a d-frame $\mathcal{L}$ in relation to that of the ordered collection $\mathsf{S}(\mathcal{L})$ of all its d-quotients. In [25] it was shown that this is a complete lattice. We notice that for a d-frame $\mathcal{L}$ since the spatialization $\phi_{\mathcal{L}} : \mathcal{L} \to \mathsf{d}\Omega(\mathsf{dpt}(\mathcal{L}))$ is a surjection it must have a quotient map $q_{\phi_{\mathcal{L}}} : \mathcal{L} \to \mathcal{L}/\mathcal{R}_\phi$ isomorphic to it. Let us then define a map $\mathsf{sp} : \mathsf{S}(\mathcal{L}) \to \mathsf{S}(\mathcal{L})$ as assigning to each d-quotient $\mathcal{L}/\mathcal{R}$ the quotient corresponding to its spatialization, which we will refer to as $\mathsf{sp}(\mathcal{L}/\mathcal{R})$. We denote as $\mathsf{sp}[\mathsf{S}(\mathcal{L})]$ the forward image of all of $\mathsf{S}(\mathcal{L})$ under spatialization. This is isomorphic to the ordered collection of fixpoints of sp. The following are two classical lattice-theoretical results.

Lemma 3.1.1. *Suppose that L is a complete lattice, and that $\iota : L \to L$ is an interior operator. Then, the fixpoints of ι are closed under arbitrary joins. Furthermore, the assignment $\iota : L \twoheadrightarrow \iota[L]$ preserves all meets.*

Lemma 3.1.2. *If L is a complete lattice and $I \subseteq L$ is a collection closed under arbitrary joins, the operator $\iota : L \to L$ defined as $\iota : x \mapsto \bigvee\{y \in I : y \leq x\}$ is an interior operator.*

We will now start viewing spatialization as an interior operator on the collection of all d-quotients of some d-frame $\mathcal{L}$. Recall that in the lattice $\mathsf{S}(\mathcal{L})$ of d-quotients, quotients are in bijective correspondence with reasonable congruence structures. In particular, the quotient $\mathcal{L}/\mathcal{R}$ corresponds to $\mathcal{R}^*$, the smallest reasonable congruence structure on $\mathcal{L}$ containing $\mathcal{R}$ componentwise. Under this correspondence the order on $\mathsf{S}(\mathcal{L})$ is the opposite of componentwise set inclusion for reasonable congruence structures. To explore the order structure of $\mathsf{S}(\mathcal{L})$ without having explicit forms of quotients of reasonable congruence structures we can reason as follows. If there is a surjection $q_{SR} : \mathcal{L}/\mathcal{S} \twoheadrightarrow \mathcal{L}/\mathcal{R}$, this means that the relation structure $\mathcal{R}$ makes more identifications, or increases the con or the tot components more than $\mathcal{S}$ does. We

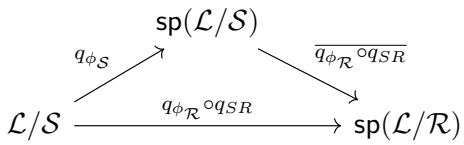
then have $\mathcal{S}^* \subseteq \mathcal{R}^*$ and, as a consequence, $\mathcal{L}/\mathcal{R} \leq \mathcal{L}/\mathcal{S}$. Then, having a d-frame surjection $\mathcal{L}/\mathcal{S} \twoheadrightarrow \mathcal{L}/\mathcal{R}$ is enough to deduce $\mathcal{L}/\mathcal{R} \leq \mathcal{L}/\mathcal{S}$. Let us state this as a lemma.

Lemma 3.1.3. *For a d-frame $\mathcal{L}$, if there is a d-frame surjection $q : \mathcal{L}/\mathcal{R} \twoheadrightarrow \mathcal{L}/\mathcal{S}$, we have that $\mathcal{L}/\mathcal{S} \leq \mathcal{L}/\mathcal{R}$ in the lattice $\mathsf{S}(\mathcal{L})$.*

Proposition 3.1.4. *The operator $\mathsf{sp} : \mathsf{S}(\mathcal{L}) \to \mathsf{S}(\mathcal{L})$ is an interior operator. Then,*

- *the lattice $\mathsf{sp}[\mathsf{S}(\mathcal{L})]$ is a complete join sublattice of $\mathsf{S}(\mathcal{L})$,*
- *the spatialization map $\mathsf{sp} : \mathsf{S}(\mathcal{L}) \twoheadrightarrow \mathsf{sp}[\mathsf{S}(\mathcal{L})]$ preserves arbitrary meets.*

Proof. Let $\mathcal{L}$ be a d-frame. Let us show that spatialization is an interior operator on the lattice of quotients of $\mathcal{L}$. Suppose that $\mathcal{L}/\mathcal{R} \leq \mathcal{L}/\mathcal{S}$. This means, in particular, that there is a surjection $q_{SR} : \mathcal{L}/\mathcal{S} \to \mathcal{L}/\mathcal{R}$. Then, there is also a surjection $q_{\phi_\mathcal{R}} \circ q_{SR} : \mathcal{L}/\mathcal{S} \twoheadrightarrow \mathsf{sp}(\mathcal{L}/\mathcal{R})$. By the universal property of the spatialization of a d-frame, we have that $\overline{q_{\phi_\mathcal{R}} \circ q_{SR}}$ is the unique map making the following triangle commute.



Here $q_{\phi_\mathcal{S}}$ is quotient map corresponding to the spatialization of $\mathcal{L}/\mathcal{S}$. Since $q_{\phi_\mathcal{R}} \circ q_{SR}$ is a d-frame surjection, the map $\overline{q_{\phi_\mathcal{R}} \circ q_{SR}}$ must be a surjection too by commutativity of the diagram. The existence of a d-frame surjection $\mathsf{sp}(\mathcal{L}/\mathcal{S}) \twoheadrightarrow \mathsf{sp}(\mathcal{L}/\mathcal{R})$ means that $\mathsf{sp}(\mathcal{L}/\mathcal{R}) \leq \mathsf{sp}(\mathcal{L}/\mathcal{S})$. The operator sp is then monotone. The fact that this is decreasing follows directly from the fact that we always have a canonical surjection $\mathcal{L}/\mathcal{R} \twoheadrightarrow \mathsf{sp}(\mathcal{L}/\mathcal{R})$ corresponding to the spatialization surjection. For idempotency, we notice that for any quotient $\mathcal{L}/\mathcal{R}$ the canonical surjection $\mathsf{sp}(\mathcal{L}/\mathcal{R}) \twoheadrightarrow \mathsf{sp}(\mathsf{sp}(\mathcal{L}/\mathcal{R}))$ is simply the identity: if a quotient $\mathcal{L}/\mathcal{S}$ is already a spatial d-frame, then the spatialization quotient $q_{\phi_\mathcal{S}} : \mathcal{L}/\mathcal{S} \twoheadrightarrow \mathsf{sp}(\mathcal{L}/\mathcal{S})$ does not make any nontrivial identification, nor does it increase the con and the tot components. Then, the operator $\mathsf{sp} : \mathsf{S}(\mathcal{L}) \to \mathsf{S}(\mathcal{L})$ is an interior operator. The rest of the claim follows from Lemma 3.1.1. □

We now work towards our main goal of showing that the poset of spatial d-frame quotients of a d-frame $\mathcal{L}$ is isomorphic to the poset of all d-sober subspaces of its spectrum $\mathcal{L}$. To do this, we will use some abstract properties of the adjunction $\mathsf{d}\Omega : \mathbf{BiTop} \leftrightarrows \mathbf{dFrm}^{op} : \mathsf{dpt}$. Let us then explore these properties.

For every adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ we know how to restrict this to an equivalence of categories. An adjunction is *idempotent* if every object in $\mathcal{D}$ of the form $F(X)$ is a fixpoint, and similarly for the other category.

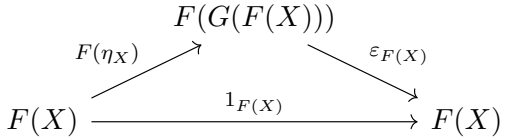
The spatial-sober adjunction for frames and that for d-frames both satisfy this special property. Indeed, the category $\mathsf{Fix}(\mathbf{Top})$ is the full subcategory of sober spaces, and this coincides with the full subcategory of **Top** determined by objects $\{\mathsf{pt}(L) : L \in \mathsf{Obj}(\mathbf{Frm})\}$. Similarly, the spatial frames are exactly those such that they are $\Omega(X)$ for some space X. The same holds for the adjunction $\mathsf{d}\Omega : \mathbf{BiTop} \leftrightarrows \mathbf{dFrm}^{op} : \mathsf{dpt}$.

Lemma 3.1.5. *Suppose that $\mathcal{C}$ is a category, and $i : X \to Y$ is an isomorphism in $\mathcal{C}$, and $j : Y \to X$ is such that $j \circ i = 1_X$. Then $i \circ j = 1_Y$.*

Proof. Since we know that $i : X \to Y$ is an isomorphism, we know that an inverse $j' : Y \to X$ exists. Then, we must have that $j' \circ i = 1_X = j \circ i$. Since all isomorphisms are epimorphisms, this implies that $j = j'$ and, in particular, that $i \circ j = 1_Y$. □

Lemma 3.1.6. *Suppose that we have an adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ with $F \dashv G$, with $\eta : 1 \to G \circ F$ the unit and $\varepsilon : F \circ G \to 1$ the counit. Suppose also that this restricts to an equivalence of the full subcategories determined by $\{F(X) : X \in \mathsf{Obj}(\mathcal{C})\}$ and $\{G(Y) : Y \in \mathsf{Obj}(\mathcal{D})\}$. Then, for every $X \in \mathsf{Obj}(\mathcal{C})$ and every object $Y \in \mathsf{Obj}(\mathcal{D})$ the maps $F(\eta_X)$ and $G(\varepsilon_Y)$ are isomorphisms.*

Proof. Suppose that $X \in \mathsf{Obj}(\mathcal{C})$. We know by assumption that the counit $\varepsilon_{F(X)}$ is an isomorphism. Additionally, the following unit-counit triangle commutes.



Then, by Lemma 3.1.5, we must have that $F(\eta_X) \circ \varepsilon_{F(X)} = 1_{F(G(F(X)))}$. Since it is the inverse of an isomorphism, the map $F(\eta_X)$ must be an isomorphism too. The argument for maps of the form $G(\varepsilon_Y)$ is analogous. □

Let us now apply the results mentioned in this small digression and prove the desired result.

Lemma 3.1.7. *For every d-frame $\mathcal{L}$, every inclusion of a d-sober space into $\mathsf{dpt}(\mathcal{L})$ is, up to homeomorphism, the dualization of a d-frame quotient map $q_R : \mathcal{L} \twoheadrightarrow \mathcal{L}/\mathcal{R}$, with $\mathcal{L}/\mathcal{R}$ a spatial d-frame.*

Proof. Consider an inclusion of the form $i : \mathsf{dpt}(\mathcal{M}) \hookrightarrow \mathsf{dpt}(\mathcal{L})$. Dualizing this we obtain a d-frame surjection $\Omega(i) : \Omega(\mathsf{dpt}(\mathcal{L})) \twoheadrightarrow \Omega(\mathsf{dpt}(\mathcal{M}))$, and so there is also a d-frame quotient $q_{R'} : \Omega(\mathsf{dpt}(\mathcal{L})) \twoheadrightarrow \Omega(\mathsf{dpt}(\mathcal{L}))/\mathcal{R}'$ relative to this surjection. Let us denote as $q_\phi : \mathcal{L} \twoheadrightarrow \mathcal{L}/\mathcal{R}_\phi$ the spatialization quotient of $\mathcal{L}$, and note that $\mathcal{L}/\mathcal{R}_\phi$ is isomorphic to $\Omega(\mathsf{dpt}(\mathcal{L}))$. We then may compose these two quotients to obtain the quotient $q_{R'} \circ q_\phi : \mathcal{L} \twoheadrightarrow \mathcal{L}/\mathcal{R}$, with $\mathcal{R} = \mathcal{R}' \vee \mathcal{R}_\phi$. Dualizing this, we obtain the bisubspace inclusion $\mathsf{dpt}(q_{R'} \circ q_\phi) : \mathsf{dpt}(\mathcal{L}/\mathcal{R}) \hookrightarrow \mathsf{dpt}(\mathcal{L})$ and, by functoriality of dpt, this is $\mathsf{dpt}(q_{R'}) \circ \mathsf{dpt}(q_\phi)$. Since $\mathsf{dpt}(q_\phi)$ is a bihomeomorphism by Lemma 3.1.6, this bisubspace inclusion is homeomorphic to $\mathsf{dpt}(q_{R'}) : \mathsf{dpt}(\mathcal{L}/\mathcal{R}') \hookrightarrow \mathsf{dpt}(\mathcal{L})$ and, by definition of the quotient $q_{R'}$, this is also bihomeomorphic to the inclusion $i : \mathsf{dpt}(\mathcal{M}) \hookrightarrow \mathsf{dpt}(\mathcal{L})$. □

Recall that the relation structures on a d-frame $\mathcal{L}$ are naturally ordered by componentwise set inclusion. We denote as $\mathsf{RS}(\mathcal{L})$ this ordered collection. In $\mathsf{RS}(\mathcal{L})$ the meets and joins are simply componentwise intersections and unions, respectively. Whenever we refer to meets and joins of relation structures on a d-frame, we mean that there are computed in $\mathsf{RS}(\mathcal{L})$. The next proposition is a d-frame analogue of a known topological result stating that sober subspaces of a sober space are closed under arbitrary intersections. Just as we showed that spatial quotients are an interior system in the collection $\mathsf{S}(\mathcal{L})$ of pointfree bisubspaces of $\mathcal{L}$, we now show that the d-sober subspaces are a closure system in the collection $\mathcal{P}(\mathsf{dpt}(\mathcal{L}))$ the point-set subspaces of its spectrum. We prove this by first showing that these are closed under arbitrary intersections.

Proposition 3.1.8. *D-sober subspaces of a d-sober space are closed under arbitrary intersections. In particular, for a collection of d-quotients $\mathcal{L}/\mathcal{R}_i$, we have $\mathsf{dpt}(\mathcal{L}/\bigvee_i \mathcal{R}_i) = \bigcap_i \mathsf{dpt}(\mathcal{L}/\mathcal{R}_i)$.*

Proof. By Lemma 3.1.7, the subspace inclusion of any d-sober subspace into $\mathsf{dpt}(\mathcal{L})$ is homeomorphic to an inclusion of the form $\mathsf{dpt}(\mathcal{L}/\mathcal{R}) \hookrightarrow \mathsf{dpt}(\mathcal{L})$ for some relation structure $\mathcal{R}$. Given any family of d-sober subspaces of $\mathsf{dpt}(\mathcal{L})$ we can assume that this is a collection of subsets of $\mathsf{dpt}(\mathcal{L})$ of the form $\mathsf{dpt}(\mathcal{L}/\mathcal{R}_i)$. The points of $\mathcal{L}/\bigvee_i \mathcal{R}_i$ correspond to d-frame maps $f : \mathcal{L} \to \mathbf{2}$ such that they satisfy the conditions of all relation structures $\mathcal{R}_i$. The underlying sets of points of $\mathsf{dpt}(\mathcal{L}/\bigvee_i \mathcal{R}_i)$ and that of $\bigcap_i \mathsf{dpt}(\mathcal{L}/\mathcal{R}_i)$ are then the same. The two bisubspaces also coincide, as they both inherit the two topologies from $\mathsf{dpt}(\mathcal{L})$. □

Corollary 3.1.9. *For any d-frame $\mathcal{L}$ the map $\mathsf{sob} : \mathcal{P}(\mathsf{dpt}(\mathcal{L})) \to \mathcal{P}(\mathsf{dpt}(\mathcal{L}))$ defined as $Y \mapsto \bigcap\{Z \subseteq \mathsf{dpt}(\mathcal{L}) : Z \text{ is d-sober}\}$ is a closure operator.*

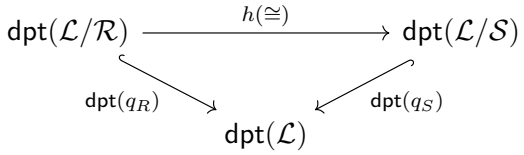
Proof. By Proposition 3.1.8 we know that the d-sober bisubspaces of $\mathsf{dpt}(\mathcal{L})$ form a collection closed under arbitrary intersections. The desired result follows from this and from dualizing Lemma 3.1.2. □

With the following results we will look in detail at how the equivalence of categories between spatial d-frames and sober bispaces establishes an isomorphism between the lattices of spatial d-quotients $\mathsf{sp}[\mathsf{S}(\mathcal{L})]$ of a d-frame and the lattice $\mathsf{sob}[\mathcal{P}(\mathsf{dpt}(\mathcal{L}))]$ of d-sober bisubspaces of its spectrum.

Proposition 3.1.10. *For any d-frame $\mathcal{L}$, we have a lattice isomorphism*

$$\begin{aligned}\mathsf{sp}[\mathsf{S}(\mathcal{L})] &\cong \mathsf{sob}[\mathcal{P}(\mathsf{dpt}(\mathcal{L}))]\\ \mathcal{L}/\mathcal{R} &\mapsto \mathsf{dpt}(q_R)[\mathsf{dpt}(\mathcal{L}/\mathcal{R})].\end{aligned}$$

Proof. We begin by pointing out that, since the functor dpt establishes a dual equivalence of categories between spatial d-frames and sober bispaces, when seen as a map between these two categories it is full, faithful, and essentially surjective. Let us show that the map defined in the claim is injective. We need to show that if the subspace inclusions $\mathsf{dpt}(q_R) : \mathsf{dpt}(\mathcal{L}/\mathcal{R}) \hookrightarrow \mathsf{dpt}(\mathcal{L})$ and $\mathsf{dpt}(q_S) : \mathsf{dpt}(\mathcal{L}/\mathcal{S}) \hookrightarrow \mathsf{dpt}(\mathcal{L})$ correspond to the same subspace of $\mathsf{dpt}(\mathcal{L})$ for two canonical quotient maps $q_R : \mathcal{L} \twoheadrightarrow \mathcal{L}/\mathcal{R}$ and $q_S : \mathcal{L} \twoheadrightarrow \mathcal{L}/\mathcal{S}$ to spatial d-frames, we must also have $q_R = q_S$. Suppose, then, that the antecedent holds. Then, there must be an isomorphism $h : \mathsf{dpt}(\mathcal{L}/\mathcal{R}) \cong \mathsf{dpt}(\mathcal{L}/\mathcal{S})$ such that the following triangle commutes.



Suppose also that $q_\phi : \mathcal{L} \to \mathsf{sp}(\mathcal{L})$ is the spatialization quotient map. Let $\overline{q_R} : \mathsf{sp}(\mathcal{L}) \to \mathcal{L}/\mathcal{R}$ and $\overline{q_S} : \mathsf{sp}(\mathcal{L}) \to \mathcal{L}/\mathcal{S}$ be the unique maps such that $\overline{q_R} \circ q_\phi = q_R$ and $\overline{q_S} \circ q_\phi = q_S$, which exist as we have assumed the two quotients to be spatial. We start from the equality $\mathsf{dpt}(q_R) \circ h = \mathsf{dpt}(q_S)$, which is true by commutativity of the triangle above, and deduce that $\mathsf{dpt}(\overline{q_R} \circ q_\phi) \circ h = \mathsf{dpt}(\overline{q_S} \circ q_\phi)$, that is $\mathsf{dpt}(\overline{q_R}) \circ \mathsf{dpt}(q_\phi) \circ h = \mathsf{dpt}(\overline{q_S}) \circ \mathsf{dpt}(q_\phi)$. By Lemma 3.1.6, the map $\mathsf{dpt}(q_\phi)$ is an isomorphism. Since isomorphisms are always epimorphisms, we deduce $\mathsf{dpt}(\overline{q_R}) = \mathsf{dpt}(\overline{q_S})$. Since $\overline{q_R}$ and $\overline{q_S}$ are maps between spatial d-frames, we may use faithfulness of dpt to deduce $\overline{q_R} = \overline{q_S}$. Then, $\overline{q_R} \circ q_\phi = \overline{q_S} \circ q_\phi$, that is, $q_R = q_S$.

On the other hand, by Lemma 3.1.7 every subspace inclusion $i : \mathsf{dpt}(\mathcal{M}) \hookrightarrow \mathsf{dpt}(\mathcal{L})$ is, up to bihomeomorphism, a bisubspace inclusion of the form $\mathsf{dpt}(q_R) : \mathsf{dpt}(\mathcal{L}/\mathcal{R}) \to \mathsf{dpt}(\mathcal{L})$, for

some relation structure $\mathcal{R}$ and for $\mathcal{L}/\mathcal{R}$ spatial. Since the d-spectrum functor is contravariant, the assignment from spatial d-quotients to d-sober subspaces that we have defined is order preserving and reflecting. □

We have already seen that spatialization is an interior operator on the collection of point-free bisubspaces of a frame $\mathcal{L}$, and that sobrification is a closure on the collection $\mathcal{P}(\mathsf{dpt}(\mathcal{L}))$ of the point-set subspaces of its spectrum. We could also rephrase this result and highlight this symmetry differently. Combining Proposition 3.1.4 and Lemma 3.1.8, we obtain that, for any d-frame $\mathcal{L}$, the following diagram in the category of complete lattices with $\bigwedge$-preserving maps commutes up to isomorphism.

$$\begin{array}{ccc} \mathsf{sp}[\mathsf{S}(\mathcal{L})] & \xrightarrow{\mathsf{dpt}(\cong)} & \mathsf{sob}[\mathcal{P}(\mathsf{dpt}(\mathcal{L}))] \\ \Big\uparrow{\scriptstyle \mathsf{sp}} & & \Big\downarrow \\ \mathsf{S}(\mathcal{L}) & \xrightarrow{\mathsf{dpt}} & \mathcal{P}(\mathsf{dpt}(\mathcal{L})) \end{array}$$

The top arrow is an isomorphism, by Proposition 3.1.10. We will come back to this picture in Chapter 8, which is an exploration of the relation between bisubspaces and d-quotients.

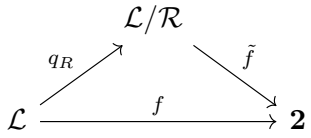
Next, we will specialize our analysis and see how different kinds of relation structures on a d-frame $\mathcal{L}$ determine different bisubspaces of $\mathsf{dpt}(\mathcal{L})$. Let us introduce some abbreviations. Whenever we quotient a d-frame $\mathcal{L}$ by a relation structure increasing one component only, we identify notationally this relation structure and the update itself. For instance, if c is a con update, the d-quotient $\mathcal{L}/c$ is that obtained quotienting $\mathcal{L}$ by the relation structure $(\mathsf{id}^+, \mathsf{id}^-, c, t_m)$. We will say that a d-frame map $f : \mathcal{L} \to \mathcal{M}$ *respects* a relation structure $\mathcal{R}$ on $\mathcal{L}$ when it factors through it, that is, when the following conditions hold.

- Whenever $a^+ R^+ b^+$ we have $f^+(a^+) \leq f^+(b^+)$.
- Whenever $a^- R^- b^-$ we have $f^-(a^-) \leq f^-(b^-)$.
- Whenever $(x^+, x^-) \in c_R$ we have $(f^+(x^+), f^-(x^-)) \in c_M$.
- Whenever $(x^+, x^-) \in t_R$ we have $(f^+(x^+), f^-(x^-)) \in t_M$.

The following proposition will be very useful for simplifying calculations.

Proposition 3.1.11. *For a d-frame $\mathcal{L}$ and a relation structure $\mathcal{R}$ on $\mathcal{L}$, consider the canonical bijection $f \mapsto \tilde{f}$ assigning to each $\mathcal{R}$ respecting map $f : \mathcal{L} \to 2$ the unique map $\tilde{f} : \mathcal{L}/\mathcal{R} \to \mathbf{2}$*

making the following triangle commute.



This map is a bihomeomorphism between the bisubspace $\{f \in \mathsf{dpt}(\mathcal{L}) : f \text{ respects } \mathcal{R}\} \subseteq \mathsf{dpt}(\mathcal{L})$ *and* $\mathsf{dpt}(\mathcal{L}/\mathcal{R})$.

Proof. Let us call β the assignment described in the statement. The map is a bijection by the universal property of d-frame quotients. Let us now show that β is such that its forward image map as well as its preimage map preserve the property of being a positive open. To show that the forward image of open sets is open, consider a typical positive open set of $\{f \in \mathsf{dpt}(\mathcal{L}) : f \text{ respects } \mathcal{R}\} \subseteq \mathsf{dpt}(\mathcal{L})$. This is a set of the form $\phi^+_{\mathcal{L}}(a^+) \cap \{f \in \mathsf{dpt}(\mathcal{L}) : f \text{ respects } \mathcal{R}\}$. That is, it is $\{f \in \mathsf{dpt}(\mathcal{L}) : f \text{ respects } \mathcal{R} \text{ and } f^+(a^+) = 1^+\}$. We have that $f^+(a^+) = 1^+$ if and only if $\tilde{f}^+(\ulcorner a^+ \urcorner) = 1^+$ by commutativity of the triangle above. Then,

$$\beta[\{f \in \mathsf{dpt}(\mathcal{L}) : f \text{ respects } \mathcal{R} \text{ and } f^+(a^+) = 1^+\}] = \phi^+_{\mathcal{L}/\mathcal{R}}(\ulcorner a^+ \urcorner).$$

For preimages, we use once again the fact that $f^+(a^+) = 1^+$ if and only if $\tilde{f}^+(\ulcorner a^+ \urcorner) = 1^+$ for every $a^+ \in L^+$, and deduce that whenever $\ulcorner a^+ \urcorner$ is an element in the positive frame component of $\mathcal{L}/\mathcal{R}$ we have

$$\beta^{-1}(\phi^+_{\mathcal{L}/\mathcal{R}}(\ulcorner a^+ \urcorner)) = \{f \in \mathsf{dpt}(\mathcal{L}) : \text{ respects } \mathcal{R} \text{ and } f^+(a^+) = 1^+\}. \qquad \square$$

Throughout this thesis, we will often encounter proofs that tacitly identify bihomeomorphic bispaces as those in the proposition above. When that happens, nonetheless, we will always warn the reader before the proof itself. Let us, then, proceed to look at the first instance where this simplification helps us.

Lemma 3.1.12. *The following are true for a d-frame* $\mathcal{L}$.

1. *Whenever* c *is a* con *update,*

$$\mathsf{dpt}(\mathcal{L}/c) = \{f \in \mathsf{dpt}(\mathcal{L}) : f^+(x^+) = 0^+ \text{ or } f^-(x^-) = 0^- \text{ for all } (x^+, x^-) \in c\}.$$

2. *Whenever* t *is a* tot *update,*

$$\mathsf{dpt}(\mathcal{L}/t) = \{f \in \mathsf{dpt}(\mathcal{L}) : f^+(x^+) = 1^+ \text{ or } f^-(x^-) = 1^- \text{ for all } (x^+, x^-) \in t\}.$$

3. *Whenever R^+ is a relation on L^+,*

$$\mathsf{dpt}(\mathcal{L}/C^+) = \{f \in \mathsf{dpt}(\mathcal{L}) : f^+(a^+) = 0^+ \text{ or } f^+(b^+) = 1^+ \text{ whenever } a^+ R^+ b^+\}.$$

4. *When c, d are downsets in $L^+ \times L^-$ and we treat them as* con *updates, we have that*

$$\mathsf{dpt}(\mathcal{L}/(c \cap d)) = \mathsf{dpt}(\mathcal{L}/c) \cup \mathsf{dpt}(\mathcal{L}/d).$$

5. *Similarly, if t, u are upsets in $L^+ \times L^-$ and we treat them as* tot *updates, we have that*

$$\mathsf{dpt}(\mathcal{L}/(t \cap u)) = \mathsf{dpt}(\mathcal{L}/t) \cup \mathsf{dpt}(\mathcal{L}/u).$$

6. *For basic congruences $A_1^+ = \nabla(a_1^+) \cap \Delta(a_2^+)$ and $B^+ = \nabla(b_1^+) \cap \Delta(b_2^+)$ on L^+, we have*

$$\mathsf{dpt}(\mathcal{L}/(A^+ \cap B^+)) = \mathsf{dpt}(\mathcal{L}/A^+) \cup \mathsf{dpt}(\mathcal{L}/B^+).$$

7. *For arbitrary congruences C^+ and D^+ on L^+, we have*

$$\mathsf{dpt}(\mathcal{L}/(C^+ \cap D^+)) = \mathsf{dpt}(\mathcal{L}/C^+) \cup \mathsf{dpt}(\mathcal{L}/D^+).$$

Proof. Let us prove each item separately.

1. We identify the points of $\mathcal{L}/c$ with those d-frame maps $f : \mathcal{L} \to \mathbf{2}$ respecting the relation structure $(\mathsf{id}^+, \mathsf{id}^-, c, t_m)$. These are those such that $(f^+(a^+), f^-(a^-)) \in c_2$ whenever $(a^+, a^-) \in c$. The only element of 2×2 which is not in the con component of $\mathbf{2}$ is $(1^+, 1^-)$, and so the condition $(f^+(a^+), f^-(a^-)) \in c_2$ is equivalent to $(f^+(a^+), f^-(a^-)) \neq (1^+, 1^-)$.

2. The argument is analogous to that used for proving item (1).

3. Recall that a congruence $\nabla(a^+) \cap \Delta(b^+)$ on L^+ is the congruence on L^+ generated by the relation $\{(a^+, b^+)\}$. The points of $\mathcal{L}/(\nabla(a^+) \cap \Delta(b^+))$, then, are those points of $\mathcal{L}$ such that they respect the relation structure $((a^+, b^+), \mathsf{id}^-, c_m, t_m)$ on $\mathcal{L}$. These are the d-frame maps $f : \mathcal{L} \to \mathbf{2}$ such that $f^+(a^+) \leq f^+(b^+)$. The only d-frame maps f not satisfying this condition are those such that $f^+(b^+) = 0^+$ and $f^+(a^+) = 1^+$. The desired result follows from applying Proposition 3.1.8 to the join $C^+ = \bigvee\{\nabla(a^+) \cap \Delta(b^+) : \nabla(a^+) \cap \Delta(b^+) \subseteq C^+\}$.

4. The set inclusion $\mathsf{dpt}(\mathcal{L}/c) \cup \mathsf{dpt}(\mathcal{L}/d) \subseteq \mathsf{dpt}(\mathcal{L}/(c \cap d))$ is clear: factoring through $\mathcal{L}/c$ and $\mathcal{L}/\lceil$ are both stronger conditions than factoring through $\mathcal{L}/(c \cap d)$. To prove our claim, we show that if there is $f \in \mathsf{dpt}(\mathcal{L}/(c \cap d))$ such that $f \notin \mathsf{dpt}(\mathcal{L}/c)$ then we must have $f \in \mathsf{dpt}(\mathcal{L}/d)$. Let us assume, then, the antecedent of this claim. Observe that, since c and d are assumed to be downsets, the elements of $c \cap d$ are exactly the pairs of the form $(c^+ \wedge d^+, c^- \wedge d^-)$, for some $(c^+, c^-) \in c$ and some $(d^+, d^-) \in d$. By item (1) our assumption that $f \notin \mathcal{L}/c$ means that there must be some pair $(c^+, c^-) \in c$ with $f^+(c^+) = 1^+$ and $f^-(c^-) = 1^-$. Whenever $(d^+, d^-) \in d$, by our assumption that $f \in \mathsf{dpt}(\mathcal{L}/(c \cap d))$ we must have that either $f^+(c^+ \wedge d^+) = 0^+$ or $f^-(c^- \wedge d^-) = 0^-$. Since the elements 0^+ and 0^- are meet prime in the copy of 2 where they live, and since $f^+(c^+) \neq 0^+$ and $f^-(c^-) \neq 0^-$, we must have that if the former holds then $f^+(d^+) = 0^+$, and if the latter holds $f^-(d^-) = 0^-$. In any case, either $f^+(d^+) = 0^+$ or $f^-(d^-) = 0^-$. By item (1), then, that $f \in \mathsf{dpt}(\mathcal{L}/d)$.

5. The proof is dual to that of item (4).

6. Consider two basic congruences $\nabla(a_1^+) \cap \Delta(a_2^+)$ and $\nabla(b_1^+) \cap \Delta(b_2^+)$ on L^+. Now, recall that Δ reverses the lattice operations of L^+, and that ∇ preserves them. Then, we have that the intersection of our two basic congruences is $\nabla(a_1^+ \wedge b_1^+) \cap \Delta(a_2^+ \vee b_2^+)$. The equality that we want to prove, then, amounts to

$$\mathsf{dpt}(\mathcal{L}/(\nabla(a_1^+) \cap \Delta(a_2^+))) \cup \mathsf{dpt}(\mathcal{L}/(\nabla(b_1^+) \cap \Delta(b_2^+))) = \mathsf{dpt}(\mathcal{L}/(\nabla(a_1^+ \wedge b_1^+) \cap \Delta(a_2^+ \vee b_2^+))).$$

   As in item (4), the subset inclusion of the left hand side into the right hand side is clear. To show the reverse set inclusion, we assume

$$\begin{cases} f \in \mathsf{dpt}(\mathcal{L}/(\nabla(a_1^+ \wedge b_1^+) \cap \Delta(a_2^+ \vee b_2^+))) & (\in) \\ f \notin \mathsf{dpt}(\mathcal{L}/(\nabla(a_1^+) \cap \Delta(a_2^+))), & (\notin) \end{cases}$$

   and we show that this implies $f \in \mathsf{dpt}(\mathcal{L}/(\nabla(b_1^+) \cap \Delta(b_2^+)))$. By assumption $(\notin)$ and by item (3), we deduce that both $f^+(a_1^+) = 1^+$ and $f^+(a_2^+) = 0^+$. On the other hand, by assumption $(\in)$ and item (3), we have that either $f^+(a_1^+ \wedge b_1^+) = 0^+$ or $f^+(a_2^+ \vee b_2^+) = 1^+$. If the first option holds, since $f^+(a_1^+) = 1^+$, and since 0^+ is meet prime, we must deduce $f^+(b_1^+) = 0^+$. If the second option holds, by the fact that $f^+(a_2^+) = 0^+$ we must deduce that $f^+(b_2^+) = 1^+$. In any case this means, by item (3), that $f \in \mathsf{dpt}(\mathcal{L}/(\nabla(b_1^+) \cap \Delta(b_2^+)))$, as desired.

7. Recall that every frame congruence on a frame L is the join of the basic frame congruences $\nabla(b) \cap \Delta(a)$ below it. So, suppose that for congruences C^+ and D^+ we have that

$\bigvee_i C_i^+$ and $\bigvee_j D_j^+$ are their respective forms as joins of basic congruences. By Lemma 3.1.8 we have that $\mathsf{dpt}(\mathcal{L}/C^+) = \bigcap_i \mathsf{dpt}(\mathcal{L}/C_i^+)$, and similarly for the other congruence. Since frame congruences on a frame form a frame, we have that $C^+ \cap D^+ = \bigvee_{i,j} C_i^+ \cap D_j^+$. The equality we need to prove, then, amounts to

$$\bigcap_i \mathsf{dpt}(\mathcal{L}/C_i^+) \cup \bigcap_j \mathsf{dpt}(\mathcal{L}/D_j^+) = \bigcap_{i,j} \mathsf{dpt}(\mathcal{L}/C_i^+ \cap D_j^+).$$

Since set theoretical unions and intersections distribute over one another, the claim also amounts to the following.

$$\bigcap_{i,j} \mathsf{dpt}(\mathcal{L}/C_i^+) \cup \mathsf{dpt}(\mathcal{L}/D_j^+) = \bigcap_{i,j} \mathsf{dpt}(\mathcal{L}/C_i^+ \cap D_j^+).$$

But this follows directly from the fact that, for each i and each j, by item (6) we have $\mathsf{dpt}(\mathcal{L}/C_i^+) \cup \mathsf{dpt}(\mathcal{L}/D_j^+) = \mathsf{dpt}(\mathcal{L}/(C_i^+ \cap D_j^+))$. □

Lemma 3.1.13. *For any d-frame $\mathcal{L}$ and any d-frame quotient $\mathcal{L}/\mathcal{R}$, we have that*

$$\mathsf{dpt}(\mathcal{L}/\mathcal{R}) = \mathsf{dpt}(\mathcal{L}_m/\mathcal{R}) \cap \mathsf{dpt}(\mathcal{L}).$$

Proof. We notice that $\mathcal{L}/\mathcal{R}$ is the result of quotienting the d-frame $\mathcal{L}_m$ by the relation structure $\mathcal{R} \vee (\mathsf{id}^+, \mathsf{id}^-, c_L, t_L)$. The desired result follows from Lemma 3.1.8. □

Let us summarize the information that we have on the order structure of d-sober subspaces of a d-sober bispace.

Proposition 3.1.14. *For any d-frame $\mathcal{L}$, the following collections of subsets of $\mathsf{dpt}(\mathcal{L})$ are closed under arbitrary intersections and finite unions.*

1. *The subspaces of the form $\mathsf{dpt}(\mathcal{L}/c)$ for c a con update.*
2. *The subspaces of the form $\mathsf{dpt}(\mathcal{L}/t)$ for t a tot update.*
3. *The subspaces of the form $\mathsf{dpt}(\mathcal{L}/R^\pm)$ for $R^\pm$ a relation on $L^\pm$.*

The following counterexample illustrates that d-sober subspaces are not closed under every finite union. We show that there is a space of the form $\mathsf{dpt}(\mathcal{L}/\nabla(a^+)) \cup \mathsf{dpt}(\mathcal{L}/\Delta(a^-))$ which is not d-sober.

Example 3.1.15. Consider the d-spectrum of $\mathbf{3} = (3, 3, c_m, t_m)$. This consists of four points, let them be x_1, x_2, x_3, x_4 with $x_1 \leq x_2 \leq x_4$, $x_1 \leq x_3 \leq x_4$ under the patch specialization order. It follows that the subspace $\{x_1, x_4\} \subseteq \mathsf{dpt}(\mathbf{3})$ is not d-sober, as its d-frame of opens is still $\mathbf{3}$. However, it is the union of two singletons, and singletons are always d-sober since they are, up to d-homeomorphism, the spectrum of $(2, 2, c_m, t_m)$.

Before we give the main proof of this section, there is one more observation we need to make. For this purpose, we introduce some new abbreviations. In the context of relation structures, we denote as $c_{(a^+,a^-)}$ the con update $\downarrow(a^+, a^-)$. We adopt a similar convention for tot updates.

Observation 3.1.16. *Certain quotients of the form $\mathcal{L}/C^+$ have the same spectrum as quotients of the form $\mathcal{L}/c$ or $\mathcal{L}/t$ for a con update c and a tot update t. Let us see some examples of this. If $\mathcal{L}$ is a d-frame, comparing items (1) and (3) of Lemma 3.1.12 we have that $\mathsf{dpt}(\mathcal{L}/c_{(x^+,1^-)}) = \mathsf{dpt}(\mathcal{L}/\nabla(x^+))$. If we calculate both these using this lemma, we obtain that both equal*

$$\{f \in \mathsf{dpt}(\mathcal{L}) : f^+(x^+) = 0^+\}.$$

There is a dual result linking quotients obtained from tot updates and those obtained from open congruences. More in general, $\mathsf{dpt}(\mathcal{L}/c_{(x^+,x^-)}) = \mathsf{dpt}(\mathcal{L}/\nabla(x^+) \cap \nabla(x^-)$. In fact, calculating both these using Lemma 3.1.12 yields that both equal

$$\{f \in \mathsf{dpt}(\mathcal{L}) : f^+(x^+) = 1^+ \text{ or } f^-(x^-) = 1^-\}.$$

A similar result holds for tot updates of the form $t_{(x^+,x^-)}$.

Theorem 3.1.17. *For a bispace $\mathsf{dpt}(\mathcal{L})$, the following pairs of collections generate the collection of d-sober subspaces of $\mathcal{L}$, in the sense that each pair (P_1, P_2) is such that each d-sober subspace is the intersection of binary unions of elements in $P_1 \cup P_2$.*

- *The subspaces of the form $\mathsf{dpt}(\mathcal{L}/c)$ together with those of the form $\mathsf{dpt}(\mathcal{L}/t)$, where c is a con update and t a tot update.*
- *The subspaces of the form $\mathsf{dpt}(\mathcal{L}/R^+)$ together with those of the form $\mathsf{dpt}(\mathcal{L}/R^-)$, where $R^\pm$ is a relation on $L^\pm$.*

Proof. Let us consider the spectrum $\mathsf{dpt}(\mathcal{L})$ of some d-frame $\mathcal{L}$. Each d-sober subspace is the spectrum of a d-frame of the form $\mathcal{L}/\mathcal{R}$, and we may write the relation structure $\mathcal{R}$ as a join

of the form $\bigvee_i(\nabla(a_i^+) \cap \Delta(b_i^+), \nabla(a_i^-) \cap \Delta(b_i^-), (x_i^+, x_i^-), (y_i^+, y_i^-))$. This implies, by Lemma 3.1.8, that every d-sober subspace of $\mathsf{dpt}(\mathcal{L})$ is an arbitrary intersection of sets of the form

$$(\mathsf{dpt}(\mathcal{L}/(\nabla(a^+) \cap \Delta(b^+))) \cap (\mathsf{dpt}(\mathcal{L}/(\nabla(a^-) \cap \Delta(b^-))) \cap \mathsf{dpt}(\mathcal{L}/c_{(x^+,x^-)}) \cap \mathsf{dpt}(\mathcal{L}/t_{(y^+,y^-)}).$$

To complete the argument, let us analyze the two parts of our claim separately.

- We want to show that the expression above is an intersection of finite unions of spaces of the form $\mathsf{dpt}(\mathcal{L}/c)$ and $\mathcal{L}/t$. Item (7) of Lemma 3.1.12, together with Observation 3.1.16, imply that we have the following two equalities.

  $$\mathsf{dpt}(\mathcal{L}/(\nabla(a^+) \cap \Delta(b^+)) = \mathsf{dpt}(\mathcal{L}/(c_{(a^+,1^-)}) \cup \mathsf{dpt}(\mathcal{L}/t_{(b^+,0^-)}),$$
  $$\mathsf{dpt}(\mathcal{L}/(\nabla(a^-) \cap \Delta(b^-)) = \mathsf{dpt}(\mathcal{L}/t_{(b^+,0^-)})) \cap (\mathsf{dpt}(\mathcal{L}/c_{(1^+,a^-)}).$$

  Substituting these two into said expression yields the desired result.

- Similarly, from item (7) of Lemma 3.1.12, and Observation 3.1.16, we obtain the equalities

  $$\mathsf{dpt}(\mathcal{L}/c_{(x^+,x^-)}) = \mathsf{dpt}(\mathcal{L}/\nabla(x^+)) \cup \mathsf{dpt}(\mathcal{L}/\nabla(x^-)),$$
  $$\mathsf{dpt}(\mathcal{L}/t_{(y^+,y^-)}) = \mathsf{dpt}(\mathcal{L}/\Delta(y^+)) \cup \mathsf{dpt}(\mathcal{L}/\Delta(y^-)).$$

  We can substitute these equalities into our formula again to prove the second part of the claim. □

This result is the first clue pointing towards a bitopological representation of the collection of the d-sober subspaces. However, it will not be until Chapter 7 that the concrete way in which this collection is represented by a bitopological space will be explored. Theorem 3.1.17 above, together with the isomorphism between the ordered collections of d-sober spaces and spatial d-quotients established in Proposition 3.1.10, yields the following result.

Proposition 3.1.18. *For any d-frame $\mathcal{L}$, the following pairs of collections generate the lattice of spatial d-quotients $\mathsf{sp}[\mathsf{S}(\mathcal{L})]$, in the sense that each pair (P_1, P_2) is such that each spatial d-quotient is the meet of a binary join of elements in $P_1 \cup P_2$.*

- *Those of the form $\mathsf{sp}(\mathcal{L}/c)$ together with those of the form $\mathsf{sp}(\mathcal{L}/t)$.*
- *Those of the form $\mathsf{sp}(\mathcal{L}/R^+)$ together with those of the form $\mathsf{sp}(\mathcal{L}/R^-)$.*

3.2 The Booleanization of a d-frame

Recall that a frame is a complete Heyting algebra and, as such, there is a negation operation on it. The double negation $\neg\neg$ on a frame L is a nucleus. It follows that, for any frame L, one can construct the sublocale whose elements are the fixpoints of double pseudocomplementation. Recall also that this sublocale is always a Boolean algebra, and it is referred to as *Booleanization* of L. Isbell's Density Theorem, proved in [21], characterizes the Booleanization of a frame as its smallest dense sublocale. We denote the Booleanization of a frame L as $\mathfrak{B}(L)$. The Booleanization construction cannot be the object part of a reflector relative to the inclusion $\mathbf{Bool} \hookrightarrow \mathbf{Frm}$ of the category of complete Boolean algebras into that of frames. This is shown in [10], where the authors also identify the largest subcategory $\mathbf{Frm}'$ of $\mathbf{Frm}$ such that the map $\neg\neg : L \to \mathfrak{B}(L)$ is a Boolean reflection, namely the unit of adjunction of an adjunction $\mathfrak{B} \dashv i$ with $i : \mathbf{Bool} \hookrightarrow \mathbf{Frm}'$ a subcategory inclusion. A frame morphism $f : L \to M$ is *weakly open* if $\neg\neg f(x) = f(\neg\neg x)$ for all $x \in L$. We denote as $\mathbf{Frm_{wo}}$ the category of frames with weakly open morphisms. In [10] the following is shown.

Proposition 3.2.1. *The Booleanization construction can be extended to a functor left adjoint to the inclusion $i : \mathbf{Bool} \hookrightarrow \mathbf{Frm_{wo}}$, with $\neg\neg : L \to \mathfrak{B}(L)$ the unit of adjunction evaluated at L. The category $\mathbf{Frm_{wo}}$ is the largest subcategory of $\mathbf{Frm}$ such that $\neg\neg : L \to \mathfrak{B}(L)$ is the unit of adjunction relative to a Boolean reflection.*

For d-frames, the d-pseudocomplementation operation is behaved, in many respects, as the frame theoretical pseudocomplementation map. Recall from Section 1.2.1 that, for a d-frame $\mathcal{L}$, the pseudocomplementation maps satisfy the following properties, all of which are a consequence of these maps being Galois connections, and from the balance axiom.

- The map $\sim$ turns arbitrary joins into meets.
- The map $\sim\sim$ is monotone, inflationary, and idempotent.
- If $x \in L^+ \cup L^-$ has a d-complement, then $x = \sim\sim x$.
- We have $\sim\sim\sim = \sim$.

However, crucially, there is one aspect in which they behave differently. The operation of double pseudocomplementation always preserves finite meets, but the analogue d-frame operation, in general, does not. We refer to the d-frames for which this does preserve finite meets as *de Morgan* d-frames. For d-frames, the notion of Booleanization as a Boolean

reflection and as the smallest dense d-quotient do not coincide. As we will see shortly, this mismatch comes exactly from the fact that $\sim\!\sim$ is not a nucleus in general.

For any d-frame $\mathcal{L}$ we define $\mathfrak{B}(\mathcal{L})$ to be the quotient of $\mathcal{L}$ by the relation structure

$$(\mathsf{id}^+, \mathsf{id}^-, c_m, \{(a^+, \sim a^+) : a^+ \in L^+\} \cup \{(\sim a^-, a^-) : a^- \in L^-\}).$$

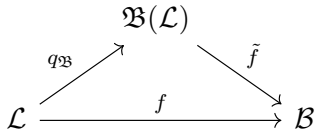
We denote as $q_{\mathfrak{B}} : \mathcal{L} \twoheadrightarrow \mathfrak{B}(\mathcal{L})$ the quotient map relative to this quotient, and we refer to the quotient itself as the *Booleanization* of the d-frame $\mathcal{L}$. Recall that a d-frame is said to be Boolean if every element of each of the two frame component has a d-complement. Then, by its construction the Booleanization of a d-frame is always Boolean. Let us denote as **dBool** the category of Boolean d-frames with d-frame morphisms. We define a map of d-frames $f : \mathcal{L} \to \mathcal{M}$ to be a *Heyting* map if it preserves pseudocomplements. We denote as $\mathbf{dFrm_H}$ the category of d-frames with Heyting d-frame maps. We now show that there is a subcategory inclusion $\mathbf{dBool} \hookrightarrow \mathbf{dFrm_H}$, to which the Booleanization construction is left adjoint, and that the unit of adjunction relative to this is the canonical quotient $q_{\mathfrak{B}}$ evaluated at each d-frame. We do not yet know whether $\mathbf{dFrm_H}$ is the largest subcategory of **dFrm** with this property.

Lemma 3.2.2. *D-frame morphisms between Boolean d-frames are Heyting morphism.*

Proof. Since d-frame maps preserve the con and the tot components, they preserve all d-complements. In particular, this means that d-frame maps between Boolean d-frames preserve all d-pseudocomplements. □

Proposition 3.2.3. *The Booleanization construction on d-frames is left adjoint to the inclusion* $\mathbf{dBool} \hookrightarrow \mathbf{dFrm_H}$*, and the unit of adjunction evaluated at* $\mathcal{L}$ *is the map* $q_{\mathfrak{B}} : \mathcal{L} \twoheadrightarrow \mathfrak{B}(\mathcal{L})$*.*

Proof. Let us first show that the quotient map $q_{\mathfrak{B}} : \mathcal{L} \to \mathfrak{B}(\mathcal{L})$ is a Heyting d-frame map. Since the pairs $(q^+_{\mathfrak{B}}(x^+), q^-_{\mathfrak{B}}(\sim x^+))$ are mapped to pairs of mutual d-complements, in particular $q^-_{\mathfrak{B}}(\sim x^+)$ must be the d-pseudocomplement of $q^+_{\mathfrak{B}}(x^+)$ by balance. To show that $\mathfrak{B}$ is a reflector, suppose that $f : \mathcal{L} \to \mathcal{B}$ is a Heyting d-frame map. We need to show that there exists a unique Heyting d-frame morphism $\tilde{f} : \mathfrak{B}(\mathcal{L}) \to \mathcal{B}$ such that the following diagram commutes.



Since $\mathcal{B}$ is Boolean it is such that $(f^+(x^+), f^-(\sim x^+)) \in t_B$ for all $x^+ \in L^+$, and similarly for elements in the negative component. Then this map factors uniquely through the quotient

map $q_{\mathfrak{B}} : \mathcal{L} \twoheadrightarrow \mathfrak{B}(\mathcal{L})$, i.e. there is a d-frame map $\tilde{f} : \mathfrak{B}(\mathcal{L}) \to \mathcal{B}$ making the triangle commute. It remains to show that this map preserves pseudocomplementation. But since both $\mathfrak{B}(\mathcal{L})$ and $\mathcal{B}$ are Boolean, this follows from Lemma 3.2.2. □

Let us now introduce another d-frame construction, analogous to the double pseudocomplementation nucleus on a frame. The first step is to analyze the notion of density for d-frame quotients. For the remaining part of this section, we will reason in terms of nuclei. Recall that each relation R on a frame L determines the smallest congruence $C(R)$ containing it, and it has an associated nucleus $\nu_R : L \to L$, defined as $\nu_R(x) = \bigvee\{y \in L : xC(R)y\}$. This establishes a bijection between congruences on L and nuclei on L. A d-frame quotient map $q_R : \mathcal{L} \twoheadrightarrow \mathcal{L}/\mathcal{R}$ is *dense* if the nuclei $\nu^{\pm}_{R^{\pm}}$ are such that $(x^+, x^-) \in c_L$ implies that $(\nu^+_{R^+}(x^+), \nu^-_{R^-}(x^-)) \in c_L$, for all pairs $(x^+, x^-) \in L^+ \times L^-$.

Remark 3.2.4. Let us spend a couple of words to justify our definition of density for d-frame morphisms. We have that a d-frame quotient map $q_R : \mathcal{L} \to \mathcal{L}/\mathcal{R}$ is dense if and only if we have that $(q^+_R(x^+), q^-_R(x^-)) \in c_{L/R}$ implies that $(x^+, x^-) \in c_L$. We claim that this is a direct translation of what it means for a biframe morphism $f : \mathcal{L} \to \mathcal{M}$ to be dense. In [48], a biframe morphism $f : \mathcal{L} \to \mathcal{M}$ is defined to be dense if and only if its main component $f : L \to M$ is a dense frame map. This in fact holds if and only if whenever a pair of the form $(f^+(\ulcorner x^+ \urcorner), f^-(\ulcorner x^- \urcorner))$ is disjoint in M, then the pair (x^+, x^-) is disjoint in L.

Recall that we say that a relation structure $\mathcal{R}$ on a d-frame $\mathcal{L}$ is *extremal* if it is of the form (R^+, R^-, c_m, t_m). With the next lemma, we see that — for d-frames too — there is a close relation between density and double pseudocomplementation.

Lemma 3.2.5. *A d-frame quotient map $q_R : \mathcal{L} \twoheadrightarrow \mathcal{L}/\mathcal{R}$ is dense if and only if $\nu^+_{R^+} \leq \sim\sim$, and similarly on the negative component.*

Proof. Suppose that $\mathcal{L}$ is a d-frame, and that $\nu^{\pm}$ are nuclei on frames $L^{\pm}$. Let as assume that the two nuclei are below $\sim\sim$. If $(x^+, x^-) \in c_L$, that is $x^+ \leq \sim x^-$, then we have that $\sim\sim x^- \leq \sim x^+$ by antitonicity of $\sim$, and so $(\sim\sim x^+, \sim\sim x^-) \in c_L$ by definition of d-pseudocomplement. Since con subsets are downward closed, this implies that $(\nu^+(x^+), \nu^-(x^-)) \in c_L$, and so the pair of nuclei corresponds to a dense extremal surjection. Conversely, suppose that the pair of nuclei is such that $(\nu^+(x^+), \nu^-(x^-)) \in c_L$ whenever $(x^+, x^-) \in c_L$. Then for each $x^+ \in L^+$, we have that $\nu^+(x^+) \leq \sim\ \nu^+(\sim\ x^+)$ since $(\nu^+(x^+), \nu^-(\sim\ x^+)) \in c_L$ by our assumption. Additionally, since ν^+ is inflationary and $\sim$ is antitone, we also have that $\sim\ \nu^+(\sim\ x^+) \leq \sim\sim x^+$. Combining these two inequalities yields the desired result that $\nu^+(x^+) \leq \sim\sim\ x^+$. □

We had mentioned in Chapter 1, when talking about d-quotients, that in general direct image quadruples of relation structures are not d-frames. The difficulties in dealing with d-frame quotients mostly stem from this fact. The next lemma shows that extremal, dense relation structures are among the favourable lucky cases where quotienting coincides with taking the direct image quadruple.

Lemma 3.2.6. *Whenever ν^+ is a nucleus on L^+ such that $\nu^+ \leq \sim\sim$, and ν^- a nucleus on L^- with the same property, the quadruple $(\nu^+[L^+], \nu^-[L^-], q_\nu[c_L], q_\nu[t_L])$ is a d-frame.*

Proof. We need to show that the set $q_\nu[c_L]$ is closed under homogeneus joins, and that the structure is balanced. We denote as $\bigvee^\nu$ the joins in $\nu[L^+]$. If $(\nu^+(x_i^+), \nu^-(x^-)) \in q_\nu[c_L]$, then by Lemma 3.2.5 we deduce that $(\nu^+(x_i^+), \nu^-(x^-)) \in c_L$, and this implies that $(\bigvee_i \nu^+(x_i^+), \nu^-(x^-)) \in c_L$, since c_L is closed under homogeneous joins. By Lemma 3.2.5 again, this also implies that $(\nu^+(\bigvee_i \nu^+(x_i^+)), \nu^-(x^-)) \in c_L$, and this is equivalent to

$$(\bigvee_i^\nu \nu^+(x_i^+), \nu^-(x^-)) \in q_{\sim\sim}[c_L].$$

To show balance, we follow a similar strategy. Suppose that $(\nu^+(x_1^+), \nu^-(x^-)) \in q_\nu[c_L]$ and that $(\nu^+(x_2^+), \nu^-(x^-)) \in q_{\sim\sim}[t_L]$. Then, by Lemma 3.2.5, we can again deduce that $(\nu^+(x_1^+), \nu^-(x^-)) \in c_L$. Since t_L is closed upwards, we also have that $(\nu^+(x_2^+), \nu^-(x^-)) \in t_L$. Then, we obtain $\nu^+(x_1^+) \leq \nu^+(x_2^+)$ by balance of $\mathcal{L}$. □

Let us now define the d-frame analogue of the double pseudocomplementation sublocale. The structure that we will present now was introduced in [34], although there it is defined in terms of sublocales of the frame components L^+ and L^-. We define $\sim\sim\mathcal{L}$ as the quotient of $\mathcal{L}$ by the relation structure

$$\bigvee\{\mathcal{R} \in \mathsf{RS}(\mathcal{L}) : \mathcal{R} \text{ is extremal and dense}\}. \tag{3.1}$$

By Lemma 3.2.5, this is the same as

$$\bigvee\{(R^+, R^-, c_m, t_m) : \nu^+_{R^+} \leq \sim\sim, \nu^-_{R^-} \leq \sim\sim\}.$$

Recall that joins in $\mathsf{RS}(\mathcal{L})$ are computed as componentwise unions. The expression above, then, is the same as

$$(\bigcup\{R^+ : \nu^+_{R^+} \leq \sim\sim\}, \bigcup\{R^- : \nu^-_{R^-} \leq \sim\sim\}, c_m, t_m).$$

Since it is a join we are computing, we obtain the same result if we replace each relation with the smallest congruence containing it. The relation structure above, then, equals

$$(\bigvee\{R^+ \in \mathsf{A}(L^+) : \nu^+_{R^+} \leq \sim\sim\}, \bigvee\{R^- \in \mathsf{A}(L^-) : \nu^-_{R^-} \leq \sim\sim\}, c_m, t_m).$$

Since there is an isomorphism between the ordered collection of frame congruences and that on nuclei on any frame, the extremal relation structure above corresponds to the pair of nuclei where each of them is the join of all nuclei below $\sim\sim$. Let us call this pair of nuclei simply $(\nu^+_{\sim\sim}, \nu^-_{\sim\sim})$. Joins of nuclei are not easy to compute, but it is known (see, for instance, [16]) that, given any closure operator on a frame L, there exists the largest nucleus on L below it. Since $\sim\sim$ is a closure operator, this means in particular that the collection of all nuclei below it has a maximum. If it has a maximum then this maximum must be the least upper bound of this family, and so $\nu_{\sim\sim} \leq \sim\sim$. By Lemma 3.2.5, then, the canonical quotient map $q_{\sim\sim} : \mathcal{L} \twoheadrightarrow \sim\sim \mathcal{L}$ is a dense surjection. By its definition, it will also be extremal. In particular, since the correspondence between relation structures and quotients is antitone, we see from our initial expression (1) that this is the smallest extremal dense quotient of $\mathcal{L}$. We then have the following result, proven in [34].

Theorem 3.2.7. *For any d-frame $\mathcal{L}$, the quotient $\sim\sim\mathcal{L}$ is the smallest dense extremal d-quotient of $\mathcal{L}$.*

In [34] the authors follow the approach of only regarding extremal d-quotients as pointfree bisubspaces of a d-frame: this is why Theorem 3.2.7, there, is regarded as a d-frame version of Isbell's Density Theorem. We will see that a smallest dense d-quotient exists if we add among our assumptions that the d-frame is a de Morgan d-frame.

The d-frame $\sim\sim\mathcal{L}$ does not in general have a simple explicit description. The difficulty is purely frame theoretical, and does not come from the interaction of the two frame components with the con and tot components. Since the quotient is dense, by Lemma 3.2.6 its image is simply the direct image quadruple. The trouble is that, for a frame L, it is not easy to describe explicitly what is the largest nucleus below a certain closure operator. Since the nucleus we are seeking is the largest below $\sim\sim$, when $\sim\sim$ happens to be a nucleus the issue is solved.

Lemma 3.2.8. *Whenever $\mathcal{L}$ is a de Morgan d-frame, the d-frame $\sim\sim\mathcal{L}$ is the quadruple $(\sim\sim[L^+], \sim\sim[L^-], q_{\sim\sim}[c_L], q_{\sim\sim}[t_L])$.*

Proof. If $\mathcal{L}$ is a de Morgan d-frame the largest nucleus below $\sim\sim$ is simply $\sim\sim$ itself on both components. Then, since the quotient $q_{\sim\sim} : \mathcal{L} \twoheadrightarrow \sim\sim \mathcal{L}$ is always dense by Lemma 3.2.5, the corresponding quotient is the direct image quadruple, by Lemma 3.2.6. □

Since we consider all d-frame quotients as pointfree bisubspaces of a d-frame, rather than just the extremal ones, Theorem 3.2.7 is not an analogue of Isbell's Theorem according to our approach. We do not know whether the smallest dense d-quotient of a d-frame exists in general. The aim of the next results is to show that for de Morgan d-frames the smallest dense d-quotient always exists, and that it the same as $q_{\mathfrak{B}} : \mathcal{L} \twoheadrightarrow \mathfrak{B}(\mathcal{L})$. First, with the next two results we show that for certain kinds of d-frames, their Booleanization simply amounts to an increasing the their tot component.

Lemma 3.2.9. *If $\mathcal{L}$ is a d-frame such that $\sim\sim$ is the identity on both component, then the quadruple*

$$(L^+, L^-, c_L, \uparrow\{(x^+, \sim x^+) : x^+ \in L^+\})$$

is a d-frame.

Proof. First, we show that the membership condition $(x^+, x^-) \in \uparrow\{(x^+, \sim x^+) : x^+ \in L^+\}$ in amounts to $\sim x^+ \leq x^-$, and that this is also equivalent to $\sim x^- \leq x^+$. To see that these two conditions are equivalent, observe that by antitonicity of $\sim$ we have that $\sim x^- \leq x^+$ implies that $\sim x^+ \leq \sim\sim x^- = x^-$, where we used our assumption for the last equality. The argument for the converse implication is dual. Now let us show our first claim about the membership condition in $\uparrow\{(x^+, \sim x^+) : x^+ \in L^+\}$. If (x^+, x^-) is in this subset, then there must be some $y^+ \in L^+$ with $(y^+, \sim y^+) \leq (x^+, x^-)$. By antitonicity of $\sim$, this means that $\sim x^- \leq \sim\sim y^+ = y^+$. Conversely, if $\sim x^+ \leq x^-$, then we have $(x^+, \sim x^+) \leq (x^+, x^-)$ and so $(x^+, x^-) \in \uparrow\{(x^+, \sim x^+) : x^+ \in L^+\}$. Let us now show that the totality described above is well defined. The subset is clearly upwards closed. The set also contains all the pairs of the form $(x^+, 1^-)$ and $(1^+, x^-)$, i.e. the empty homogeneous meets, since $\sim 1^+ = 0^- \leq x^+$ for all $x^+ \in L^+$, and similarly for the negative component. If $(x^+, x_1^-), (x^+, x_2^-)$ is in this subset, then $\sim x^+ \leq x_1^-$ and $\sim x^+ \leq x_2^-$, and so $\sim x^+ \leq x_1^- \wedge x_2^-$ and so our subset must be closed under finite homogeneous meets. It remains to check balance. For this, notice that if $(x^+, x_1^-) \in c_L$ and $\sim x^+ \leq x_2^-$, we have the chain of inequalities $x_1^- \leq \sim x^+ \leq x_2^-$. □

Proposition 3.2.10. *If $\mathcal{L}$ is a d-frame such that $\sim\sim$ is the identity on both frame components, then*

$$\mathfrak{B}(\mathcal{L}) = (L^+, L^-, c_L, \uparrow\{(x^+, \sim x^+) : x^+ \in L^+\}).$$

Proof. It suffices to show that $\uparrow\{(x^+, \sim x^+) : x^+ \in L^+\}$ is the smallest tot subset such that it contain $t_L \cup t_{\mathfrak{B}}$, where $t_{\mathfrak{B}} = \{(x^+, \sim x^+) : x^+ \in L^+\} \cup \{(\sim x^-, \sim x^-) : x^- \in L^-\}$ is the increase in the tot component that the Booleanization quotient consists of. Let us abbreviate $t_\sim = \uparrow\{(x^+, \sim x^+) : x^+ \in L^+\}$. Let us first show the inclusion $t_L \cup t_{\mathfrak{B}} t_\sim$. If $(x^+, x^-) \in t_L$,

we must have that $\sim x^+ \leq x^-$, by balance. Then, $(x^+, \sim x^+) \leq (x^+, x^-)$ and we have $(x^+, x^-) \in t_\sim$, which tells us $t_L \subseteq t_\sim$. If $(y^+, y^-) \in t_{\mathfrak{B}}$, we have two cases. If $y^- = \sim y^+$, we immediately have $(y^+, y^-) \in t_\sim$. If $y^+ = \sim y^-$, using our assumption that $\sim\sim$ is the identity we may deduce that $(\sim y^-, \sim\sim y^-) = (y^+, y^-) \in t_\sim$. We then have that $t_L \cup t_{\mathfrak{B}} \subseteq t_\sim$. Let us show that $t_\sim$ is the smallest tot component containing these two sets. Suppose that $t_L \cup t_{\mathfrak{B}} \subseteq t$ for some tot subset $t \subseteq L^+ \times L^-$. Now, suppose that $(z^+, z^-) \in t_\sim$, in particular let $(x^+, \sim x^+) \leq (z^+, z^-)$. Since $(x^+, \sim x^+) \in t_{\mathfrak{B}}$, we have also that $(x^+, \sim x^+) \in t$. Then, we may deduce that $(z^+, z^-) \in t$, because this is an upset. It follows that $t_\sim \subseteq t$. □

The next step is to show that, for a de Morgan d-frame $\mathcal{L}$, the d-frame $\sim\sim \mathcal{L}$ has the property described in the proposition above, and as such its Booleanization consists of increasing the tot component without any extra quotienting.

Lemma 3.2.11. *Dense d-frame quotients are Heyting d-frame morphisms.*

Proof. Suppose that $\mathcal{L}$ is a d-frame and that $q_R : \mathcal{L} \twoheadrightarrow \mathcal{L}/\mathcal{R}$ is a dense d-quotient. We have that if $(f^+(a^+), f^-(a^-)) \in \mathsf{con}$ also $(a^+, a^-) \in c_L$, and so $a^- \leq \sim a^+$. This implies that $f(a^-) \leq f(\sim a^+)$. Then, $f(\sim a^+)$ must be the d-pseudocomplement of $f(a^+)$. □

Lemma 3.2.12. *For every de Morgan d-frame $\mathcal{L}$, the d-frame $\sim\sim\mathcal{L}$ is such that both the d-pseudocomplementation maps are the identities.*

Proof. If $\mathcal{L}$ is a de Morgan d-frame, by Lemma 3.2.8, the d-frame $\sim\sim\mathcal{L}$ is simply the forward image $(\sim\sim[L^+], \sim\sim[L^-], q_{\sim\sim}[c_L], q_{\sim\sim}[t_L])$. Furthermore, by Lemma 3.2.11, the canonical quotient map $q_{\sim\sim} : \mathcal{L} \twoheadrightarrow \sim\sim\mathcal{L}$ is a Heyting d-frame morphism, since it is dense. Since the quotient $q_{\sim\sim}$ preserves d-pseudocomplements, and since by its definition $q_{\sim\sim}(x^+) = q_{\sim\sim}(\sim\sim x^+)$ for each $x^+ \in L^+$, we must have that $\sim\sim q_{\sim\sim}(x^+) = q_{\sim\sim}(x^+)$ for each $x^+ \in L^+$. □

We now use the results we have got up to now to give an explicit description of the Booleanization of a de Morgan d-frame.

Proposition 3.2.13. *If $\mathcal{L}$ is a de Morgan d-frame, we have*

$$\mathfrak{B}(\mathcal{L}) = \mathfrak{B}(\sim\sim\mathcal{L}) = (\sim\sim[L^+], \sim\sim[L^-], q_{\sim\sim}[c_L], \uparrow\{(\sim\sim x^+, \sim x^+) : x^+ \in L^+\}).$$

Proof. Suppose that $\mathcal{L}$ is a de Morgan d-frame. Recall that by Lemma 3.2.12 double d-pseudocomplementation maps of $\sim\sim\mathcal{L}$ are both identities. This implies, by 3.2.10, that $\mathfrak{B}(\sim\sim\mathcal{L})$ is the d-frame in the statement. We now show that $\mathfrak{B}(\mathcal{L}) = \mathfrak{B}(\sim\sim\mathcal{L})$ by showing that a d-frame map $f : \mathcal{L} \to \mathcal{M}$ that factors through one also factor through the other, and vice versa. For every d-frame $\mathcal{M}$, we have the following.

- A map $f : \mathcal{L} \to \mathcal{M}$ factors through $\mathfrak{B}(\sim\sim\mathcal{L})$ if and only if $f^+(x^+) = f^+(\sim\sim x^+)$ for every $x^+ \in L^+$, and similarly for the negative component, and also $(f^+(\sim\sim x^+), f^+(\sim x^+)) \in t_M$.
- A map $f : \mathcal{L} \to \mathcal{M}$ factors through $\mathfrak{B}(\mathcal{L})$ if and only if $(f^+(x^+), f^-(\sim x^+)) \in t_M$ and $(f^+(\sim x^-), f^-(x^-)) \in t_M$ for every $x^+ \in L^+$ and every $x^- \in L^-$.

Suppose, then, that a map $f : \mathcal{L} \to \mathcal{M}$ factors through $\mathfrak{B}(\sim\sim\mathcal{L})$. For $x^+ \in L^+$, we combine the two conditions that this amounts to and we deduce that $(f^+(x^+), f^-(\sim x^+)) \in t_M$ for every $x^+ \in L^+$. For $x^- \in L^-$, we have that $(f^+(\sim\sim\sim x^-), f^-(\sim\sim x^-)) \in t_M$ by assumption. Using the fact that $f^-(x^-) = f^-(\sim\sim x^-)$, we obtain that $(f^+(\sim x^-), f^-(x^-)) \in t_M$. Then, the map factors through $\mathfrak{B}(\mathcal{L})$.

For the converse, suppose that f factors through $\mathfrak{B}(\mathcal{L})$. We only need to show that f identifies elements with their double d-pseudocomplement. We claim that $f^+(\sim\sim x^+)$ and $f^+(x^+)$ are both d-complements of $f^-(\sim x^+)$ in $\mathcal{M}$. On the one hand both $(\sim\sim x^+, \sim x^+) \in c_L$ and $(x^+, \sim x^+) \in c_L$. Because f is a d-frame map, then, both $(f^+(x^+), f^-(\sim x^+)) \in c_M$ and $(f^+(\sim\sim x^+), f^-(\sim x^+)) \in c_M$. On the other hand, by our assumption that f factors through $\mathfrak{B}(\mathcal{L})$ we have both $(f^+(x^+), f^-(\sim x^+)) \in t_M$ and $(f^+(\sim\sim x^+), f^-(\sim x^-)) \in t_M$. Then, indeed the elements $f^+(x^+)$ and $f^+(\sim\sim x^+)$ have the same d-complement. Since d-complements are always unique, these must in fact be the same elements. Then, our map f also factors through $\mathfrak{B}(\sim\sim\mathcal{L})$. □

The last step before we show the main theorem is to show that the Booleanization of de Morgan d-frames is, in fact, dense.

Proposition 3.2.14. *The Booleanization quotient map $q_{\mathfrak{B}} : \mathcal{L} \to \mathfrak{B}(\mathcal{L})$ of a d-frame $\mathcal{L}$ is dense if and only if $\mathcal{L}$ is a de Morgan d-frame.*

Proof. Let $q_{\mathfrak{B}} : \mathcal{L} \twoheadrightarrow \mathfrak{B}(\mathcal{L})$ be the canonical quotient map, and let $\nu^{\pm}_{\mathfrak{B}}$ be its two frame nuclei. If a d-frame $\mathcal{L}$ is de Morgan, then $\sim\sim\mathcal{L}$ is of the form described in Proposition 3.2.13. This is extremal, and so it is dense by Lemma 3.2.5, and since $\mathfrak{B}(\mathcal{L})$ only updates the tot component of $\sim\sim\mathcal{L}$, by Proposition 3.2.14, the quotient map $q_{\mathfrak{B}} : \mathcal{L} \twoheadrightarrow \mathfrak{B}(\mathcal{L})$ must be dense too. For the converse, suppose that the map $q_{\mathfrak{B}} : \mathcal{L} \to \mathfrak{B}(\mathcal{L})$ is dense. Then, by Lemma 3.2.5, we must have $\nu^+_{\mathfrak{B}} \leq \sim\sim$. We also have that, by balance, $q^+_{\mathfrak{B}}(\sim\sim x^+) = q^+_{\mathfrak{B}}(x^+)$, since these are both d-complements of $q_{\mathfrak{B}}(\sim x^+)$. Then, we must have that $\sim\sim x^+ \leq \nu_{\mathfrak{B}}(\sim\sim x^+) \leq \nu_{\mathfrak{B}}(x^+)$, and so we also have the reverse inequality $\sim\sim \leq \nu^+_{\mathfrak{B}}$. Since $\sim\sim = \nu_{\mathfrak{B}}$ is a nucleus on both frame components, the d-frame $\mathcal{L}$ must be a de Morgan d-frame. □

We are ready to prove the desired result.

Theorem 3.2.15. *If $\mathcal{L}$ is a de Morgan d-frame, $\mathfrak{B}(\mathcal{L})$ is its smallest dense d-quotient.*

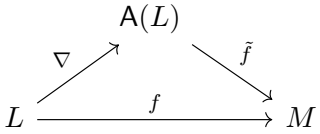
Proof. Suppose that $\mathcal{L}$ is a de Morgan d-frame, and that $q_D : \mathcal{L} \twoheadrightarrow \mathcal{L}/\mathcal{D}$ is a dense d-quotient map. For the desire claim, by Lemma 3.1.3 it suffices to show that there is a d-frame surjection $\mathcal{L}/\mathcal{D} \twoheadrightarrow \mathfrak{B}(\mathcal{L})$. Since q_D is dense, the corresponding frame nuclei ν_D^+ and ν_D^- are below $\sim\!\sim$ by Lemma 3.2.5. We also have that $\nu_{\mathfrak{B}}^+ = \sim\!\sim$ by Proposition 3.2.13. This means that $\nu_D^+ \leq \nu_{\mathfrak{B}}^+$. By the order anti-isomorphism between nuclei and quotients of a frame, we deduce that $\sim\!\sim[L^+] \leq L^+/D^+$ in the lattice of quotients of the frame L^+. Then, there must be a frame quotient $q^+ : L^+/D^+ \twoheadrightarrow \sim\!\sim[L^+]$ acting as $q_D^+(x^+) \mapsto \sim\!\sim x^+$, and a similar one on the negative component. It remains to show that the pairing of these frame quotients respects the con and tot subsets. Denote as $c_{\mathfrak{B}}$ and $t_{\mathfrak{B}}$ the consistency and the totality components of $\mathfrak{B}(\mathcal{L})$, respectively. If $(q_D^+(x^+), q_D^-(x^-)) \in c_D$ then, by density, $(x^+, x^-) \in c_L$ and so $(\sim\!\sim x^+, \sim\!\sim x^-) \in c_{\mathfrak{B}}$. If $(q_D^+(x^+), q_D^-(x^-)) \in t_D$, by balance, we have that $\sim q_D^-(x^-) \leq q_D^+(x^+)$. Since dense morphisms preserves d-pseudocomplementation, by 3.2.11, this means that $q_D^-(\sim x^-) \leq q_D^+(x^+)$. Since q^+ and q^- are monotone, this implies that $q^+(q_D^-(\sim x^-)) \leq q^+(q_D^+(x^+))$, and by definition of the quotient q^+ this means that $\sim x^- \leq \sim\!\sim x^+$. Recall that the tot component of the Booleanization is defined as $t_{\mathfrak{B}} = \uparrow\{(\sim\!\sim x^+, \sim x^+) : x^+ \in L^+\}$. We have that $(\sim x^-, \sim\!\sim x^-) \leq (\sim\!\sim x^+, \sim\!\sim x^-)$, and so we can deduce that $(\sim\!\sim x^+, \sim\!\sim x^-) \in t_{\mathfrak{B}}$. It follows that the pairing $(q^+, q^-) : \mathcal{L}/\mathcal{D} \to \mathfrak{B}(\mathcal{L})$ is a d-frame surjection, and so we must have $\mathfrak{B}(\mathcal{L}) \leq \mathcal{L}/\mathcal{D}$. □

Chapter 4

The frame assembly as a d-frame

Introduction

The collection of $\mathsf{S}(L)$ of all sublocales of a frame L is known to always be a coframe. Then, its opposite $\mathsf{S}(L)^{op}$ is a frame. In the context of frame theory, one considers the isomorphic frame $\mathsf{A}(L)$ of frame congruences of L. This is called the *assembly* of the frame L. The map $\nabla : L \to \mathsf{A}(L)$, assigning to each element of L the corresponding closed congruence, is a frame embedding. Joyal and Tierney proved in [28] that the assembly of a frame L has the following universal property. Whenever $f : L \to M$ is a frame map such that $f(a)$ is complemented for each $a \in L$, then there is a unique frame map $\tilde{f} : \mathsf{A}(L) \to M$ making the following triangle commute.



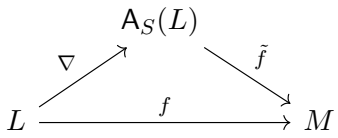
In this chapter we explore how this universal property makes it apparent that the assembly of a frame is bitopological in nature. In [31] it is shown that the assembly of any frame L is the patch of a d-frame

$$(L, \mathsf{Filt}(L), \mathsf{Con}_L, \mathsf{Tot}_L),$$

where Con_L and Tot_L are the smallest consistency and totality subsets making the pairs $(a, \uparrow a)$ into pairs of mutual d-complements, for each $a \in L$. In the first section we will start from a construction of the assembly as a free frame having the required universal property, and derive from this that the construction in [31] indeed presents the assembly.

In [55] it is shown (Section 16.2) that there are subframes of the assembly $\mathsf{A}(L)$ of a frame L such that they have a more general version of the above universal property. In particular, it

is shown that for any frame L and any arbitrary subset S, there is a subframe $\mathsf{A}_S(L) \subseteq \mathsf{A}(L)$ such that $\nabla[L] \subseteq \mathsf{A}_S(L)$, and such that $\nabla : L \to \mathsf{A}_S(L)$ has the following universal property. Whenever $f : L \to M$ is a frame map such that $f(s)$ is complemented for each $s \in S$, then there is a unique frame map $\tilde{f} : \mathsf{A}_S(L) \to M$ making the following triangle commute.



We call such frames *partial assemblies* of the frame L. We adapt the technique of [31] to show that partial assemblies, too, can be presented by d-frames.

In Section 4.2 we will look at the bitopological view of Priestley and Stone duality and see that this can be rephrased in terms of d-frame presentations of partial assemblies. We will give a very broad overview of Priestley and Stone duality, and provide the reader with several pointers to the literature. Our main focus is the result that the d-frames corresponding to Priestley spaces are exactly the *coherent* ones, a known result. Our contribution is to point out that the matter can be described in terms of d-frame presentations of certain partial assemblies. For completeness, we will also give explicit proofs of the bitopological phrasing of Priestley and Stone duality.

Finally, in Section 4.3 we work towards giving a broader, more categorical view of the situation. We will introduce the category **sFrm**, whose objects are pairs (L, S) where L is a frame and $S \subseteq L$ a subset. Its morphisms $f : (L, S) \to (M, T)$ are frame maps $f : L \to M$ such that $f[S] \subseteq T$. The category **CohFrm** of coherent frames with frame maps such that the image of compact elements is compact is isomorphic to the full subcategory of **sFrm** determined by the objects of the form $(L, \mathsf{K}(L))$, where L is a coherent frame and $\mathsf{K}(L)$ is the collection of its compact elements. We will see that we have an endofunctor $\mathsf{sA} : \mathbf{sFrm} \to \mathbf{sFrm}$, such that on objects it takes a pair (L, S) and maps it to the pair $(\mathsf{A}_S(L), \nabla[L])$. We will also introduce a category **sTop**, whose objects are pairs (X, S) where X is a space and $S \subseteq \Omega(X)$. Similarly as for coherent frames, the category of coherent spaces is equivalent to a full subcategory of **sTop**. We will then show that there is a spectrum functor $\mathsf{spt} : \mathbf{sFrm}^{op} \to \mathbf{sTop}$ which can be described in terms of a dualizing object, as well as an open set functor $\mathsf{s}\Omega : \mathbf{sTop} \to \mathbf{sFrm}^{op}$ with $\mathsf{s}\Omega \dashv \mathsf{spt}$. Then, we will introduce a functor $sSk : \mathbf{sTop} \to \mathbf{sTop}$ and prove that it is the spatial analogue of sA in the precise sense that

the following square commutes up to natural isomorphism.

$$\begin{array}{ccc} \mathbf{sFrm}^{op} & \xrightarrow{\mathsf{sA}} & \mathbf{sFrm}^{op} \\ \Big\downarrow{\scriptstyle \mathsf{spt}} & & \Big\downarrow{\scriptstyle \mathsf{spt}} \\ \mathbf{sTop} & \xrightarrow{sSk} & \mathbf{sTop} \end{array}$$

We will then connect **sFrm** to the bitopological world in the same way. We will show that we have a functor $\mathsf{s}\mathcal{A} : \mathbf{sFrm} \to \mathbf{dFrm}$ assigning to each pair (L, S) the d-frame presentation of the partial assembly $\mathsf{A}_S(L)$. We will then find a spatial analogue $biSk_S : \mathbf{sTop} \to \mathbf{BiTop}$ of this functor, and prove that indeed it is its spatial analogue by showing that a square similar to the square above commutes up to natural isomorphism.

4.1 D-frame presentations of partial assemblies

We express the universal property of the assembly by saying that the ∇ embedding *provides complements freely* for all elements of L. Because of its universal property, the assembly of a frame may be expressed as a free structure. For a frame L, denote as $' : L \to L^{op}$ the canonical anti-isomorphism, and set

$$C = \{(\ulcorner a \urcorner \wedge \ulcorner a' \urcorner, \ulcorner 0 \urcorner) : a \in L\} \cup \{(\ulcorner 1 \urcorner, \ulcorner a \urcorner \vee \ulcorner a' \urcorner) : a \in L\},$$

where the elements of the form $\ulcorner x \urcorner$, as usual, denote the generators as mapped into in the free structure. In particular, we may write

$$\mathsf{A}(L) \cong \mathbf{Fr}\langle L \text{ qua frame}, L^{op} \text{ qua set} | C \rangle.$$

Forcing the elements of L^{op} to behave as complements of those in L means that the poset L^{op} will be mapped into the free frame above as a distributive lattice, hence we may also write

$$\mathsf{A}(L) \cong \mathbf{Fr}\langle L \text{ qua frame}, L^{op} \text{ qua DL} | C \rangle.$$

We may also deduce

$$\mathsf{A}(L) \cong \mathbf{Fr}\langle L \text{ qua frame}, \mathsf{Idl}(L^{op}) \text{ qua frame} | C \rangle,$$

where we have abused notation and identified relation C with the relation making elements in pairs $(\ulcorner a \urcorner, \ulcorner \downarrow a' \urcorner)$ into mutual complements. We deduce that, for any frame L,

$$\mathsf{A}(L) \cong (L \oplus \mathsf{Filt}(L))/C. \tag{4.1}$$

Because we have constructed the assembly as a free structure, the canonical embedding is the map $L \to \mathsf{A}(L)$ acting as $a \mapsto \ulcorner a \urcorner$ for each $a \in L$, mapping each generator to the corresponding syntactic expression of $\mathsf{A}(L)$. From the fact that the assembly of a frame can be expressed as (4.1), we see that it is bitopological in nature. Since quotienting by the relation C only involves making pairs of elements disjoint and covering - as we will see shortly - it may be expressed as the patch of a d-frame. In [31] this is proven, and an explicit description of this d-frame is given. In particular, it is shown that the assembly of a frame L is isomorphic to the patch of the d-frame

$$(L, \mathsf{Filt}(L), \{(x, F) : x \leq \bigwedge F\}, \{(x, F) : x \in F\}),$$

We call this d-frame $\mathcal{A}(L)$. By the main result in [31], then, we know that the canonical mapping $\ulcorner - \urcorner : L \to \Pi(\mathcal{A}(L))$ provides complements freely to elements of L. We then know that the patch of $\mathcal{A}(L)$ is isomorphic to the assembly $\mathsf{A}(L)$ because they have the same universal property. We now want to turn this result into an explicit isomorphism $\Pi(\mathcal{A}(L)) \cong \mathsf{A}(L)$, and it is not difficult to do so. Explicitly, the isomorphism between these is $\alpha : \Pi(\mathcal{A}(L)) \cong \mathsf{A}(L)$ defined on generators as $\ulcorner x \urcorner \mapsto \nabla(x)$ and $\alpha(\ulcorner \uparrow x \urcorner) = \Delta(x)$ for every $x \in L$.

Remark 4.1.1. Recall that we had warned the reader in Chapter 1 (see Remark 1.2.11) that when we have a d-frame $\mathcal{L}$ the two maps $\ulcorner - \urcorner : L^+ \to \Pi(\mathcal{L})$ and $\ulcorner - \urcorner : L^- \to \Pi(\mathcal{L})$ are not guaranteed to be embeddings. We have encountered an example of this. We have just noticed that the map $\ulcorner - \urcorner : \mathsf{Filt}(L) \to \Pi(\mathcal{A}(L))$ is essentially the map

$$\begin{aligned} &\mathsf{Filt}(L) \to \mathsf{A}(L), \\ &F \mapsto \bigvee_{f \in F} \Delta(f). \end{aligned}$$

In terms of sublocales, this corresponds to sending a filter F to the fitted sublocale $\bigcap_f \mathfrak{o}(f)$. From our analysis in subsection 2.2.1 we know that this is not injective, in general.

In [55], the author generalizes the result of Joyal and Tierney, and proves that for an arbitrary subset S of a frame L there is an embedding of L into a frame providing complements freely for elements of S. Additionally, it is proven that this frame is always a subframe of the assembly $\mathsf{A}(L)$. For every frame L and every subset $S \subseteq L$, denote as $\nabla_S : L \hookrightarrow \mathsf{A}_S(L)$ the canonical embedding into the frame providing complements freely for the elements of S. By

a *partial assembly* of a frame L we understand a frame of the form $\mathsf{A}_S(L)$ for some subset $S \subseteq L$. A partial assembly is a free construction, in particular

$$\mathsf{A}_S(L) \cong \mathbf{Fr}\langle L \text{ qua frame}, S^{op} \text{ qua set } |C_S\rangle,$$

where C_S is the restriction of the relation C to $\ulcorner S \urcorner \times \ulcorner S^{op} \urcorner$. Because complemented elements are closed under the lattice operations, we do not lose any generality if we assume that S is a sublattice of L. For a frame L and a subset $S \subseteq L$, we denote as $D_S[L]$ the distributive sublattice of L generated by S. Similarly as in the assembly case, we can formulate the following isomorphism. For any frame L and any subset $S \subseteq L$,

$$\mathsf{A}_S(L) \cong (L \oplus \mathsf{Filt}(D_S[L]))/C_S. \tag{4.2}$$

We would now like to have d-frame presentations of partial assemblies based on this result. To simplify our description, we want to turn the frame $\mathsf{Filt}(D_S[L])$ into a frame of filters of L. For any subset $S \subseteq L$, we refer as $\mathsf{Filt}_S(L)$ to the subframe of $\mathsf{Filt}(L)$ generated by $\{\uparrow s : s \in S\}$. For any frame L and any subset $S \subseteq L$ we have mutually inverse isomorphisms

$$\begin{aligned} &i : \mathsf{Filt}_S(L) \leftrightarrows \mathsf{Filt}(D_S[L]) : j \\ &F \mapsto F \cap D_S[L] \\ &\{x \in L : g \leq x \text{ for some } g \in G\} \hookleftarrow G. \end{aligned}$$

To see that these are mutually inverse, we notice that since $D_S[L]$ is closed under finite meets, for a filter in $F \in \mathsf{Filt}_S(L)$ taking $F \cap D_S[L]$ allows us to completely recover F by taking the upward closure of this set. For the other composition, notice that when we have a filter G of $D_S[L]$ and we take its upward closure in L we cannot possibly add any new elements from $D_S[L]$, since G is already upward closed in $D_S[L]$. Then, $i \circ j$ is the identity on $\mathsf{Filt}(D_S[L])$. We observe that the isomorphism j above simply maps a principal upset $\uparrow x \subseteq D_S[L]$ to the corresponding principal upset $\uparrow x \subseteq L$. We use this fact to deduce that for a frame L and a subset $S \subseteq L$ the partial assembly $\mathsf{A}_S(L)$ may be expressed as

$$(L \oplus \mathsf{Filt}_S(L))/C_S,$$

where C_S is the relation making the pairs $(\ulcorner s \urcorner, \ulcorner \uparrow s \urcorner)$, for $s \in S$, into mutual complements. Let us now construct our d-frame presentation.

Lemma 4.1.2. *For any frame L and subset $S \subseteq L$, the quadruple*

$$(L, \mathsf{Filt}_S(L), \{(x, F) : x \leq \bigwedge F\}, \{(x, F) : x \in F\})$$

is a d-frame.

Proof. If (x, F) is such that $x \leq \bigwedge F$, and if $y \leq x$ and $G \subseteq F$, then clearly we have $y \leq x \leq \bigwedge F \leq \bigwedge G$, and so the con component of this quadruple is a downset. Closure under homogeneous joins is similarly clear, since $x_i \leq \bigwedge F$ implies $\bigvee_i \leq \bigwedge F$. As $\mathsf{Filt}_S(L) \subseteq \mathsf{Filt}(L)$ is a subframe, the joins in these two frames are computed in the same way, and so $x \leq \bigwedge F_i$ implies that we have the chain of inequalities $x \leq \bigwedge_i \bigwedge F_i = \bigwedge(\bigcup_i F_i) = \bigwedge \bigvee_i F_i$. For the tot component, upward closure is clear: if $x \in F$ and $x \leq y$ and $F \subseteq G$ then we have $y \in F \subseteq G$, since filters are closed upwards. Closure under finite homogeneous meets on the left holds because filters are closed under meets. Since $\mathsf{Filt}_S(L)$ is a subframe of $\mathsf{Filt}(L)$, finite meets are computed as set theoretical intersections, and so homogeneous closure on the other component is clear. For balance on one side, suppose that $x \leq \bigwedge F$ and that $x \in G$. We then must have $F \subseteq {\uparrow}x \subseteq G$. For balance on the other side, suppose that $x \leq \bigwedge F$ and that $y \in F$. We then have that $x \leq \bigwedge F \leq y$. □

Proposition 4.1.3. *For any frame L and any arbitrary subset $S \subseteq L$, the consistency $\{(x, F) : x \leq \bigwedge F\}$ and the totality $\{(x, F) : f \leq x \text{ for some } f \in F\})$ on $L \times \mathsf{Filt}_S(L)$ are the smallest ones making the pairs $\{(s, {\uparrow}s) : s \in S\}$ into pairs of mutual d-complements.*

Proof. Let Con_S and Tot_S be, respectively, the smallest consistency and totality making the pairs $(s, {\uparrow}s)$, for $s \in S$, into mutual d-complements. The consistency $\{(x, F) : x \leq \bigwedge F\}$ and the totality $\{(x, F) : x \in F\}$ are such that every pair $(s, {\uparrow}s)$ is in both of them, so we have the inclusions $\mathsf{Con}_S \subseteq \{(x, F) : x \leq \bigwedge F\}$ and $\mathsf{Tot}_S \subseteq \{(x, F) : x \in F\}$. For the reverse inclusions, suppose that (x, F) is such that $x \leq \bigwedge F$. Since $F \in \mathsf{Filt}_S(L)$ we may write $F = \bigvee_i {\uparrow}s_1 \cap ... \cap {\uparrow}s_{n_i}$ for some family $\bigcup_i \{s_1, ...s_{n_i}\} \subseteq S$, and so it is also true that $\bigwedge F = \bigwedge\{s_1 \vee ... \vee s_{n_i} : i \in I\}$. For each $i \in I$, we then have $x \leq s_1 \vee ... \vee s_{n_i}$. Then, since Con_S contains each pair $(s_{m_j}, {\uparrow}s_{m_j})$, it must also contain pairs $(s_1 \vee ... \vee s_{n_i}, {\uparrow}(s_1 \vee ... \vee s_{n_i}))$, using both downward closure and closure under homogeneous joins. By downwards closure again, $(x, {\uparrow}s_1 \vee ... \vee s_{n_i}) \in \mathsf{Con}_S$; and by homogeneous closure again, $(x, F) \in \mathsf{Con}_S$. We have obtained the desired set inclusion. Now, suppose that $x \in F$. Let us write $F = \bigvee_i {\uparrow}s_1 \cap ... \cap {\uparrow}s_{n_i}$ for each $s_{m_j} \in S$. There must then be some finite subset $J \subseteq I$ with $\bigwedge_{j \in J} s_1 \vee ... \vee s_{n_j} \leq x$. From the fact that $(s_{m_l}, {\uparrow}s_{m_l}) \in \mathsf{Tot}_S$, and using both upward closure and homogeneous closure of Tot_S, we obtain $(s_1 \vee ... \vee s_{n_i}, {\uparrow}(s_1 \vee ... \vee s_{n_i})) \in \mathsf{Tot}_S$ for every $i \in I$. Using both these properties again, we deduce that $(\bigwedge_{j \in J} s_1 \vee ... \vee s_{n_j}, \bigvee_{j \in J} {\uparrow}(s_1 \vee ... \vee s_{n_j})) \in \mathsf{Tot}_S$. Finally, we obtain the desired inequality by noticing that upward closure implies $(x, \bigvee_{j \in J} {\uparrow}(s_1 \vee ... \vee s_{n_j})) \leq (x, F) \in \mathsf{Tot}_S$. □

For any frame L and any subset $S \subseteq L$, we call $\mathcal{A}_S(L)$ the d-frame

$$(L, \mathsf{Filt}_S(L), \{(x, F) : x \leq \bigwedge F\}, \{(x, F) : x \in F\}).$$

We will sometimes abbreviate this as $(L, \mathsf{Filt}_S(L), \mathsf{Con}_S, \mathsf{Tot}_S)$. We recall Proposition 1.3.6. If we have a d-frame $\mathcal{L}$ and a relation structure $\mathcal{R}_{CT} = (\mathsf{id}^+, \mathsf{id}^-, C, T)$, we may consider the smallest consistency C' and totality T' such that they contain $c_L \cup C$ and $t_L \cup T$, respectively. Proposition 1.3.6 states that if the pair (C', T') is balanced then $\mathcal{L}/\mathcal{R}_{CT} = (L^+, L^-, C', T')$. This fact together with Lemmas 4.1.3 and 4.1.2 tells us the following.

Proposition 4.1.4. *For a frame L and an arbitrary subset $S \subseteq L$ the d-frame $\mathcal{A}_S(L)$ is the result of quotienting $(L, \mathsf{Filt}_S(L), c_m, t_m)$ by the relation structure*

$$(\mathsf{id}^+, \mathsf{id}^-, \{(s, {\uparrow}s) : s \in S\}, \{(s, {\uparrow}s) : s \in S\}).$$

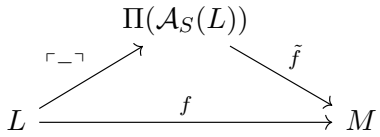
Before we show that the patches of such d-frames present partial assemblies, we need a few more intermediate results.

Lemma 4.1.5. *For a frame L and an arbitrary subset $S \subseteq L$, if a frame map $f : L \to M$ is such that $f(s)$ is complemented for all $s \in S$, there is a frame map $f_F : \mathsf{Filt}_S(L) \to M$ defined as $f({\uparrow}s) = \neg f(s)$ for all $s \in S$.*

Proof. By the universal property of the ideal completion of a distributive lattice, it suffices to show that the map $\neg f : D_S[L]^{op} \to M$ is a lattice map. Indeed, since all elements $f(s)$ are complemented in M, so are their finite meets and finite joins and - in particular - complementation reverses the lattice operations. An arbitrary element of $D_S[L]$ is of the form $\bigwedge_n \bigvee_m s_{s,m}$ of $D_S[L]$; by the reasoning above we know that the element $f(\bigwedge_n \bigvee_m s_{(n,m)})$ is complemented with complement $\bigvee_n \bigwedge_m \neg f(s_{(n,m)})$. □

We are now ready to prove the desired theorem.

Theorem 4.1.6. *For any frame L and any subset $S \subseteq L$ we have that whenever $f : L \to M$ is a frame map such that $f(s)$ is complemented for every $s \in S$ there is a unique frame morphism $\tilde{f} : \Pi(\mathcal{A}_S(L)) \to M$ making the following triangle commute.*



Proof. We know by Lemma 1.3.11 and by Proposition 4.1.4 that the patch $\Pi(\mathcal{A}_S(L))$ is the result of quotienting $L \oplus \mathsf{Filt}_S(L)$ by the relation

$$C_S = \{(\ulcorner s \urcorner \wedge \ulcorner \uparrow s \urcorner, 0) : s \in S\} \cup \{(1, \ulcorner s \urcorner \vee \ulcorner \uparrow s \urcorner) : s \in S\}.$$

There is then a canonical map $L \to \Pi(\mathcal{A}_S(L))$ defined as $a \mapsto \ulcorner a \urcorner$ for every $a \in L$. To show the desired claim, then, it suffices to show that this canonical map into this frame quotient satisfies the same universal property as the canonical embedding $\nabla_S : L \to \mathsf{A}_S(L)$. Suppose, then, that $f : L \to M$ is a frame map such that $f(s)$ is complemented in M for each $s \in S$. By Lemma 4.1.5, there is a map $f_F : \mathsf{Filt}_S(L) \to M$ defined as $f(\uparrow s) = \neg f(s)$. The pairing $f \oplus f_F : L \oplus \mathsf{Filt}_S(L) \to M$ factors through the quotienting by C_S, since indeed the elements $f(s)$ and $f_F(\uparrow s)$ are complements of one another in M. Then, there is a unique map $\tilde{f} : (L \oplus \mathsf{Filt}_S(L))/C_S \to M$ such that $\tilde{f}(\ulcorner a \urcorner) = f(a)$ for each $a \in L$ and $\tilde{f}(\ulcorner \uparrow s \urcorner)$ for every $s \in S$. We have not shown, yet, that this is also unique with the desired property, i.e. the formally weaker property that $\tilde{f}(\ulcorner a \urcorner) = f(a)$ for all $a \in L$. If there is a map $\tilde{g} : (L \oplus \mathsf{Filt}_S(L))/C_S \to M$ with this property, then - by the relation C_S - it is also such that $\tilde{g}(\ulcorner \uparrow s \urcorner) = \neg f(s)$ for all $s \in S$, and so it equals $\tilde{f}$ on the generators from $\mathsf{Filt}_S(L)$, too. We then have $\tilde{f} = \tilde{g}$. □

For a partial assembly $\mathsf{A}_S(L)$, we refer to the d-frame $\mathcal{A}_S(L)$ as the *canonical* d-frame presentation of $\mathsf{A}_S(L)$. We now know that we have a frame isomorphism $\Pi(\mathcal{A}_S(L)) \cong \mathsf{A}_S(L)$ and, as in the assembly case, we may make this explicit by looking at the canonical embeddings $\ulcorner - \urcorner : L \to \Pi(\mathcal{A}_S(L))$ and $\nabla_S : L \to \mathsf{A}_S(L)$. The following result is known, and in fact is the definition of partial assembly given in [55] (in Section 16.2).

Proposition 4.1.7. *For any frame L and any subset $S \subseteq L$, the partial assembly $\mathsf{A}_S(L)$ is the subframe of $\mathsf{A}(L)$ generated by the collection*

$$\{\nabla(x) : x \in L\} \cup \{\Delta(s) : s \in S\}$$

We now come back to a structure that we introduced at the end of Chapter 1 and show that this is an example of a d-frame presentation of some partial assembly. Recall that, for any two frames L^+ and L^-, there is a d-frame

$$(L^+ \oplus L^-, \mathsf{filt}(L^+ \oplus L^-), \mathsf{Con}, \mathsf{Tot}),$$

where $\mathsf{filt}(L^+ \oplus L^-)$ denotes the collection of finitary filters of $L^+ \oplus L^-$, and Con and Tot are the smallest consistency and totality subsets making the pairs $(\ulcorner x^+ \urcorner, \uparrow \ulcorner x^+ \urcorner)$ and $(\ulcorner x^- \urcorner, \uparrow \ulcorner x^- \urcorner)$

into pairs of mutual d-complements. If we denote as $\mathsf{A}_{\mathsf{fin}}(L^+ \oplus L^-)$ the partial assembly of $L^+ \oplus L^-$ with the universal property that it provides complements freely to the finitary elements of $L^+ \oplus L^-$, we have the following.

Proposition 4.1.8. *For any two frames L^+ and L^-, we have*

$$\mathcal{A}_{\mathsf{fin}}(L^+ \oplus L^-) = (L^+ \oplus L^-, \mathsf{filt}(L^+ \oplus L^-), \mathsf{Con}, \mathsf{Tot}).$$

4.1.1 Bitopological spectra of partial assemblies

It is clear that partial assemblies have a bitopological structure, and by presenting them via d-frames we have made it explicit. Let us now do the same with the spectra of partial assemblies, by examining the d-spectra of their canonical d-frame presentations.

For a topological space X, its *Skula* space is the space with $|X|$ as its underlying set of points, and whose opens are the topology generated by the open sets of X together with the closed ones. It is well known that the spectrum of the assembly of a frame L is the Skula space of $\mathsf{pt}(L)$ (see for instance [37], Chapter 6, Proposition 7.3.1). We will focus on this result in detail in Chapter 7, where we will look at a bitopological generalization of a Skula space. Just as a Skula space "provides complements freely" for the starting topology by adding the complements of the opens to the topology, spectra of partial assemblies add to the topology only the complements of a selected collection of opens. We will now show this, or rather, the bitopological refinement of this result which we get by considering these Skula-like spaces as bispaces. Let us start by showing that the points of a frame L are in bijective correspondence with the d-points of any d-frame of the form $(L, \mathsf{Filt}(D_S[L]), \mathsf{Con}_S, \mathsf{Tot}_S)$. Recall that by Lemma 4.1.5, for every frame map $f : L \to 2$ there is a frame map $\mathsf{Filt}_\neg : \mathsf{Filt}_S(L) \to 2$ defined on generators as $\mathsf{Filt}_\neg(\uparrow a) = \neg f(a)$ for every $s \in S$. We keep this notation for the following proof.

Lemma 4.1.9. *Let L be a frame and $S \subseteq L$ an arbitrary subset. Then, there is a bijection $|\mathsf{pt}(L)| \cong |\mathsf{dpt}(\mathcal{A}_S(L))|$ assigning to each frame map $f : L \to 2$ the d-frame map $(f, \mathsf{Filt}_\neg(f))$.*

Proof. Suppose that $f : L \to 2$ is a frame map. In the d-frame $\mathcal{A}_S(L)$ we have that the consistency and totality are the smallest making the pairs $(s, \uparrow s)$ into pairs of mutual d-complements, for each $s \in S$. To show that the pair of frame maps $(f, \mathsf{Filt}_\neg(f)) : \mathcal{A}_S(L) \to \mathbf{2}$ form a d-frame map it suffices to show that $(f^+(s), f^-(\uparrow s))$ is a pair of mutual d-complements whenever $s \in S$. This indeed holds for our pair of frame maps, and so the assignment $f \mapsto (f, \mathsf{Filt}_\neg(f))$ is then a well-defined map $|\mathsf{pt}(L)| \to |\mathsf{dpt}(\mathcal{A}_S(L))|$. The assignment is clearly injective. For surjectivity, consider any point $(f^+, f^-) : \mathcal{A}_S(L) \to \mathbf{2}$. We need to show that

the map $f^- : \mathsf{Filt}_S(L) \to 2$ is defined on generators as $f^-(\uparrow s) = \neg f(s)$ for every $a \in S$. Since for every $s \in S$ the pair $(s, \uparrow s)$ is a pair of mutual d-complements, we have that $(f^+(s), f^-(\uparrow s)) \in c_2 \cap t_2$. The only two elements of 2×2 that are in $c_2 \cap t_2$ are $(1, 0)$ and $(0, 1)$. It follows that either $f^+(s) = 1$ and $f^-(\uparrow s) = 0$ or $f^+(s) = 0$ and $f^-(\uparrow s) = 1$. In both cases, $f^-(\uparrow s) = \neg f^+(s)$. Then, we have that $(f^+, f^-) = (f^+, \mathsf{Filt}_\neg(f^+))$. The assignment $f \mapsto (f, \mathsf{Filt}_\neg(f))$ is then surjective, too. □

Recall that, for a frame L, we refer to its spatialization map as $\phi_L : L \twoheadrightarrow \Omega(\mathsf{pt}(L))$.

Proposition 4.1.10. *For any frame L and any subset $S \subseteq L$, the d-spectrum of $\mathcal{A}_S(L)$ is bihomeomorphic to the bispace*

- *with $|\mathsf{pt}(L)|$ as its underlying set of points;*
- *whose positive opens are the opens of $\mathsf{pt}(L)$; and*
- *whose negative opens are the topology generated by $\{\phi_L(s)^c : s \in S\}$.*

Proof. Let $\alpha : |\mathsf{dpt}(\mathcal{L})| \cong |\mathsf{pt}(L)|$ be the bijection of Lemma 4.1.9, assigning to each frame map $f : L \to 2$ the d-frame map $(f, \mathsf{Filt}_\neg(f)) : \mathcal{A}_S(L) \to \mathbf{2}$. First, we show that α is such that the positive opens of $\mathsf{dpt}(\mathcal{A}_S(L))$ are exactly the forward images of opens of the space we described in our claim. By Lemma 4.1.9, we know that every point in $\mathsf{dpt}(\mathcal{A}_S(L))$ is of the form $(f, \mathsf{Filt}_\neg(f))$ for some frame map $f : L \to 2$. A subset of $\mathcal{A}_S(L)$ is a positive open if and only if it is of the form $\{(f, \mathsf{Filt}_\neg(f)) \in \mathsf{dpt}(\mathcal{A}_S(L)) : f(a) = 1\}$ for some $a \in L$. By definition of our map α, this is $\alpha[\phi_L(a)]$. Let us show that the subbasic negative opens of $\mathsf{dpt}(\mathcal{A}_S(L))$ are exactly the forward images of subbasic negative opens. We notice that for that $\mathsf{Filt}_\neg(f)(\uparrow s) = 1$ if and only if $f(s) = 0$ for every $s \in S$. We use this fact to deduce that a subset of $\mathsf{dpt}(\mathcal{A}_S(L))$ is a subbasic negative open if and only if it is of the form

$$\phi^+_{\mathcal{A}_S(L)}(\uparrow s) = \{(f, \mathsf{Filt}_\neg(f)) \in \mathsf{dpt}(\mathcal{A}_S(L)) : f(s) = 0\}$$

for some $s \in S$. By definition of the map α this equals $\alpha[\phi_L(s)^c]$. Then, this map is a homeomorphism according to the negative topology. □

4.2 An example: coherent d-frames

We refer the reader to the introduction of this thesis for a review of why bitopological point-free analogues of spaces are needed for understanding Stone duality. Recall that *coherent* frames are those that have a basis of compact elements which is closed under finite meets.

Alternatively, a frame is coherent if and only if it is the ideal completion of a distributive lattice. Taking the ideal completion of a distributive lattice is, in fact, the object part of a functor $\mathsf{Idl} : \mathbf{Distr} \cong \mathbf{CohFr}$ giving an equivalence of categories between the category of distributive lattices and that of coherent frames. Its inverse $K : \mathbf{CohFr} \cong \mathbf{Distr}$ assigns to each frame the ordered collection of its compact elements. We denote as $\mathbf{CohFr_0}$ the full subcategory of **CohFr** consisting of the zero dimensional coherent frames. The spatial-sober duality restricts to an equivalence $\mathbf{CohTop} \cong \mathbf{CohFr}^{op}$ and this, in turn, restricts to an equivalence $\mathbf{Stone} \cong \mathbf{CohFr_0}^{op}$. We should note that **CohFr** is not a full subcategory of **Frm**: morphisms of coherent frames are frame morphisms such that the image of compact elements is compact. This restriction on morphisms is the frame theoretical translation of the restriction of morphisms in **CohTop**. We summarize the picture with the following diagram.

$$\begin{array}{ccccc}
\mathbf{CohTop} & \underset{\mathsf{pt}}{\overset{\Omega}{\underset{\cong}{\rightleftarrows}}} & \mathbf{CohFr}^{op} & \underset{\mathsf{Idl}}{\overset{K}{\underset{\cong}{\rightleftarrows}}} & \mathbf{Distr} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{Stone} & \underset{\longleftarrow}{\overset{\longrightarrow}{\cong}} & \mathbf{CohFr_0}^{op} & \underset{\longleftarrow}{\overset{\longrightarrow}{\cong}} & \mathbf{Bool}
\end{array}$$

Our question is now how to account bitopologically for the duality between coherent spaces and coherent frames. Bitopological versions of coherent frames are needed, and here is where the theory of d-frames comes into play.

In [29] the connection between Priestley spaces and d-frames is outlined, while in Chapter 2 of [22] the dualities of Priestley and Stone are proved in terms of d-frames, by taking *coherent* d-frames bitopological versions of coherent frames. There, the duality is proven as a special case of the duality between compact regular d-frames and pairwise compact and regular bispaces, and in the same section coherent d-frames are explicitly proven to be equivalent to distributive lattices. Our next aim is to re-prove this bitopological duality by proving directly the duality between coherent d-frames and pairwise Stone spaces. Some of the results obtained in the previous sections will help to simplify some of the proofs. We say that a d-frame is a *coherent* d-frame if it is compact and zero dimensional. For a distributive lattice, consider the canonical d-frame presentation

$$(\mathsf{Idl}(D), \mathsf{Filt}(D), \mathsf{Con}_D, \mathsf{Tot}_D)$$

of the partial assembly $\mathsf{A}_D(\mathsf{Idl}(D))$. Here, we have identified the lattice D with its embedding in $\mathsf{Idl}(D)$. Recall that, by construction of d-frame presentations of partial assemblies (see Proposition 4.1.3), we have that $(I, F) \in \mathsf{Con}_D$ if and only if $I \subseteq {\downarrow}d$ whenever $d \in F$, and $(I, F) \in \mathsf{Tot}_D$ if and only if there is $d \in F$ such that ${\downarrow}d \subseteq I$. Alternatively, Con_D and

Tot_D are the smallest consistency and totality making the pairs $(\downarrow d, \uparrow d)$ into pairs of mutual d-complements for each $d \in D$.

Lemma 4.2.1. *If $\mathcal{L}$ is a d-frame, then the d-complemented elements of L^+ and those of L^- form sublattices of the frame they live in. The two sublattices of d-complemented elements are anti-isomorphic to each another.*

Proof. Suppose that $a^+, b^+ \in L^+$ are d-complemented, with d-complements a^- and b^-, respectively. Since con_L is a downset, and closed under finite homogeneous joins, we obtain $(a^+ \vee b^+, a^- \wedge b^-) \in \mathsf{con}_L$ and reasoning dually we obtain $(a^+ \vee b^+, a^- \wedge b^-) \in \mathsf{tot}_L$. The argument to show that finite meets of d-complemented elements are still d-complemented is analogous. It is easy to check that the d-pseudocomplementation map determines an anti-isomorphism between the lattices of d-complemented elements. □

Lemma 4.2.2. *In a coherent d-frame $\mathcal{L}$, the d-complemented elements of each frame coincide with the compact ones.*

Proof. Suppose that $\mathcal{L}$ is a coherent d-frame. By zero dimensionality of $\mathcal{L}$ we deduce that if a^+ is a compact element then it must be a finite join of d-complemented element, and so, by Lemma 4.2.1, it is d-complemented. Conversely, suppose that a^+ has a d-complement a^-. Let us write a^+ as $\bigvee_i b_i^+$, with each b_i^+ a d-complemented element. By compactness of $\mathcal{L}$ there must be a finite $F \subseteq I$ such that $(\bigvee_{f \in F} b_f^+, a^-) \in \mathsf{tot}_L$. Since con_L is a downset, we also have $(\bigvee_{f \in F} b_f^+, a^-) \in \mathsf{con}_L$ and since d-complements are unique, this means that $a^+ = \bigvee_{f \in F} b_f^+$. □

Lemma 4.2.3. *If a zero dimensional d-frame $\mathcal{L}$ is such that the d-complemented elements of each frame coincide with the compact ones, then $\mathcal{L}$ must be of the form*

$$(\mathsf{Idl}(D), \mathsf{Filt}(D), \mathsf{Con}_D, \mathsf{Tot}_D)$$

for some distributive lattice D.

Proof. By Lemma 4.2.1 we deduce that L^+ and L^- are frames such that their compact elements form a sublattice that generates the frame, and such that these two sublattices are anti-isomorphic. Then, the two frame components are ideal completions of anti-isomorphic distributive lattices. Then, we must have $L^+ \cong \mathsf{Idl}(D)$ and $L^- \cong \mathsf{Filt}(D)$ for some distributive lattice D. We may then identify L^+ and L^- with $\mathsf{Idl}(D)$ and $\mathsf{Filt}(D)$, and the two lattices of complemented elements as the ordered collections $\{\downarrow d : d \in D\}$ and $\{\uparrow d : d \in D\}$. By construction, we also know that $(\downarrow d, \uparrow d) \in \mathsf{con}_L \cap \mathsf{tot}_L$ for each $d \in L$, and so $\mathsf{Con}_D \subseteq \mathsf{con}_L$

and $\mathsf{Tot}_D \subseteq \mathsf{tot}_L$. If we have $(I, F) \in \mathsf{con}_L$ with $(I, F) \notin \mathsf{Con}_D$, then $d_1 \not\leq d_2$ for some $d_1 \in I$ and some $d_2 \in F$. We have both $(\downarrow d_1, \uparrow d_2) \in \mathsf{con}_L$ and $(\downarrow d_1, \uparrow d_1) \in \mathsf{tot}_L$. By balance, this implies that $d_1 \leq d_2$, which is a contradiction. Then, we must have that $\mathsf{con}_L = \mathsf{Con}_D$. With an analogous argument by contradiction one shows $\mathsf{tot}_L = \mathsf{Tot}_D$. □

Proposition 4.2.4. *The coherent d-frames are exactly those of the form*

$$(\mathsf{Idl}(D), \mathsf{Filt}(D), \mathsf{Con}_D, \mathsf{Tot}_D)$$

for some distributive lattice D.

Proof. D-frames of the form described in the statement above are such that, in both components, directed unions are set theoretical unions. For compactness, suppose that $(\bigcup_i I_i, \bigcup_i F_i) \in \mathsf{Tot}_D$ and so there is $f \in F_j$ with $f \in I_k$ for some $j, k \in I$. Then, $(I_k, F_j) \in \mathsf{Tot}_D$. For zero dimensionality, observe that all principal ideals and all principal filters are d-complemented.

For the converse, we notice that if a d-frame $\mathcal{L}$ is coherent, then by Lemma 4.2.2 the d-complemented and the compact elements of each component coincide. Then, by Lemma 4.2.3, the d-frame must be of the form described in the statement. □

Let us now start proving the desired bitopological duality. In [29] it is proven that all compact regular d-frames are spatial. Since zero dimensional d-frames are regular, this implies that coherent d-frames are spatial. Let us see an explicit proof of this.

Lemma 4.2.5. *Coherent d-frames are spatial.*

Proof. Coherent d-frames, by Proposition 4.2.4, are such that both their frame components are spatial. Suppose that $\mathcal{L} = (\mathsf{Idl}(D), \mathsf{Filt}(D), \mathsf{Con}_D, \mathsf{Tot}_D)$ is a coherent d-frame and that $(I, F) \notin \mathsf{Con}_D$. In particular, suppose that $x \in I$ and $y \in F$ are such that $i \not\leq f$. There is a lattice map $f : D \to 2$ such that $f(z) = 0$ if and only if $z \leq y$, and in particular $f(x) = 1$ and $f(y) = 0$. By the universal property of the ideal completion of a distributive lattice, this can be lifted to a map $\tilde{f}^+ : \mathsf{Idl}(D) \to 2$. By Lemma 4.1.9 we may extend this frame map to a d-frame map $(\tilde{f}^+, \tilde{f}^-) : \mathcal{L} \to \mathbf{2}$, such that $\tilde{f}^+(\downarrow x) = 1$ if and only if $\tilde{f}^-(\uparrow x) = 0$ for all $x \in D$. This map is such that it maps (I, F) to $(1, 1)$.

Suppose now that $(I, F) \notin \mathsf{Tot}_D$, that is, that $I \cap F = \emptyset$. Then, there is a lattice map $f : D \to 2$ such that $f(x) = 1$ if and only if $x \in F$. We may again lift this to a frame map $\tilde{f}^+ : \mathsf{Idl}(D) \to 2$ from the ideal completion of D; notice that this map satisfies $\tilde{f}^+(I) = 0$. Again we extend this to a d-frame map $(\tilde{f}^+, \tilde{f}^-) : \mathcal{L} \to \mathbf{2}$, which maps (I, F) to $(0, 0)$. □

Lemma 4.2.6. *The d-spectrum of a coherent d-frame* $(\mathsf{Idl}(D), \mathsf{Filt}(D), \mathsf{Con}_D, \mathsf{Tot}_D)$ *is homeomorphic to the bispace such that*

- *its underlying set of points is* $|\mathsf{pt}(\mathsf{Idl}(D))|$,
- *its positive opens are the opens of* $\mathsf{pt}(\mathsf{Idl}(D))$,
- *its negative opens are the topology generated by the complements of the compact opens of* $\mathsf{pt}(\mathsf{Idl}(L))$.

Proof. Consider a coherent d-frame, which we may assume of the form

$$(\mathsf{Idl}(D), \mathsf{Filt}(D), \mathsf{Con}_D, \mathsf{Tot}_D)$$

by Proposition 4.2.4. By Proposition 4.1.10, its d-spectrum is bihomeomorphic to the bispace with $|\mathsf{pt}(\mathsf{Idl}(D))|$ as its underlying set of points, whose positive opens are the opens of $\mathsf{Idl}(D)$, and whose negative opens are the topology generated by the sets of the form $\phi_{\mathsf{Idl}(D)}(d)^c$, for $d \in D$. Since the spectrum of any coherent frame $\mathsf{Idl}(E)$, the compact opens are exactly the opens of the form $\phi_{\mathsf{Idl}(E)}(\downarrow e)$, with $e \in E$, we are done. □

Lemma 4.2.7. *The spectrum of a coherent d-frame is a pairwise Stone space.*

Proof. Since coherent d-frames are spatial, by Lemma 4.2.5, they are isomorphic to the d-frames of opens of their d-spectra. Since pairwise compactness of a bispace is equivalent to compactness of its d-frame of opens, d-spectra of coherent d-frames are pairwise compact. For pairwise Hausdorffness, we notice that by Lemma 4.2.6 the d-spectrum of a coherent d-frame is of the form $(X, \Omega(X), \Omega^{\mathsf{k}^c}(X))$ for some coherent space $(X, \Omega(X))$. Then, the bispace must be pairwise Hausdorff, since any two points can be separated by a basic compact open of $\Omega(X)$ and its complement. Zero dimensionality is clear, since both topologies are generated by a basis of compact sets which have complements in the other topology. □

Lemma 4.2.8. *Pairwise Stone spaces are d-sober.*

Proof. Suppose that X is a pairwise Stone space. Since it is pairwise Hausdorff, no two points can have the same pair of neighborhood filters, and then injectivity of the d-sobrification map is immediate. For surjectivity, we look at the prime d-sobrification map $\psi'_X : X \to \mathsf{dpt}'(\mathsf{d}\Omega(X))$ to the prime spectrum of $\mathcal{L}$. Suppose that $(P^+, P^-) \in \Omega^+(X) \times \Omega^-(X)$ is a pair of meet primes in their respective topology such that (1) whenever $U^+ \cap U^- = \emptyset$, either $U^+ \subseteq P^+$ or $U^- \subseteq P^-$, and (2) whenever $U^+ \cup U^- = X$, either $U^+ \nsubseteq P^+$ or $U^- \nsubseteq P^-$. Because of condition (2), we cannot have $P^+ \cup P^- = X$, and so we may fix $x \notin P^+ \cup P^-$. Let P^+_x and

P_x^- be the meet prime elements determined by the point x, the meet prime elements, that is, such that $x \notin U^+$ if and only if $U^+ \subseteq P^+$ for each $U^+ \in \Omega^+(X)$, and similarly for the negative side. We immediately get the inequalities $P^+ \subseteq P_x^+$ and $P^- \subseteq P_x^-$. To obtain the reverse inequalities, suppose towards contradiction that $y \in P_x^+$ and $y \notin P^+$. Since the space X is pairwise zero dimensional, there is some positive open $K^+ \subseteq P_x^+$ such that $y \in K^+$ and such that K^+ has a negative open complement K^-. Because K^+ and K^- are disjoint, by condition (1) we deduce that either $K^+ \subseteq P^+$, or $K^- \subseteq P^-$. This is a contradiction, since $y \in K^+ \backslash P^+$ and $x \in K^- \backslash P^-$. □

Lemma 4.2.9. *The d-frame of opens of a pairwise Stone space is a coherent d-frame.*

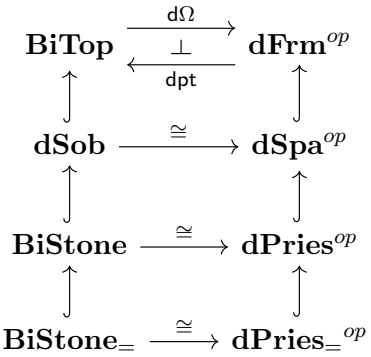
Proof. It is immediate from the definitions that the d-frame of opens of a pairwise compact bispace is compact, and that the same holds for zero dimensionality. □

We have proved the following fact.

Theorem 4.2.10. *The categories of coherent d-frames and pairwise Stone spaces are equivalent.*

Let us now see how this bitopological rephrasing of Priestley duality restricts to the bitopological rephrasing of Stone duality. We denote as $\mathbf{BiStone}_=$ the category of pairwise Stone spaces such that the two topologies are equal. Let us also denote as $\mathbf{dPries}_=$ the full subcategory of **dPries** consisting of the coherent d-frames where the two frame components are isomorphic. The equivalence of categories $\mathbf{BiStone} \cong \mathbf{dPries}^{op}$ restricts to an equivalence $\mathbf{BiStone}_= \cong \mathbf{dPries}_={}^{op}$. Clearly we have that a coherent d-frame $(\mathsf{Idl}(D), \mathsf{Filt}(D), \mathsf{Con}_D, \mathsf{Tot}_D)$ is an object of $\mathbf{Pries}_=$ if and only if D is a Boolean algebra.

To conclude this analysis of the classical dualities, let us point out that the advantage of the bitopological approach is also that it gives a way of viewing the dualities of Priestley and Stone as natural restrictions of a broader spatial-sober duality. The situation is depicted in the diagram below, in which all inclusions are full. The equivalences from the second to the last row are restrictions of the functor $\mathsf{d}\Omega : \mathbf{BiTop} \to \mathbf{dFrm}^{op}$.



The third and fourth rows represent, respectively, the duality of Priestley and Stone. One cannot get a similar picture by adopting the monotopological view. The inclusion of the category of coherent spaces into that of sober spaces is not full, giving a picture which is somewhat less natural.

4.3 Partial assemblies as a functor

Let us broaden our view of the matter by considering the assembly and the partial assembly constructions as functors. It is very well known that assigning to a frame its assembly is the object part of an endofunctor on **Frm**. This functor assigns to a frame morphism $f : L \to M$ the frame map $\mathsf{A}(f) : \mathsf{A}(L) \to \mathsf{A}(M)$ defined on generators as $\mathsf{A}(f)(\nabla(x)) = \nabla(f(x))$ and $\mathsf{A}(f)(\Delta(x)) = \Delta(f(x))$ for every $x \in L$. We next ask how one can extend the partial assembly construction to a functor. The category **Frm** is not the correct setting to do this. Informally, we will call this category **sFrm** is that of "frames with selected subsets". We note that the category **CohFrm** is isomorphic to the full subcategory of **sFrm** whose objects are the pairs $(L, \mathsf{K}(L))$, where L is a coherent frame and $\mathsf{K}(L)$ the collection of its compact elements.

Let us now look at the topological dual of the category **sFrm** of frames with selected subsets. Define the category **sTop** to be the category whose objects are pairs (X, S) where X is a topological space, and S is a collection of opens of X which includes X and the empty set $\emptyset$. We define a morphism $f : (X, S) \to (Y, T)$ in this category to be a continuous map $f : X \to Y$ such that preimages of elements in T are in S. The identity for each object simply corresponds to the identity function, and composition of morphisms is function composition. We observe that the category **CohTop** is isomorphic to the full subcategory of **sTop** whose objects are the pairs $(X, \mathsf{K}(X))$, with X a coherent space and $\mathsf{K}(X)$ the collection of its compact open sets.

Let us introduce a notion of spectrum connecting this category and the category **sFrm**.

For $(L, S) \in \mathsf{Obj}(\mathbf{sFrm})$, let us define

$$\mathsf{spt}(L, S) = (\mathsf{pt}(L), \phi_L[S]).$$

As usual here ϕ_L is the frame spatialization map $\phi_L : L \twoheadrightarrow \Omega(\mathsf{pt}(L))$. Since this is a frame map, it preserves both top and bottom elements. In particular, then $\phi_L(1) = \mathsf{pt}(L)$ and $\phi_L(0) = \emptyset$. Since the empty set and the whole space are in $\phi_L[S]$, the pair $\mathsf{spt}(L, S)$ is indeed an object of the category **sTop**. This shows us that we have a well-defined map $\mathsf{spt} : \mathsf{Obj}(\mathbf{sFrm}) \to \mathsf{Obj}(\mathbf{sTop})$. To extend this assignment to morphisms, let us set simply set $\mathsf{spt}(f) = \mathsf{pt}(f)$ for every morphism f of **sFrm**. By functoriality of the classical spectrum functor, we know that whenever $f : (L, S) \to (M, T)$ is a morphism in **sFrm** the function $\mathsf{pt}(f) : \mathsf{pt}(M) \to \mathsf{pt}(L)$ is continuous. Let us show that this map also matches the selected subsets.

Lemma 4.3.1. *For every morphism $f : (L, S) \to (M, T)$ of* **sFrm***, the map*

$$\mathsf{pt}(f) : (\mathsf{pt}(M), \phi_M[T]) \to (\mathsf{pt}(L), \phi_L[S])$$

is a morphism of **sTop**.

Proof. Suppose that $f : (L, S) \to (M, T)$ is a morphism in **sFrm**. We have already observed that we know that $\mathsf{pt}(f) : \mathsf{pt}(M) \to \mathsf{pt}(L)$ is a continuous map. To check that this constitutes a morphism on **sTop**, we check that it matches the selected subsets of opens. Consider an open $\phi_L(s)$ of $\mathsf{pt}(L)$ with $s \in S$. Recall that $\mathsf{pt}(f)$ is precomposition $- \circ f$. Expanding the definition of preimage we have

$$\begin{aligned}
&\mathsf{pt}(f)^{-1}(\phi_L(s)) = \\
&= \{g \in \mathsf{pt}(M) : g \circ f \in \phi_L(s)\} = \\
&= \{g \in \mathsf{pt}(M) : g(f(s)) = 1\} = \\
&= \phi_M(f(s)).
\end{aligned}$$

Since by assumption on the map f we have $f(s) \in T$, this is an open in the collection $\phi_M[T]$. Then, indeed $\mathsf{pt}(f)$ is a morphism in **sTop**. □

We have shown that there is a well-defined map $\mathsf{spt} : \mathsf{Mor}(\mathbf{sFrm}^{op}) \to \mathsf{Mor}(\mathbf{sTop})$ such that it respects domain and codomain. In the category **sTop** identities and compositions are simply identity functions and function compositions, and so the fact that spt respects this data is inherited from functoriality of the classical spectrum functor pt. We will now show

that there is also an open set functor relating the categories **sFrm** and **sTop** which is a left adjoint of the spectrum functor. For every pair $(X, S) \in \mathsf{Obj}(\mathbf{sTop})$ we define

$$\mathsf{s}\Omega(X, S) = (\Omega(X), S).$$

Since S contains both $\emptyset$ and X, and these are the bottom and the top element of $\Omega(X)$, we know that this is a well-defined assignment $\mathsf{s}\Omega : \mathsf{Obj}(\mathbf{sTop}) \to \mathsf{Obj}(\mathbf{sFrm}^{op})$. For a morphism $f : (X, S) \to (Y, T)$, we define $\mathsf{s}\Omega(f) = \Omega(f)$. Since $f \in \mathsf{Mor}(\mathbf{sTop})$ we also know that the preimage of any open set in T will be in the set S. Then, indeed we have a well-defined assignment $\mathsf{s}\Omega : \mathsf{Mor}(\mathbf{sTop}) \to \mathsf{Mor}(\mathbf{sFrm}^{op})$ which respects domains and codomains. In the category **sFrm** identities and compositions are identity functions and composition of functions, and so the fact that $\mathsf{s}\Omega$ respects this data is inherited by the classical Ω functor.

Now, we would like to check that these two functors form a refinement of the classical spatial-sober adjunction. Let us, then, introduce analogues of the sobrification and the spatialization natural transformations. For every object (X, S), we define

$$\begin{aligned}&\psi_{(X,S)} : (X, S) \to \mathsf{spt}(\mathsf{s}\Omega(X)) = (\mathsf{pt}(\Omega(X)), \phi_{\Omega(X)}[S]),\\ &x \mapsto N_x.\end{aligned}$$

Here, N_x is the neighborhood map relative to x. The map $\psi_{(X,S)}$, in other words, is the classical sobrification map ψ_X.

Lemma 4.3.2. *For every object $(X, S) \in \mathsf{Obj}(\mathbf{sTop})$ the map $\psi_{(X,S)}$ is a morphism in* **sTop**.

Proof. We already know that the map $\psi_{(X,S)}$ is continuous, since it is the classical sobrification map. To show that the map is well behaved with respect to the selected subsets, we notice that a typical element of $\phi_{\Omega(X)}[S]$ is of the form $\phi_{\Omega(X)}(U)$ for some open $U \in S$. Then we have that

$$\begin{aligned}&\psi^{-1}_{(X,S)}(\phi_{\Omega(X)}(U)) =\\ &= \{x \in X : N_x \in \phi_{\Omega(X)}(U)\} =\\ &= \{x \in X : N_x(U) = 1\} =\\ &= U.\end{aligned}$$

Since by our assumption this is in S, we have shown that $\psi_{(X,S)}$ is a morphism in **sTop**. □

Naturality of this sobrification amounts to naturality of the classical one. We move on to defining the analogue of the spatialization map. Once again, this is just the frame theoretical

spatialization map. For any object $(L, S) \in \mathsf{Obj}(\mathbf{sFrm})$, we define

$$\begin{aligned}&\phi_{(L,S)} : \mathsf{s\Omega}(\mathsf{spt}(L)) = (\Omega(\mathsf{pt}(L)), \phi_L[S])\\ &a \mapsto \phi_L(a).\end{aligned}$$

To check that it is a morphism of **sFrm**, it suffices to notice that for $s \in S$ we have $\phi_{(L,S)}(s) = \phi_L(s)$, and this indeed is in the selected subset $\phi_L[S]$. Naturality of this transformation amounts to naturality of the classical spatialization. We are now ready to prove that the data that we have built comes together to form an adjunction. But commutativity of the unit-counit triangles simply reduces to commutativity of the unit-counit triangle for the classical duality. We have the following.

Theorem 4.3.3. *The pair of functors* $\mathsf{s\Omega} : \mathbf{sTop} \leftrightarrows \mathbf{sFrm}^{op} : \mathsf{spt}$ *constitutes an adjunction, with* ψ *as the unit and* φ *as the counit.*

Recall that the category **CohFrm** is defined to be the category of coherent frames with maps such that the image of compact elements is compact. Likewise, the category **CohTop** is the category whose objects are the coherent spaces with continuous maps such that the preimages of compact open sets are compact. The two categories are dually equivalent, and the equivalence is $\Omega : \mathbf{CohTop} \cong \mathbf{CohFrm}^{op} : \mathsf{pt}$, the restriction of the classical spatial-sober adjunctions between frames and spaces. This is a restriction of the spatial-sober duality along inclusions which are not full.

We now show that the adjunction $\mathsf{s\Omega} : \mathbf{sTop} \leftrightarrows \mathbf{sFrm}^{op} : \mathsf{spt}$ gives a setting in which the classical duality $\mathbf{CohTop} \cong \mathbf{CohFrm}^{op}$ is the restriction of a spatial-sober adjunction along a full inclusion, giving perhaps a more natural picture. We have already observed that the category **CohFrm** is isomorphic to the full subcategory of **sFrm** determined by the pairs $(L, \mathsf{K}(L))$, where L is a coherent frame and $\mathsf{K}(L)$ the collection of its compact elements. We need to show that the classical $\Omega \dashv \mathsf{pt}$ functors are the restrictions of the functors $\mathsf{s\Omega} \dashv \mathsf{spt}$. That is, we have to show that for a coherent frame $(L, K(L))$ its image under spt is the same as $\mathsf{pt}(L)$ when seen as an element of **sTop**, and that a similar result holds for the other functor. With the next two lemmas we show that this indeed is the case.

Lemma 4.3.4. *If* L *is a coherent frame, we have*

$$\mathsf{spt}(L, \mathsf{K}(L)) = (\mathsf{pt}(L), \mathsf{K}(\mathsf{pt}(L))).$$

Proof. By definition of the functor spt we have that $\mathsf{spt}(L, \mathsf{K}(L)) = (\mathsf{pt}(L), \phi_L[\mathsf{K}(L)])$. The claim we need to prove, then, amounts to $\phi_L[\mathsf{K}(L)] = \mathsf{K}(\mathsf{pt}(L))$. Since we have that the frame

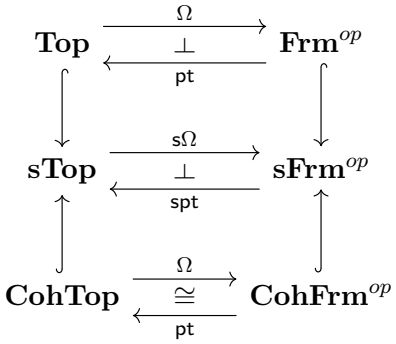
L is spatial, an element $a \in L$ is compact if and only if the element $\phi_L(a)$ is compact as an open of $\mathsf{pt}(L)$. The desired result follows. □

Lemma 4.3.5. *If X is a coherent space, we have*

$$\mathsf{s}\Omega(X, \mathsf{K}(X)) = (\Omega(X), \mathsf{K}(\Omega(X))).$$

Proof. By definition of the functor $\mathsf{s}\Omega$ we have that $\mathsf{s}\Omega(X, \mathsf{K}(X)) = (\Omega(X), \mathsf{K}(L))$. The desired result follows. □

Then, indeed the classical dual equivalence between coherent frames and coherent spaces is a restriction of the adjunction $\mathsf{s}\Omega : \mathbf{sTop} \leftrightarrows \mathbf{sFrm}^{op} : \mathsf{spt}$. The spatial-sober adjunction for frames and spaces, too, can be seen as a restriction of this adjunction. It suffices to notice that the category **Frm** is equivalent to the full subcategory of **sFrm** determined by the collection of objects $\{(L, L) : L \in \mathsf{Obj}(\mathbf{Frm})\}$. The classical adjunction is indeed a restriction of the adjunction $\mathsf{s}\Omega \dashv \mathsf{spt}$. Let us see why. For a pair (L, L) we have that by definition $\mathsf{spt}(L, L) = (\mathsf{pt}(L), \phi_L[L])$. By definition of the topology on $\mathsf{pt}(L)$ this is the same as $\mathsf{pt}(L)$ when seen as an object of **sTop**. For a space (X, X) its image under $\mathsf{s}\Omega$ is $(\Omega(X), \Omega(X))$ which is indeed $\Omega(X)$ seen as an element of **sFrm**. We have the following picture. All the inclusions appearing in the diagram are full, and the top and bottom adjunctions are restrictions of the middle one along these inclusions.



4.3.1 The functor $\mathsf{sA} : \mathbf{sFrm} \to \mathbf{sFrm}$ and its spatial analogue

We claim that the setup that we have built enables us to see the partial assembly constructions as being given by a single functor. In particular, we claim that there is endofunctor on the category **sFrm** which is analogue of the assembly functor, and which assigns to every object (L, S) the object $(\mathsf{A}_S(L), \nabla_S[S])$. We define the functor $\mathsf{sA} : \mathbf{sFrm} \to \mathbf{sFrm}$ on objects as $(L, S) \mapsto (\mathsf{A}_S(L), \nabla_S[S])$. This, indeed, is an object in this category, since the top and

the bottom element of $\mathsf{A}_S(L)$ are both closed congruences, as they are $\nabla_S(1)$ and $\nabla_S(0)$, respectively.

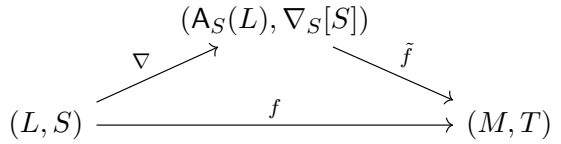
Lemma 4.3.6. *If $f : (L, S) \to (M, T)$ is a morphism in* **sFrm***, the following map, which we define on generators, is a morphism in* **sFrm***.*

$$\begin{aligned}
&\mathsf{sA}(f) : (\mathsf{A}_S(L), \nabla_S[S]) \to (\mathsf{A}_T(M), \nabla_T[T]) \\
&\nabla(x) \mapsto \nabla(f(x)) \\
&\Delta(s) \mapsto \Delta(f(s))
\end{aligned}$$

Proof. The frame map we have defined is the restriction of the map $\mathsf{A}(f) : \mathsf{A}(L) \to \mathsf{A}(M)$ to the subframe $\mathsf{A}_S(L)$. We have to show that $\mathsf{A}(f)[\mathsf{A}_S(L)] \subseteq \mathsf{A}_T(M)$. To show our claim, it suffices to show that $\mathsf{A}(f)$ maps each of these subbasic elements into $\mathsf{A}_T(M)$ whenever $x \in L$ and $s \in S$. For $\nabla(x)$, this is easy to see, since $\mathsf{A}(f)(\nabla(x)) = \nabla(f(x))$. Suppose that $s \in S$. By assumption on the map f, we have that $f(s) \in T$, and by definition of partial assemblies we must have $\Delta(f(s)) \in \mathsf{A}_T(M)$. Then, the assignment we have defined is indeed a frame map. It is also a morphism in **sFrm**: the map assigns to each closed congruence $\nabla(s)$ for $s \in S$ the closed congruence $\nabla(f(x))$, and $f(x) \in T$ by assumption on f. □

The lemma we have just proven tells us that the assignment $\mathsf{sA} : \mathsf{Mor}(\mathbf{sFrm}) \to \mathsf{Mor}(\mathbf{sFrm})$ defined as $f \mapsto \mathsf{A}(f)$ is well defined. We then take this to be the definition of sA on morphisms. We deduce that we have a functor $\mathsf{sA} : \mathbf{sFrm} \to \mathbf{sFrm}$. We check that this generalized version of the assembly functor gives on objects constructions with a universal property similar to that of the classical assembly.

Proposition 4.3.7. *For an object (L, S) of* **sFrm***, whenever we have a morphism $f : (L, S) \to (M, T)$ such that $f(s)$ is complemented in M for every $s \in S$, there is a unique morphism $\tilde{f} : (\mathsf{A}_S(L), \nabla[S]) \to (M, T)$ making the following triangle commute.*



Proof. Suppose that $f : (L, S) \to (M, T)$ is a morphism in **sFrm** such that $f(s)$ is complemented whenever $s \in S$. By the universal property of partial assemblies in the category **Frm**, we know that there is a unique frame map $\tilde{f} : \mathsf{A}_S(L) \to M$ such that $\tilde{f} \circ \nabla_S = f$. We only need to check that this map matches the selected subsets. Indeed, if we have a closed congruence $\nabla(s)$ with $s \in S$, then $\tilde{f}(\nabla(s)) = f(s) \in T$, by the fact that f is a morphism in **sFrm**. □

Let us now define a construction on the category **sTop** which is a spatial counterpart of the partial assembly construction in **sFrm**. For a topological space X and a collection $S \subseteq \Omega(X)$, define the space $Sk_S(X)$ as the space with the same underlying set of points as X, and whose opens are the topology generated by the opens of X together with the complements of the elements in S. For an object $(X, S) \in \mathsf{Obj}(\mathbf{sTop})$ we define

$$sSk(X, S) = (Sk_S(X), S).$$

Every open of X is also an open in $Sk_S(X)$, and so S is a subset of the open sets of $Sk_S(X)$. Then, we have a well-defined assignment $sSk : \mathsf{Obj}(\mathbf{sTop}) \to \mathsf{Obj}(\mathbf{sTop})$. On morphisms, we map each function to itself. On morphisms, we map each function to itself. Every function which is a morphism $f : (X, S) \to (Y, T)$ is such that preimages of elements in T are in S, and so sSk maps morphisms to continuous maps $f : Sk_S(X) \to SK_T(Y)$. The fact that these are also morphisms in **sTop** is immediate. Then, we have a functor $sSk : \mathbf{sTop} \to \mathbf{sTop}$.

Let us work towards showing that this is nothing but the spatial counterpart of the functor $\mathsf{sA} : \mathbf{sFrm} \to \mathbf{sFrm}$. Observe that every point of (L, S) provides complements for all elements of S. Then, by the universal property on **sFrm** of the assembly construction, there is a unique morphism $(A_S(L), \nabla_S[L]) \to (2, 2)$ such that $\tilde{f} \circ \nabla = f$. Then we have a bijection between the points of (L, S) and those of its assembly. We call this bijection $\alpha_{(L,S)}$. We show that this, as in the classical case, determines an isomorphism in **sTop** between the Skula spaces of spectra and the spectra of assemblies.

Lemma 4.3.8. *In the category* **sTop***, the isomorphisms $i : (X, S) \cong (Y, T)$ are exactly the homeomorphisms $i : X \cong Y$ such that they are morphisms in* **sTop** *and such that $i[U] \in T$ whenever $U \in S$.*

Proof. For one direction if the implication, suppose that we have a homeomorphism $i : X \cong Y$ such that it is a morphism $i : (X, S) \to (Y, T)$ and such that forward images of opens in S are opens in T. Let $j : Y \cong X$ be its inverse. For $U \in S$, we have that

$$\begin{aligned} &j^{-1}(U) = \\ &= \{y \in Y : j(y) \in U\} = \\ &= \{i(x) : x \in X, j(i(x)) \in U\} = \\ &= i[U]. \end{aligned}$$

By assumption, $i[U] \in T$ and so indeed j is a morphism in **sTop**. For the converse, suppose that $i : (X, S) \cong (Y, T)$ is an isomorphism in **sTop**. Then, let j be its inverse. Then, we

have a pair of continuous maps $i : X \cong Y : j$ which are mutually inverse, and so i must be a homeomorphism. Additionally, since $j : (Y, T) \to (X, S)$ is a morphism in **sTop**, we also have that if $U \in S$ then $i[U] = j^{-1}(U) \in T$. □

Lemma 4.3.9. *For every $(L, S) \in \mathsf{Obj}(\mathbf{sFrm})$ we have a homeomorphism*

$$\alpha_{(L,S)} : Sk_S(\mathsf{pt}(L), \phi_L[S]) \cong \mathsf{pt}(\mathsf{A}_S(L))$$
$$f \mapsto \tilde{f}.$$

Here, $\tilde{f} : \mathsf{A}_S(L) \to 2$ is the unique frame map with $\tilde{f} \circ \nabla = f$.

Proof. We claim that the subbasic open sets of $\mathsf{pt}(\mathsf{A}_S(L))$ are exactly the forward images of the subbasic open sets of $Sk_S(\mathsf{pt}(L))$. A subbasis for the opens of $\mathsf{pt}(\mathsf{A}_S(L))$ is given by the collection

$$\{\phi_{\mathsf{A}_S(L)}(\nabla(x)) : x \in L\} \cup \{\phi_{\mathsf{A}_S(L)}(\Delta(s)) : s \in S\}.$$

On the other hand, a subbasis for $Sk_S(\mathsf{pt}(L))$ is given by $\{\phi_L(x) : x \in L\} \cup \{\phi_L(s)^c : s \in S\}$. Since we have that $\tilde{f}(\nabla(x)) = 1$ if and only if $f(x) = 1$ for every $x \in L$, and using the fact that every point of $\mathsf{A}_S(L)$ is of the form $\tilde{f}$ for some $f \in \mathsf{pt}(L)$ we have that for every $x \in L$

$$\phi_{\mathsf{A}_S(L)}(x) = \{\tilde{f} \in \mathsf{pt}(\mathsf{A}_S(L)) : f(x) = 1\} = \alpha[\phi_L(x)].$$

On the other hand, since for every $s \in S$ we have that $\nabla(s)$ and $\Delta(s)$ are complements of each other in $\mathsf{A}_S(L)$, we also have that $\tilde{f}(\Delta(s)) = 1$ if and only if $f(s) = 0$. Then, for every $s \in S$ we have

$$\phi_{\mathsf{A}_S(L)}(\nabla(s)) = \{\tilde{f} \in \mathsf{pt}(\mathsf{A}_S(L)) : f(s) = 0\} = \alpha[\phi_L(s)^c].$$ □

Lemma 4.3.10. *For every object (L, S) of* **sFrm**, *the following is an isomorphism in* **sTop**.

$$\alpha_{(L,S)} : sSk(\mathsf{spt}(L, S)) \cong \mathsf{spt}(\mathsf{sA}(L, S))$$
$$f \mapsto \tilde{f}.$$

Here, $\tilde{f} : \mathsf{A}_S(L) \to 2$ is the unique frame map with $\tilde{f} \circ \nabla = f$.

Proof. We have already checked (see Lemma 4.3.9) that $\alpha_{(L,S)}$ is a homeomorphism between the two relevant topological spaces. We now need to show that this homeomorphism is an isomorphism in the category **sTop**. For this, by Lemma 4.3.8, it suffices to show that forward images of elements in $\phi_L[S]$ are in $\phi_{\mathsf{A}_S(L)}[\nabla_S[S]]$, and that preimages of elements in $\phi_{\mathsf{A}_S(L)}[\nabla[S]]$ are in $\phi_L[S]$. For forward images we notice that for $s \in S$ we have that $f(s) = 1$ if and only if $\tilde{f}(\nabla(s)) = 1$ and so $\alpha[\phi_L(s)] = \phi_{\mathsf{A}_S(L)}(\nabla(s))$. For preimages, we notice that for the same reason $\alpha^{-1}(\phi_{\mathsf{A}_S(L)}(\nabla(s))) = \phi_L(s)$. □

Pasting all these isomorphisms together yields a natural transformation.

Theorem 4.3.11. *The following square commutes up to natural isomorphism.*

$$\begin{array}{ccc} \mathbf{Frm}_S^{op} & \xrightarrow{\mathsf{sA}} & \mathbf{Frm}_S^{op} \\ \Big\downarrow{\scriptstyle \mathsf{spt}} & & \Big\downarrow{\scriptstyle \mathsf{spt}} \\ \mathbf{sTop} & \xrightarrow{sSk} & \mathbf{sTop} \end{array}$$

The natural isomorphism is given by the map $\alpha : \mathsf{Obj}(\mathbf{Frm}_S^{op}) \to \mathsf{Mor}(\mathbf{sTop})$.

Proof. Lemma 4.3.10 shows that the assignment is an isomorphism when evaluated at each object. It remains to show naturality. Using the definition of the functors involved, this means that for every map $f : (L, S) \to (M, T)$ we need to check that the following square commutes in **Top**.

$$\begin{array}{ccc} Sk_S(\mathsf{pt}(L)) & \xrightarrow{\alpha_{(L,S)}} & \mathsf{pt}(\mathsf{A}_S(L)) \\ \Big\downarrow{\scriptstyle \mathsf{pt}(f)} & & \Big\downarrow{\scriptstyle \mathsf{pt}(\mathsf{A}(f))} \\ Sk_T(\mathsf{pt}(M)) & \xrightarrow{\alpha_{(M,T)}} & \mathsf{pt}(\mathsf{A}_T(M)) \end{array}$$

For a frame map $f : L \to 2$ and a subset $S \subseteq L$ we denote as $\tilde{f} : \mathsf{A}_S(L) \to 2$ the unique map such that $\tilde{f} \circ \nabla_S = f$. For every $g \in \mathsf{pt}(M)$, then, we have the following chain of equalities.

$$\begin{aligned} &\mathsf{pt}(\mathsf{A}(f))(\alpha_{(L,S)}(g)) = \\ &= \mathsf{pt}(\mathsf{A}(f))(\tilde{g}) = \\ &= \tilde{g} \circ \mathsf{A}(f). \end{aligned}$$

On the other hand, expanding the other half of the square yields

$$\begin{aligned} &\alpha_{(M,T)}(\mathsf{pt}(f)(g)) = \\ &= \widetilde{\mathsf{pt}(f)(g)} = \\ &= \widetilde{g \circ f}. \end{aligned}$$

To show that the last lines of both derivations correspond to the same point of $\mathsf{A}_S(L)$, it suffices to show that the two frame maps agree on where they map the closed congruences.

Let us then consider an arbitrary $x \in L$. We have

$$\begin{aligned}
&\tilde{g}(\mathsf{A}(f)(\nabla(x))) = \\
&= \tilde{g}(\nabla(f(x))) = \\
&= g(f(x)) \\
&= \widetilde{g \circ f}(\nabla(x)).
\end{aligned}$$

Then, these two points of $\mathsf{A}_T(M)$ are the same. □

4.3.2 The functor $\mathsf{s}\mathcal{A} : \mathbf{sFrm} \to \mathbf{dFrm}$ and its spatial analogue

In the first section of this chapter, we have seen that for every frame L and any subset $S \subseteq L$ there is a d-frame $\mathcal{A}_S(L)$ whose patch is the partial assembly $\mathsf{A}_S(L)$. It is then natural to ask if this can be turned into a functor $\mathbf{sFrm} \to \mathbf{dFrm}$ in such a way that it acts on objects as $(L, S) \mapsto \mathcal{A}_S(L)$. In this subsection, we will show that this is indeed possible, and we will also define a functor $sbiSk : \mathbf{sTop} \to \mathbf{BiTop}$ as the spatial analogue of this.

This assignment $(L, S) \mapsto \mathcal{A}_S(L)$ clearly maps objects to objects. Let us extend this definition to morphisms. We observe that every frame map $f : L \to M$ between frames L and M can also be seen as a lattice map $f : L^{op} \to M^{op}$. By functoriality of the ideal completion construction, then, there is a frame map $\mathsf{Filt}(f) : \mathsf{Filt}(L) \to \mathsf{Filt}(M)$ such that it acts on generators as $\uparrow a \mapsto \uparrow f(a)$. Recall that $\mathsf{Filt}_S(L) \subseteq \mathsf{Filt}(L)$ is defined to be the subframe of $\mathsf{Filt}(L)$ generated by principal filters of elements from S.

Lemma 4.3.12. *If $f : (L, S) \to (M, T)$ is a morphism in $\mathbf{sFrm}$, the map $\mathsf{sFilt}(f) : \mathsf{Filt}_S(L) \to \mathsf{Filt}_T(M)$ defined on generators as $\uparrow s \mapsto \uparrow f(s)$ for every $s \in S$ is a well-defined frame map.*

Proof. To show our claim, it suffices to show that the map $\mathsf{Filt}(f) : \mathsf{Filt}(L) \to \mathsf{Filt}(M)$ maps filters in $\mathsf{Filt}_S(L)$ to filters in $\mathsf{Filt}_T(M)$. An arbitrary filter $F \in \mathsf{Filt}_S(L)$ is of the form $\bigvee_i \uparrow s_1 \cap ... \cap \uparrow s_{n_i}$ for some collection of elements $s_m \in S$. Since the map $\mathsf{Filt}(f)$ preserves the frame operations, we have that $\mathsf{Filt}(f)(F) = \bigvee_i \uparrow f(s_1) \cap ... \cap \uparrow f(s_{n_i})$. By assumption that f is a morphism in the category $\mathbf{sFrm}$, for every s_m we have that $f(s_m) \in T$. Then, $\mathsf{Filt}(f)(F) \in \mathsf{Filt}_T(M)$. □

Now, for every morphism $f : (L, S) \to (M, T)$ in the category $\mathbf{sFrm}$, let us define the map $\mathsf{s}\mathcal{A}(f) : \mathcal{A}_S(L) \to \mathcal{A}_T(M)$ as the pair of frame maps $(f, \mathsf{sFilt}(f))$.

Lemma 4.3.13. *The assignment $\mathsf{s}\mathcal{A} : \mathbf{sFrm} \to \mathbf{dFrm}$ is a functor.*

Proof. Let us show that for any map $f : (L, S) \to (M, T)$ the map

$$(f, \mathsf{sFilt}(f)) : \mathcal{A}_S(L) \to \mathcal{A}_T(M)$$

is a d-frame map. By Lemma 4.3.12 both components of (f, sFilt) are frame maps. Let us check that they respect the con and tot components. Recall that $\mathcal{A}_S(L)$ is the quotient of $(L, \mathsf{Filt}_S(L), c_m, t_m)$ by the relation structures making the pairs $(s, \uparrow s)$ into mutual d-complements for each $s \in S$. Then, it suffices to show that our pair of maps, when seen as maps $(f, \mathsf{sFilt}(f)) : (L, \mathsf{Filt}_S(L), c_m, t_m) \to \mathcal{A}_T(M)$, factor through this relation structure. If $s \in S$, we have $(f(s), \mathsf{Filt}(f)(\uparrow s) = (f(s), \uparrow f(s))$. Since $f(s) \in T$ by assumption on the map f, we have that $(f(s), \uparrow f(s))$ is a d-complemented pair in $\mathcal{A}_T(L)$. □

Let us now introduce the spatial version of this functor. For an object (X, S) of the category **sTop** let us define the bispace $biSk_S(X)$ as the bispace $(X, \Omega(X), \Omega^{S^c}(X))$, where $\Omega^{S^c}(X)$ is the topology on X generated by the complements of opens in S. Let us denote as $\mathsf{s}biSk$ this assignment of objects. For a morphism $f : (X, S) \to (Y, T)$ let us define the map $\mathsf{s}biSk(f)$ simply as the map $f : biSk_S(X) \to biSk_T(Y)$. By continuity of f, this map is continuous with respect to the positive topology. Since f is also a morphism in **sTop**, we have that preimages of complements of opens in T are complements of opens in S. Then, this determines a well-defined assignment $\mathsf{s}biSk : \mathsf{Mor}(\mathbf{sTop}) \to \mathsf{Mor}(\mathbf{sTop})$. Since this assignment simply maps each function to itself and since compositions and identities on **sTop** are function compositions and identity functions we know that this assignment respects this data. Recall that by Proposition 4.1.5 for every frame L and every point $f : L \to 2$ there is a map $\mathsf{Filt}_\neg : \mathsf{Filt}(L) \to 2$ defined on generators as $\mathsf{Filt}_\neg(\uparrow a) = \neg f(a)$ for every $a \in L$. Recall that at the end of the first section we showed (Proposition 4.1.10) the following result, which we can now rephrase in terms of our functor $\mathsf{s}biSk$.

Proposition 4.3.14. *For every frame L and any subset $S \subseteq L$ we have a bihomeomorphism*

$$\alpha_{(L,S)} : biSk_S(\mathsf{pt}(L), \phi_L[S]) \cong \mathsf{dpt}(\mathcal{A}_S(L)),$$
$$f \mapsto (f, \mathsf{Filt}_\neg(f)).$$

Let us now show that this is the evaluation at an object (L, S) of a natural isomorphism.

Lemma 4.3.15. *Whenever we have frame maps $f : L \to M$ and $g : M \to 2$, we have that $\mathsf{Filt}_\neg(g) \circ \mathsf{Filt}(f) = \mathsf{Filt}_\neg(g \circ f)$.*

Proof. Suppose that we have $a \in L$. Expanding the definitions, on the one hand we have $\mathsf{Filt}_\neg(g)(\mathsf{Filt}(f)(\uparrow a)) = \mathsf{Filt}_\neg(g)(\uparrow f(a)) = \neg g(f(a))$. On the other hand, we also have $\mathsf{Filt}_\neg(g \circ f)(\uparrow a) = \neg g(f(a))$. □

Theorem 4.3.16. *The following square commutes up to natural isomorphism.*

$$\begin{array}{ccc} \mathbf{sFrm}^{op} & \xrightarrow{\mathsf{s}\mathcal{A}} & \mathbf{dFrm}^{op} \\ \downarrow{\scriptstyle \mathsf{spt}} & & \downarrow{\scriptstyle \mathsf{dpt}} \\ \mathbf{sTop} & \xrightarrow{biSk_S} & \mathbf{BiTop} \end{array}$$

Proof. To show our claim it suffices to show that $\alpha : \mathsf{Obj}(\mathbf{sFrm}) \to \mathsf{Mot}(\mathbf{BiTop})$ is a natural transformation. Suppose, then, that $f : (L, S) \to (M, T)$ is a morphism in **sFrm**. The naturality square is the following square in **BiTop**.

$$\begin{array}{ccc} \mathsf{s}biSk(\mathsf{spt}(L,S)) & \xrightarrow{\alpha_{(L,S)}} & \mathsf{dpt}(\mathcal{A}_S(L)) \\ \downarrow{\scriptstyle biSk_S(\mathsf{spt}(f))} & & \downarrow{\scriptstyle \mathsf{dpt}(\mathsf{s}\mathcal{A}(f))} \\ \mathsf{s}biSk(\mathsf{spt}(M,T)) & \xrightarrow{\alpha_{(M,T)}} & \mathsf{dpt}(\mathcal{A}_T(M)) \end{array}$$

Since we only need to show equality of two functions, we can consider just the underlying sets of the relevant bitopological spaces and simplify this to the following square in **Set**. We have also simplified using the fact that both the functors spt and dpt map morphisms to their respective precomposition maps.

$$\begin{array}{ccc} \mathsf{pt}(L) & \xrightarrow{\alpha_{(L,S)}} & \mathsf{dpt}(\mathcal{A}_S(L)) \\ \downarrow{\scriptstyle -\circ f} & & \downarrow{\scriptstyle -\circ \mathsf{s}\mathcal{A}(f)} \\ \mathsf{pt}(M) & \xrightarrow{\alpha_{(M,T)}} & \mathsf{dpt}(\mathcal{A}_T(M)) \end{array}$$

Suppose, then, that we have $g \in \mathsf{pt}(L)$. We have the following derivation, in which we use Lemma 4.3.15.

$$\begin{aligned} &\alpha_{(L,S)}(g) \circ \mathsf{s}\mathcal{A}(f)) = \\ &= (g, \mathsf{Filt}_\neg(g)) \circ (f, \mathsf{Filt}(f)) = \\ &= (g \circ f, \mathsf{Filt}_\neg(g) \circ \mathsf{Filt}(f)) = \\ &= (g \circ f, \mathsf{Filt}_\neg(g \circ f)) \\ &= \alpha_{(M,T)}(g \circ f). \end{aligned}$$

This shows that the naturality square commutes. □

Let us now connect the functors $\mathsf{s}\mathcal{A} : \mathbf{sFrm} \to \mathbf{dFrm}$ and $\mathsf{s}biSk : \mathbf{sTop} \to \mathbf{BiTop}$ with a simplification of these two functors $\mathcal{A} : \mathbf{Frm} \to \mathbf{dFrm}$ and $biSk : \mathbf{Top} \to \mathbf{BiTop}$. For the Skula functor, we observe that the Skula space X of a space X has a natural bitopological description as $(X, \Omega(X), \Omega^c(X))$. Here $\Omega^c(X)$ is the topology on X generated by the closed sets of X. We will refer to this bispace as $biSk(X)$. We may extend this assignment to a functor $biSk : \mathbf{Top} \to \mathbf{BiTop}$ acting on morphisms by mapping a function to itself. Continuous maps $f : X \to Y$ are such that the preimage of closed sets is closed, and so the map $f : biSk(X) \to biSk(Y)$ is bicontinuous: continuity on the positive topology is equivalent to continuity of $f : X \to Y$, and continuity for the negative topology holds because preimages of basic negative opens (i.e. closed sets of Y) are also basic opens in $biSk(X)$. Since the assignment $biSk$ maps each function to itself, it respects both identities and compositions. Then we have a functor $biSk : \mathbf{Top} \to \mathbf{biTop}$.

For the assembly functor, let us recall that for a frame L the d-frame $\mathcal{A}(L)$ is defined as $(L, \mathsf{Filt}(L), \mathsf{Con}, \mathsf{Tot})$ where, as usual, Con and Tot are the smallest consistency and totality making the pairs $(x, \uparrow x)$ into mutual d-complements. For a frame map $f : L \to M$, we may define a map $\mathcal{A}(f) : \mathcal{A}(L) \to \mathcal{A}(M)$ as the pairing $(f, \mathsf{Filt}(f))$. This is a d-frame map by an argument completely analogous to that of Lemma 4.3.13. Still by an argument analogous to that of 4.3.13 the assignment $\mathcal{A}$ is functorial.

This functor has already been defined in [31]. We would like to show that up to these equivalences the functors $\mathcal{A}$ and $biSk$ are restrictions of the functors $\mathsf{s}\mathcal{A}$ and $\mathsf{s}biSk$, respectively; in the sense that $\mathsf{s}\mathcal{A} \circ e_F = \mathcal{A}$ and $\mathsf{s}biSk \circ e_T = biSk$. That is, we want to check that

$$\mathcal{A}(L) = \mathsf{s}\mathcal{A}(L, L),$$
$$BiSk(X) = \mathsf{s}biSk(X, \Omega(X)).$$

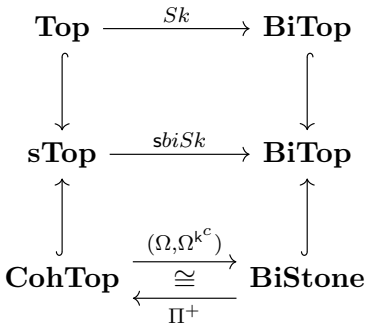
For the assembly, we notice that by definition $\mathsf{s}\mathcal{A}(L, L) = (L, \mathsf{Filt}(L), \mathsf{Con}_L, \mathsf{Tot}_L)$, where Con_L and Tot_L are the smallest con and tot subsets making the elements $(x, \uparrow x)$ into mutual d-complements for each $x \in L$. By definition, this is $\mathcal{A}(L)$. It is immediate from their definition that the functors $\mathsf{s}biSk$ and $biSk$ agree: for a space X, we have that the space $biSk(X)$, by definition, is $Sk_{\Omega(X)}(X)$ and this, in turn, is $\mathsf{s}BiSk(X, \Omega(X))$.

This can be connected very naturally to the bitopological view of Priestley duality that we have outlined in Section 4.2. Recall that, as we have explained in Section 4.2, in [12] it is proven that the categories of coherent spaces and that of pairwise Stone spaces are equivalent. The equivalence is given by a functor $(\Omega, \Omega^{\mathsf{k}^c}) : \mathbf{CohTop} \to \mathbf{BiStone}$, which assigns to each coherent space the bispace whose underlying set is X, whose positive topology is the same as

the topology in X and whose negative topology is that generated by the complements of the compact opens of X. We now claim that the functor $(\Omega, \Omega^{\mathsf{k}^c})$ is the restriction of the functor $\mathsf{s}biSk$. We need to show that for every coherent space X we have

$$(X, \Omega(X), \Omega^{\mathsf{k}^c}(X)) = \mathsf{s}biSk(X, \mathsf{K}(X)).$$

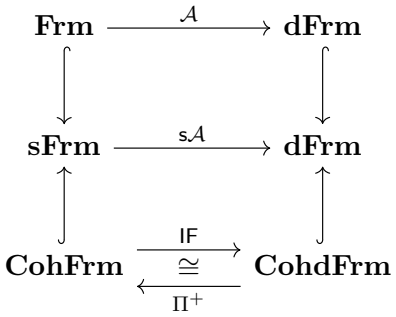
This holds directly by the definition of the functor $\mathsf{s}biSk$. Integrating this with the facts about $biSk : \mathbf{Top} \to \mathbf{BiTop}$, we obtain the following picture. All the inclusions appearing here are full, and the top and the bottom functors are restrictions of the middle one along these inclusions. The bottom functor has an inverse, which we also depict in the picture.



On the pointfree side, to have the same picture we exploit an equivalence which, so far, we have only alluded to. It is known that the categories **CohFrm** and the category **CohdFrm** of coherent d-frames are equivalent. The equivalence is given by mapping each coherent frame $\mathsf{Idl}(D)$ to the d-frame $(\mathsf{Idl}(D), \mathsf{Filt}(D), \mathsf{Con}_D, \mathsf{Tot}_D)$. Its inverse maps a coherent d-frame, which we know is always a d-frame of the form $(\mathsf{Idl}(D), \mathsf{Filt}(D), \mathsf{Con}_D, \mathsf{Tot}_D)$, to the frame $\mathsf{Idl}(D)$. We call the first assignment IF and the second one Π^+. Our next claim is that the functor $\mathsf{s}\mathcal{A}$ restricted to coherent frames is IF; in the sense that $\mathsf{s}\mathcal{A}(e_{CF}(\mathsf{Idl}(D))) = \mathsf{IF}(\mathsf{Idl}(D))$ for every coherent frame $\mathsf{Idl}(D)$. We need to show, that is, that for every coherent frame $\mathsf{Idl}(D)$ we have

$$(\mathsf{Idl}(D), \mathsf{Filt}(D), \mathsf{Con}_D, \mathsf{Tot}_D) = \mathsf{s}\mathcal{A}(\mathsf{Idl}(D), D).$$

The second d-frame, by definition of the functor $\mathsf{s}\mathcal{A}$, is the canonical presentation of the partial assembly $\mathsf{A}_D(\mathsf{Idl}(D))$, where we have abused notation and considered $D \subseteq \mathsf{Idl}(D)$ by identifying the isomorphic lattices $D \cong \{\uparrow d : d \in D\}$. This, indeed, equals the left-hand side, as we observed in Section 4.2 just before Lemma 4.2.1. We have then obtained the pointfree counterpart of the previous diagram. Here, too, all of the inclusions are full.


Frm
$\mathcal{A}$
dFrm
sFrm
$\mathsf{s}\mathcal{A}$
dFrm
CohFrm
IF
$\cong$
Π^+
CohdFrm

Chapter 5

The d-frame assembly

Introduction

This thesis is concerned, among other things, with exploring the relation between the two properties of the assembly of a frame: on the one hand, it has the universal property proved by Joyal and Tierney which we have explored in Chapter 3 and, on the other, it is also the ordered collection of sublocales of the locale corresponding to said frame. With this chapter, we begin our exploration of what these two properties become, in the category of d-frames and biframes, and how these two properties relate to each other. It will soon become clear that in the bitopological world these two notions no longer coincide in any obvious way.

In the coming chapter we will introduce the *assembly* of a d-frame, by which we mean an object $\mathsf{A}(\mathcal{L})$, for each d-frame $\mathcal{L}$, satisfying a universal property analogous to that of the assembly of a frame. It is not surprising that the assembly of a d-frame exists, by the fact that we may present d-frames by generators and relations (see [23]). Then, we might define the assembly of a d-frame as a free structure. In this chapter we will adapt the results in [31] (which we have analyzed in the previous chapter) to show that the assembly $\mathsf{A}(\mathcal{L})$ of a d-frame $\mathcal{L}$ can be obtained as a structure of the form

$$(\mathcal{L} \oplus \mathsf{Filt}(\mathcal{L}))/\mathcal{C},$$

where $\mathsf{Filt}(-)$ is a construction analogous to taking the filters of a distributive lattice, and $\mathcal{C}$ is a relation structure which only updates the con and tot components. We also show that there is another way of presenting the assembly of a d-frame which is, so to speak, "orthogonal" to this. The second way of presenting the assembly of a d-frame is

$$(\mathsf{A}(\mathcal{L}^+) \oplus \mathsf{A}(\mathcal{L}^-))/\mathcal{R},$$

where $\mathcal{R}$ is an extremal relation structure, and where $\mathcal{L}^+ = (L^+, 2, c_m, t_m)$ and $\mathcal{L}^- = (2, L^-, c_m, t_m)$. We will prove that, as expected, the assembly of a d-frame is always zero dimensional, and a d-frame is Boolean if and only if it is isomorphic to its assembly.

5.1 D-lattices and ideal completions

We say that a quadruple $\mathcal{L} = (L^+, L^-, c, t)$ is a *d-lattice* if L^+ and L^- are (bounded) lattices, $c \subseteq L^+ \times L^-$ is a downset closed under finite homogeneous joins, and $t \subseteq L^+ \times L^-$ is an upset closed under finite homogeneous meets, and c and t are balanced. We refer to c and t as the con and the tot components, defined in the same way as they are defined for d-frames. We note that for d-lattices too there are minimal con and tot components. We say that a d-lattice is *distributive* if the two lattice components are distributive. We denote with **dDistr** the category whose objects are the distributive d-lattices, and where a morphism $(f^+, f^-) : (L^+, L^-, c_L, t_L) \to (M^+, M^-, c_M, t_M)$ is a pair of (bounded) lattice maps $f^+ : L^+ \to M^+$ and $f^- : L^- \to M^-$ such that it respects the con and the tot components.

Our next aim is to prove that the category of d-frames is a reflective subcategory of that of distributive d-lattices, and the reflector is given by a construction analogous to the ideal completion of a distributive lattice. For any distributive d-lattice $\mathcal{L}$ we define the following two subsets of $\mathsf{Idl}(L^+) \times \mathsf{Idl}(L^-)$:

$$\mathsf{id}_\forall := \{(I^+, I^-) : (i^+, i^-) \in c_L \text{ for all } (i^+, i^-) \in I^+ \times I^-\},$$
$$\mathsf{id}_\exists := \{(I^+, I^-) : (i^+, i^-) \in t_L \text{ for some } (i^+, i^-) \in I^+ \times I^-\}.$$

For any distributive d-lattice $\mathcal{L}$ we refer to the quadruple $(\mathsf{Idl}(L^+), \mathsf{Idl}(L^-), \mathsf{id}_\forall, \mathsf{id}_\exists)$ as $\mathsf{Idl}(\mathcal{L})$, and we call it the *ideal completion* of the d-lattice $\mathcal{L}$.

Lemma 5.1.1. *If $\mathcal{L}$ is a d-lattice, the quadruple $\mathsf{Idl}(\mathcal{L})$ is a d-frame.*

Proof. Suppose that $\mathcal{L}$ is a d-lattice. We need to check that the con and tot components of $\mathsf{Idl}(\mathcal{L})$ satisfy the defining axioms of a d-frame, and that they are balanced. The set $\mathsf{id}_\forall$ is clearly a downset. Suppose that $(I^+, I_k^-) \in \mathsf{id}_\forall$. An arbitrary element of $\bigvee_k I_k^-$ is of the form $x_1^+ \vee ... \vee x_n^-$, with $x_m \in \bigcup_k I_k$ for each $m \leq n$. By assumption, we must have $(x^+, x_m^-) \in c_L$ for all $m \leq n$ and all $x^+ \in I^+$ and, since $\mathcal{L}$ was assumed to be a d-lattice, we also have that $(x^+, x_1^- \vee ... \vee x_n^-) \in c_L$. Since the element $x_1^- \vee ... \vee x_n^-$ was picked to be arbitrary, we also have $(x^+, x^-) \in c_L$ for all $x^- \in \bigvee_k I_k^-$ and so $(I^+, \bigvee_k I_k^-) \in \mathsf{id}_\forall$. For the totality subset, upward closure is clear. For closure under finite homogeneous joins, suppose that we have $(I^+, I_1^-), ..., (I^+, I_n^-) \in \mathsf{id}_\exists$, with witnesses $(x_n^+, x_n^-) \in (I^+ \times I_n^-) \cap t_L$. Then, the element

$(x_1^+ \vee ... \vee x_n^+, x_1^- \wedge ... \wedge x_n^-) \in t_L$ is a witness for $(I^+, I_1^- \cap ... \cap I_n^-) \in \mathsf{id}_\exists$. Finally, we check that balance holds. Suppose, then, that $(I^+, I_1^-) \in \mathsf{id}_\forall$ and that $(I^+, I_2^-) \in \mathsf{id}_\exists$, and in particular that we have $(x^+, x_2^-) \in (I^+ \times I_2^-) \cap t_L$. If $x_1^- \in I_1^-$, by our assumption, $(x^+, x_1^-) \in c_L$ and so, by balance of $\mathcal{L}$, we must have $x_1^- \leq x_2^-$ and, since I_2^- is downwards closed, also $x_1^- \in I_2^-$. Then $I_1^- \subseteq I_2^-$ as desired. □

For any two lattices L^+ and L^- we still have that the subset $c_m = \{(x^+, x^-) : x^+ = 0^+ \text{ or } x^- = 0^-\}$ is a consistency subset, and similarly $t_m = \{(x^+, x^-) : x^+ = 1^+ \text{ or } x^- = 1^-\}$ is a totality subset; furthermore these are balanced. The proof is analogous to the d-frame case. Let us check that the ideal completion of a distributive d-lattice such that its con and tot components are minimal still has minimal con and tot components.

Lemma 5.1.2. *For a d-lattice $\mathcal{L} = (L^+, L^-, c_m, t_m)$ its ideal completion $\mathsf{Idl}(\mathcal{L}_m)$ is the d-frame*

$$(\mathsf{Idl}(L^+), \mathsf{Idl}(L^-), c_m, t_m).$$

Proof. Suppose that L^+ and L^- are lattices. The consistency subset $Id(c_m)$ on $\mathsf{Idl}(L^+) \times \mathsf{Idl}(L^-)$ is the set of those pairs (I^+, I^-) such that for all elements $(i^+, i^-) \in I^+ \times I^-$ we have either $i^+ = 0^+$ or $i^- = 0^-$. This means that one of the two ideals must consist just of the bottom element of its frame. Conversely, by definition of the set $\mathsf{Idl}(c_m)$, we have that all pairs of the form $(\{0^+\}, I^-)$ and $(I^+, \{0^-\})$ are in this set. Then, $\mathsf{Idl}(c_m)$ is the minimal consistency subset of $\mathsf{Idl}(L^+) \times \mathsf{Idl}(L^-)$. The argument for the analogous result for the totality subset is symmetrical. □

With the following lemma, we introduce a simpler way of viewing the ideal completion of a d-lattice.

Lemma 5.1.3. *For a distributive d-lattice $\mathcal{L}$, its ideal completion is the d-frame obtained by quotienting the quadruple $\mathsf{Idl}(\mathcal{L}_m) = (\mathsf{Idl}(L^+), \mathsf{Idl}(L^-), c_m, t_m)$ by the relation structure*

$$(\mathsf{id}^+, \mathsf{id}^-, \{(\downarrow a^+, \downarrow a^-) : (a^+, a^-) \in c_L\}, \{(\downarrow a^+, \downarrow a^-) : (a^+, a^-) \in t_L\}).$$

Proof. To prove the claim, it suffices to show that for any distributive d-lattice the sets $\mathsf{id}_\forall$ and $\mathsf{id}_\exists$ are, respectively, the smallest consistency containing the set $\{(\downarrow a^+, \downarrow a^-) : (a^+, a^-) \in c_L\}$ and the smallest totality containing $\{(\downarrow a^+, \downarrow a^-) : (a^+, a^-) \in t_L\}$. Suppose then that C and T are a consistency and a totality, respectively, such that they contain these sets. Suppose that $(I^+, I^-) \in \mathsf{id}_\forall$. For all $x^+ \in I^+$ and all $x^- \in I^-$, we must have $(\downarrow x^+, \downarrow x^-) \in C$ and so, by closure under arbitrary homogeneous joins, also $(I^+, I^-) \in C$. Then $\mathsf{id}_\forall \subseteq C$. For the totality component, suppose that $(I^+, I^-) \in \mathsf{id}_\exists$ and, in particular, that $(x^+, x^-) \in (I^+ \times I^-) \cap t_L$.

We must then have that $(\downarrow x^+, \downarrow x^-) \in T$ and, by upward closure of T, also $(I^+, I^-) \in T$, and so $\mathsf{id}_\exists \subseteq T$. $\square$

We will come back to the following remark several times in the coming chapters.

Remark 5.1.4. The ideal completion of a d-frame is an example of a kind of free construction with a certain property, which we will encounter several times. Let us phrase this property in full generality. Suppose that for every d-frame $\mathcal{L}$ there is some free d-frame $\mathsf{F}(\mathcal{L})$ which is generated by $\mathcal{L}$ as a d-frame, and let $(\ulcorner - \urcorner, \ulcorner - \urcorner) : \mathcal{L}_m \to \mathsf{F}(\mathcal{L}_m)$ be the canonical d-frame map from generators. The free constructions we will encounter will often satisfy the property that for any d-frame $\mathcal{L}$

$$\mathsf{F}(\mathcal{L}) = \mathsf{F}(\mathcal{L}_m)/(\mathsf{id}^+, \mathsf{id}^-, \ulcorner c_L \urcorner, \ulcorner t_L \urcorner),$$

where $\ulcorner c_L \urcorner = \{(\ulcorner x^+ \urcorner, \ulcorner x^- \urcorner) : (x^+, x^-) \in c_L\}$, and $\ulcorner t_L \urcorner$ is defined similarly. Lemma 5.1.3 can be seen as telling us that the ideal completion is one of the free constructions with this property, i.e. that we have

$$\mathsf{Idl}(\mathcal{L}) = \mathsf{Idl}(\mathcal{L}_m)/(\mathsf{id}^+, \mathsf{id}^-, \ulcorner c_L \urcorner, \ulcorner t_L \urcorner),$$

Another way of viewing this property of free constructions is that, for free constructions with this special property, we can obtain $\mathsf{F}(\mathcal{L})$ from $\mathsf{F}(\mathcal{L}_m)$ by increasing its consistency and totality. It suffices to increase them just enough so that the pair of frame maps $(\ulcorner - \urcorner, \ulcorner - \urcorner) : \mathcal{L} \to \mathsf{F}(\mathcal{L}_m)$, seen as a map between quadruples, becomes a d-frame map.

To phrase the desired adjunction in detail, let us first make $\mathcal{L} \mapsto \mathsf{Idl}(\mathcal{L})$ functorial. For a d-lattice map $f : \mathcal{L} \to \mathcal{M}$ we define the map $\mathsf{Idl}(f) : \mathsf{Idl}(\mathcal{L}) \to \mathsf{Idl}(\mathcal{M})$ as simply the pairing of the maps $\mathsf{Idl}(f^+)$ and $\mathsf{Idl}(f^-)$. It would be easy to check explicitly that this is a d-frame map, but let us instead prove it as an instance of a much more general fact that we will use several times in this thesis. The situation we are dealing with is that if we have a d-lattice map $f : \mathcal{L} \to \mathcal{M}$ then we surely have a d-frame map

$$(\mathsf{Idl}(f^+), \mathsf{Idl}(f^-)) : \mathsf{Idl}(\mathcal{L}_m) \to \mathsf{Idl}(\mathcal{M}_m).$$

We want to — roughly speaking — shift this map along the quotienting by $(\mathsf{id}^+, \mathsf{id}^-, \ulcorner c_L \urcorner, \ulcorner t_L \urcorner)$ and $(\mathsf{id}^+, \mathsf{id}^-, \ulcorner c_M \urcorner, \ulcorner t_M \urcorner)$. Given d-frames $\mathcal{L}$ and $\mathcal{M}$ and relation structures $\mathcal{R}$ on $\mathcal{L}$ and $\mathcal{S}$ on $\mathcal{M}$, what we would particularly like is a general result which tells us when is it that a d-frame map $f : \mathcal{L} \to \mathcal{M}$ can be shifted along the pair of d-frame quotients (q_R, q_S), in the sense of

there being a d-frame map $f_{RS} : \mathcal{L}/\mathcal{R} \to \mathcal{M}/\mathcal{S}$ making the following square commute.

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{f} & \mathcal{M} \\ {\scriptstyle q_R}\downarrow & & \downarrow{\scriptstyle q_S} \\ \mathcal{L}/\mathcal{R} & \xrightarrow{f_{RS}} & \mathcal{M}/\mathcal{S} \end{array}$$

From the next proof onward, we will slightly abuse our notation and when we have a map $f : \mathcal{L} \to \mathcal{M}$ and a relation structure $\mathcal{R}$ on $\mathcal{L}$ we refer to the forward image quadruple of $\mathcal{R}$ under the map f as simply $f[\mathcal{R}]$. Recall also that for relation structures $\mathcal{R}$ and $\mathcal{S}$ on the same d-frame we write that $\mathcal{R} \leq \mathcal{S}$ to denote componentwise set inclusion of $\mathcal{R}$ into $\mathcal{S}$.

Lemma 5.1.5. *Suppose that $\mathcal{L}$ and $\mathcal{M}$ are d-frames, $\mathcal{R}$ a relation structure on $\mathcal{L}$, and $\mathcal{S}$ is a relation structure on $\mathcal{M}$. If we have a d-frame map $f : \mathcal{L} \to \mathcal{M}$ such that $f[\mathcal{R}] \leq \mathcal{S}$, then there is a unique d-frame map $f_{RS} : \mathcal{L}/\mathcal{R} \to \mathcal{M}/\mathcal{S}$ making the following square commute.*

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{f} & \mathcal{M} \\ {\scriptstyle q_R}\downarrow & & \downarrow{\scriptstyle q_S} \\ \mathcal{L}/\mathcal{R} & \xrightarrow{f_{RS}} & \mathcal{M}/\mathcal{S} \end{array}$$

Proof. To show our claim, it suffices to show that the map $q_S \circ f$ respects the relation structure $\mathcal{R}$. Once we have shown this, the desired map f_{RS} will be the universal map coming from the universal property of the quotient $\mathcal{L}/\mathcal{R}$. We have that if $a^+ R^+ b^+$ then by our assumption we also have $f^+(a^+) S^+ f^+(b^+)$, and so $q_S^+(f^+(a^+)) \leq q_S^+(f^+(b^+))$ by definition of d-frame quotients. The argument to show that $(x^+, x^-) \in c_R$ implies that $(q_S^+(x^+), q_S^-(x^-)) \in c_{M/S}$ and that $(x^+, x^-) \in t_R$ implies that $(q_S^+(x^+), q_S^-(x^-)) \in t_{M/S}$ is similar. Then, indeed the map $q_S \circ f$ respects the relation structure $\mathcal{R}$. This means that there is a map $\widetilde{q_S \circ f} =: f_{RS}$ making the following diagram commute.

$$\begin{array}{ccc} & \mathcal{L}/\mathcal{R} & \\ {\scriptstyle q_R}\nearrow & & \searrow{\scriptstyle f_{RS}} \\ \mathcal{L} & \xrightarrow{q_S \circ f} & \mathcal{M}/\mathcal{S} \end{array}$$

This means that the map f_{RS} is as desired. □

Let us apply Lemma 5.1.5 above to show functoriality of our assignment Idl.

Lemma 5.1.6. *If $f : \mathcal{L} \to \mathcal{M}$ is a d-lattice map, the map $\mathsf{Idl}(f) : \mathsf{Idl}(\mathcal{L}) \to \mathsf{Idl}(\mathcal{M})$ is a d-frame map.*

Proof. We claim that $\mathsf{Idl}(f) = (\mathsf{Idl}(f^+), \mathsf{Idl}(f^-))$ constitutes a d-frame map $\mathsf{Idl}(\mathcal{L}) \to \mathsf{Idl}(\mathcal{M})$. In order to do this, by Lemma 5.1.5 it suffices to show that we have the componentwise inclusion $(\mathsf{id}^+, \mathsf{id}^-, f[\ulcorner c_L \urcorner], f[\ulcorner t_L \urcorner]) \leq (\mathsf{id}^+, \mathsf{id}^-, \ulcorner c_M \urcorner, \ulcorner t_M \urcorner)$. But this follows directly from $f : \mathcal{L} \to \mathcal{M}$ being a d-lattice map. □

Since the assignment Idl sends each pair of lattice maps to the pairing of their forward image maps, this assignment respects identities and composition. Then, it is functorial. Now, we claim that there is an adjunction $\mathsf{Idl} : \mathbf{dLat} \leftrightarrows \mathbf{dFrm} : i$ where i is simply subcategory inclusion. Let us now look at the unit and counit of the adjunction.

Lemma 5.1.7. *For any d-lattice $\mathcal{L}$ the pairing $(\downarrow, \downarrow) : \mathcal{L} \to i(\mathsf{Idl}(\mathcal{L}))$ is a d-lattice map.*

Proof. It suffices to show that the pair of lattice maps respects consistency and totality. We notice that if $(x^+, x^-) \in c_L$ then $(y^+, y^-) \in c_L$ whenever $y^+ \leq x^+$ and $y^- \leq x^-$, and so we also have that $(\downarrow x^+, \downarrow x^-) \in \mathsf{id}_\forall$. Then, the map $(\downarrow, \downarrow)$ preserves the con component. For the tot component we observe that if $(x^+, x^-) \in t_L$, then $(\downarrow x^+, \downarrow x^-) \in \mathsf{id}_\exists$ with witness (x^+, x^-). □

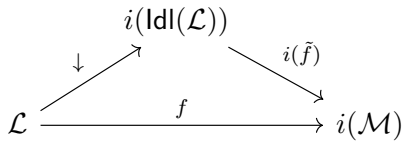
Lemma 5.1.8. *For any d-frame $\mathcal{L}$ the pairing $(\bigvee(-), \bigvee(-)) : \mathsf{Idl}(i(\mathcal{L})) \to \mathcal{L}$ is a d-frame map.*

Proof. It suffices to show that the pair of frame maps respects consistency and totality. If a pair of ideals (I^+, I^-) is such that all elements of $I^+ \times I^-$ are in the c_L subset, then by closure of c_L under homogeneous joins we have $(\bigvee I^+, i^-) \in c_L$ for every $i^- \in I^-$ and applying closure under homogeneous joins again we also obtain $(\bigvee I^+, \bigvee I^-) \in c_L$. For the tot component, suppose that $(I^+, I^-) \in \mathsf{id}_\exists$ for witness (i^+, i^-). Since we have that $(i^+, i^-) \leq (\bigvee I^+, \bigvee I^-)$ we also have $(\bigvee I^+, \bigvee I^-) \in t_L$. Then, the map $(\bigvee(-), \bigvee(-))$ respects the tot component, too. □

We omit the proof that the two maps are natural transformations, since the commutativity of the naturality squares amounts to the commutativity of the corresponding naturality squares for the unit and the counit of the adjunction $\mathsf{Idl} : \mathbf{Distr} \leftrightarrows \mathbf{Frm} : i$. Commutativity of the relevant unit-counit triangles of the adjunction $\mathsf{Idl} : \mathbf{dDistr} \leftrightarrows \mathbf{dFrm} : i$, too, amounts to commutativity of the corresponding unit-counit triangles of the adjunction $\mathsf{Idl} : \mathbf{Distr} \leftrightarrows \mathbf{Frm} : i$.

Theorem 5.1.9. *There is an adjunction* $\mathsf{Idl} : \mathbf{dDistr} \leftrightarrows \mathbf{dFrm} : i$ *with* $\mathsf{Idl} \dashv i$*; whose unit is the assignment* $\mathsf{Obj}(\mathbf{dDistr}) \to \mathsf{Mor}(\mathbf{dDistr})$ *assigning to a distributive d-lattice* $\mathcal{L}$ *the map* $\downarrow : \mathcal{L} \to i(\mathsf{Idl}(\mathcal{L}))$*; and whose counit is the assignment* $\mathsf{Obj}(\mathbf{dFrm}) \to \mathsf{Mor}(\mathbf{dFrm})$ *assigning to each d-frame* $\mathcal{L}$ *the map* $(\bigvee(-), \bigvee(-)) : \mathsf{Idl}(i(\mathcal{L})) \to \mathcal{L}$*.*

Proposition 5.1.10. *For any distributive d-lattice* $\mathcal{L}$*, the d-lattice map* $(\downarrow^+, \downarrow^-) : \mathcal{L} \to \mathsf{Idl}(\mathcal{L})$ *is such that for any d-frame* $\mathcal{M}$ *and every d-lattice map* $f : \mathcal{L} \to i(\mathcal{M})$ *there is a unique d-frame map* $\tilde{f} : \mathsf{Idl}(\mathcal{L}) \to \mathcal{M}$ *such that the following triangle commutes.*



Proof. By definition of adjoint functors, and by Theorem 5.1.9, we know that there is a bijection between d-lattice maps $\mathcal{L} \to i(\mathcal{M})$ and d-frame maps $\mathsf{Idl}(\mathcal{L}) \to \mathcal{M}$. Furthermore this bijection, which we denote as $f \mapsto \tilde{f}$, is such that $f = i(\tilde{f}) \circ \downarrow$ still by definition of adjunction. □

We also have a notion of filter completion of a distributive d-lattice. For a d-lattice $\mathcal{L}$ we call its *opposite* the quadruple (L^{-op}, L^{+op}, t, c) and we denote it as $\mathcal{L}^{op}$. Notice that the con and the tot components have swapped places. We denote with $\mathsf{Filt}(\mathcal{L})$ the d-frame $\mathsf{Idl}(\mathcal{L}^{op})$, and we call it the *filter completion* of $\mathcal{L}$. Explicitly, for a distributive d-lattice $\mathcal{L}$, the filter completion $\mathsf{Filt}(\mathcal{L})$ is the quadruple

$$(\mathsf{Filt}(L^-), \mathsf{Filt}(L^+), \mathsf{id}_\exists, \mathsf{id}_\forall).$$

Remark 5.1.11. The following remark largely consists of a paraphrase of ideas in the nLab page [1]. It is a known categorical fact that an adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : i$ with $F \dashv i$ and $i : \mathcal{D} \hookrightarrow \mathcal{C}$ a subcategory inclusion is such that the functor $i \circ F$ is idempotent if and only if the inclusion $i : \mathcal{D} \hookrightarrow \mathcal{C}$ is full. Intuitively, one can make sense of this by reasoning in terms of the contrast between *property* and *structure*. If i is a full inclusion, it means that objects in $\mathcal{D}$ may be seen as objects of $\mathcal{C}$ satisfying a certain property. The functor $i \circ F$ is the most efficient solution to the problem of adding the relevant property to each object of $\mathcal{C}$. Since for each object $X \in \mathsf{Obj}(\mathcal{L})$ the object $i(F(X))$ already has the desired property, the most efficient solution to the problem is just to leave it as it is. On the other hand, suppose that the inclusion $i : \mathcal{D} \hookrightarrow \mathcal{C}$ is not full. The restriction on morphisms means that the objects of the category $\mathcal{D}$ also have some extra structure. Then, the functor $i \circ F$ is the most efficient

solution to adding the missing structure to objects of $\mathcal{C}$. "Most efficient", in this case, also means that the missing structure has to be added *freely*, and this is why these are usually called free completions. When we have an object $X \in \mathsf{Obj}(\mathcal{C})$, the object $i(F(X))$ does have the missing structure, but the structure is not free enough to be able to consider $i(F(X))$ its own free completion. In general, then, we will have $i(F(i(F(X)))) \ncong i(F(X))$.

In the case of the adjunction $\mathsf{Idl} : \mathbf{Distr} \leftrightarrows \mathbf{Frm} : i$, the inclusion is not full, and so a distributive lattice which is already a frame is not in general isomorphic to its ideal completion. The ideal completion of a d-lattice is a free completion, too.

5.2 The assembly of a d-frame

In the construction of the assembly of a d-frame, d-frame coproducts will be involved. An explicit way of constructing the coproduct of d-frames is carried out in [22]. Here, we are not interested in looking into its explicit structure, but rather, we want to point out that the coproduct of d-frames, too, has a concise description as a free construction similar to those introduced in Remark 5.1.4. From the universal property of the d-frame coproduct alone, we can see that the d-frame coproduct of d-frames with minimal consistency and totality is simply computed as

$$\mathcal{L}_m \oplus \mathcal{M}_m = (L^+ \oplus M^+, L^- \oplus M^-, c_m, t_m).$$

To obtain the general d-frame coproduct of two d-frames $\mathcal{L}$ and $\mathcal{M}$ it suffices to quotient $\mathcal{L}_m \oplus \mathcal{M}_m$ by a relation structure $\mathcal{R}$ which is just large enough that the pairings of the canonical frame embeddings $(e_L^+, e_L^-) : \mathcal{L} \to (\mathcal{L}_m \oplus \mathcal{M}_m)/\mathcal{R}$ become d-frame maps. Let us abbreviate $(\mathsf{id}^+, \mathsf{id}^-, c_L, t_L)$ as $\mathcal{R}_{ct}$, so that we have $\mathcal{L} = \mathcal{L}_m/\mathcal{R}_{ct}$. Let us also abbreviate $(\mathsf{id}^+, \mathsf{id}^-, \ulcorner c_L \urcorner, \ulcorner t_L \urcorner)$ as $\ulcorner \mathcal{R}_L \urcorner$. What we want to show, stated in another way, amounts to the equality

$$(\mathcal{L}_m/\mathcal{R}_L) \oplus (\mathcal{M}_m/\mathcal{R}_M) = (\mathcal{L}_m \oplus \mathcal{M}_m)/(\ulcorner \mathcal{R}_L \urcorner \cup \ulcorner \mathcal{R}_M \urcorner).$$

In a very informal way, we may say that we want to show that the relation structures of the form $(\mathsf{id}^+, \mathsf{id}^-, c_L, t_L)$ can go through coproducts undisturbed. We can actually prove something stronger, i.e. that all relation structures have this property. Suppose that we have two d-frames given as d-frame quotients $\mathcal{L}/\mathcal{R}$ and $\mathcal{M}/\mathcal{S}$. We denote as $\ulcorner \mathcal{R} \urcorner$ the lifting of the relation structure along the coproduct embedding $e_L : \mathcal{L} \to \mathcal{L} \oplus \mathcal{M}$. That is, we define $\ulcorner \mathcal{R} \urcorner$ tas he quadruple

$$\begin{aligned}&(\{(\ulcorner a^+ \urcorner, \ulcorner b^+ \urcorner) : a^+ R^+ b^+\}, \{(\ulcorner a^- \urcorner, \ulcorner b^- \urcorner) : a^- R^- b^-\},\\ &\{(\ulcorner x^+ \urcorner, \ulcorner x^- \urcorner) : (x^+, x^-) \in c_R\}, \{(\ulcorner x^+ \urcorner, \ulcorner x^- \urcorner) : (x^+, x^-) \in t_R\})\end{aligned}$$

on the d-frame $\mathcal{L} \oplus \mathcal{M}$.

Lemma 5.2.1. *Suppose that we have d-frames $\mathcal{L}$ and $\mathcal{M}$, and that $\mathcal{R}$ and $\mathcal{S}$ are relation structures on $\mathcal{L}$ and on $\mathcal{M}$, respectively. Then we have that*

$$(\mathcal{L}/\mathcal{R}) \oplus (\mathcal{M}/\mathcal{S}) \cong (\mathcal{L} \oplus \mathcal{M})/(\ulcorner \mathcal{R} \urcorner \cup \ulcorner \mathcal{S} \urcorner).$$

Proof. Let us show that the two structures are isomorphic. To show this, we will show that they have the same universal property. First, let us notice that by the universal property of the d-frame coproduct the d-frame maps $f : \mathcal{L} \oplus \mathcal{M} \to \mathcal{N}$ can be identified with the pairing $f_L \oplus f_M$ of pairs of frame maps from the two frames $\mathcal{L}$ and $\mathcal{M}$ to $\mathcal{N}$. Still reasoning in terms of universal properties, we see that if we have a relation structure $\mathcal{R}$ on $\mathcal{L}$, then the map $f_L \oplus f_M$ respects $\ulcorner \mathcal{R} \urcorner$ if and only if f_L respects $\mathcal{R}$. For any d-frame $\mathcal{N}$, we have the following. We omit the proofs that all bijections are natural.

$$\begin{aligned}
&\mathbf{dFrm}((\mathcal{L}/\mathcal{R}) \oplus (\mathcal{M}/\mathcal{S}), \mathcal{N}) \cong \mathbf{dFrm}(\mathcal{L}/\mathcal{R}, \mathcal{N}) \times \mathbf{dFrm}(\mathcal{M}/\mathcal{S}, \mathcal{N}) \cong \\
&\cong \{f \in \mathbf{dFrm}(\mathcal{L}, \mathcal{N}) : f \text{ respects } \mathcal{R}\} \times \{f \in \mathbf{dFrm}(\mathcal{M}, \mathcal{N}) : f \text{ respects } \mathcal{S}\} \cong \\
&\cong \{f_L \oplus f_M \in \mathbf{dFrm}(\mathcal{L} \oplus \mathcal{M}, \mathcal{N}) : f_L \text{ respects } \mathcal{R} \text{ and } f_M \text{ respects } \mathcal{S}\} \cong \\
&\cong \{f_L \oplus f_M \in \mathbf{dFrm}(\mathcal{L} \oplus \mathcal{M}, \mathcal{N}) : f_L \oplus f_M \text{ respects } \ulcorner \mathcal{R} \urcorner \vee \ulcorner \mathcal{S} \urcorner\} \cong \\
&\cong \mathbf{dFrm}((\mathcal{L} \oplus \mathcal{M})/(\ulcorner \mathcal{R} \urcorner \vee \ulcorner \mathcal{S} \urcorner), \mathcal{N}).
\end{aligned}$$

Then, the two structures must be isomorphic. □

We have now gathered enough technical tools to construct the assembly of a d-frame. We will see that this, in fact, can be constructed in two ways. The first is inspired by the d-frame construction of the assembly that we have explained in Chapter 4. We will adopt a d-frame theoretical version of the technique, and use filter completions of d-frames, d-frame coproducts, and d-frame quotients to translate the construction into a construction within the category of d-frames. Recall that in the first section of this chapter we showed that for a d-frame $\mathcal{L}$ its filter completion is a quadruple

$$(\mathsf{Filt}(L^-), \mathsf{Filt}(L^+), \mathsf{id}_\exists, \mathsf{id}_\forall).$$

For a d-frame $\mathcal{L}$, we denote the collection $\{(\ulcorner a^+ \urcorner, \ulcorner \uparrow a^+ \urcorner : \ulcorner a^+ \urcorner \in L^+)\} \cup \{(\ulcorner \uparrow a^- \urcorner, \ulcorner a^- \urcorner) : a^- \in L^-\}$ as $\mathcal{C}$. We define the *assembly* of a d-frame $\mathcal{L}$ to be the structure

$$\mathsf{A}(\mathcal{L}) = (\mathcal{L} \oplus \mathsf{Filt}(\mathcal{L}))/(\mathsf{id}^+, \mathsf{id}^-, \mathcal{C}, \mathcal{C}).$$

In other words, the assembly of a d-frame $\mathcal{L}$ is the quotient of the d-frame $\mathcal{L} \oplus \mathsf{Filt}(\mathcal{L})$ which makes it so that every element of the form $\ulcorner a^+ \urcorner$ becomes the d-complement of $\ulcorner \uparrow a^+ \urcorner$, and similarly for the elements from the negative frame. Recall that there is a canonical d-frame embedding $e_L : \mathcal{L} \hookrightarrow \mathcal{L} \oplus \mathsf{Filt}(\mathcal{L})$, and a canonical d-frame surjection $q_C : \mathcal{L} \oplus \mathsf{Filt}(\mathcal{L}) \to \mathsf{A}(\mathcal{L})$. We denote their composition as $\nabla = (\nabla^+, \nabla^-) : \mathcal{L} \to \mathsf{A}(\mathcal{L})$ and call this map the *canonical* map from a d-frame to its d-assembly. In contrast with the case of the canonical map from a frame to its assembly, we do not know if this map is an injection in general. The first step we will take is to show that this construction yields an object with the desired universal property. We first need to prove two lemmas.

Lemma 5.2.2. *If $\mathcal{L}$ is a d-lattice, and a^+ and b^+ are d-complemented, then $\sim(a^+ \wedge b^+) = \sim a^+ \vee \sim b^+$.*

Proof. The inequality $\sim a^+ \vee \sim b^+ \leq \sim(a^+ \wedge b^+)$ is immediate because $\sim$ is antitone, so let us prove the other. Since the totality subset of a d-lattice is an upset, and closed under finite homogeneous meets, we must have $(a^+ \wedge b^+, \sim a^+ \vee \sim b^+) \in t_L$ and this, by balance, implies that $\sim(a^+ \wedge b^+) \leq \sim a^+ \vee \sim b^+$. □

Lemma 5.2.3. *Whenever we have a d-frame map $f : \mathcal{L} \to \mathcal{M}$ providing d-complements for all elements of $L^+ \cup L^-$, the map $\sim f = (\sim f^-, \sim f^+) : \mathcal{L}^{op} \to \mathcal{M}$ is a d-lattice map.*

Proof. Suppose that $f : \mathcal{L} \to \mathcal{M}$ is a d-frame map such that it provides d-complements to all elements of $L^+ \cup L^-$. Recall that $\mathcal{L}^{op}$ is defined to be the quadruple $(L^{-op}, L^{+op}, t_L, c_L)$. By Lemma 5.2.2, and since $\sim$ turns joins into meets, we have that $\sim f^+ : L^{+op} \to M^-$ and $\sim f^- : L^{-op} \to M^+$ are both lattice maps. For preservation of the con and tot components, suppose first that $(a^+, a^-) \in t_L$. We have the following derivation.

$$
\begin{aligned}
&(a^+, a^-) \in t_L \\
&(f^+(a^+), f^-(a^-)) \in t_M \\
&\sim f^+(a^+) \leq f^-(a^-) \leq \sim\sim f^-(a^-) \\
&(\sim f^-(a^-), \sim f^+(a^+)) \in c_M.
\end{aligned}
$$

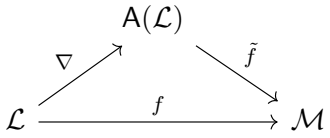
For $(a^+, a^-) \in c_L$, instead, the derivation is as follows. We use the fact that we must have

$\sim\sim f^+(a^+) = f^+(a^+)$ because $f^+(a^+)$ is complemented.

$$
\begin{aligned}
&(a^+, a^-) \in c_L \\
&(f^+(a^+), f^-(a^-)) \in c_M \\
&f^+(a^+) \leq \sim f^-(a^-) \\
&\sim\sim f^+(a^+) \leq \sim f^-(a^-) \\
&(\sim f^-(a^-), \sim f^+(a^+)) \in t_M.
\end{aligned}
$$

□

Theorem 5.2.4. *For every d-frame $\mathcal{L}$, whenever $f : \mathcal{L} \to \mathcal{M}$ is a d-frame map which provides d-complements for all elements of $L^+ \cup L^-$, there is a unique d-frame map $\tilde{f} : \mathsf{A}(\mathcal{L}) \to \mathcal{M}$ such that the following triangle commutes.*



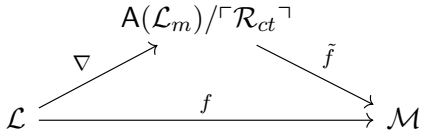
Proof. Suppose that there is a d-frame map $f : \mathcal{L} \to \mathcal{M}$ such that it provides complements for all elements in $L^+ \cup L^-$. We define a map $\tilde{f} : \mathsf{A}(\mathcal{L}) \to \mathcal{M}$ by defining a map $\mathcal{L} \oplus \mathsf{Filt}(\mathcal{L}) \to \mathcal{M}$ which factors through the quotient $\mathcal{C}$. This map is the d-frame coproduct pairing $f \oplus \mathsf{Filt}(\sim f)$, where $\mathsf{Filt}(\sim f) : \mathsf{Filt}(\mathcal{L}) \to \mathcal{M}$ is the lifting of the d-lattice map $\sim f : \mathcal{L}^{op} \to \mathcal{M}$, we know that this is a d-lattice map, by Lemma 5.2.3. We need to show that if this map factors through the quotient $\mathcal{C}$. For this it suffices to show that for each pair $(\ulcorner a^+ \urcorner, \ulcorner \uparrow a^+ \urcorner)$ is mapped by $f \oplus \mathsf{Filt}(\sim f)$ to a pair in $c_M \cap t_M$ and similarly for elements in L^-. By definition of this map, we have that it maps a pair $(\ulcorner a^+ \urcorner, \ulcorner \uparrow a^+ \urcorner)$ to the pair $(f^+(a^+), \sim f^+(a^+))$ and, since $f^+(a^+)$ is assumed to be d-complemented, this pair is both in c_M and in t_M. The argument for elements in L^- is analogous. The map $\tilde{f} : \mathsf{A}(\mathcal{L}) \to \mathcal{M}$ is then a d-frame map and, by its definition, it is such that $\tilde{f} \circ \nabla = f$, and so the triangle above commutes. □

It is worth observing that the assembly of a d-frame $\mathcal{L}$ may be presented as a quotient of the d-frame assembly of $\mathcal{L}_m$. We show that the assembly of a d-frame is one of those free constructions which we have described in Remark 5.1.4.

Lemma 5.2.5. *The assembly of a d-frame $\mathcal{L}$ is the quotient of the d-frame $\mathsf{A}(\mathcal{L}_m)$ by the relation structure $(\mathsf{id}^+, \mathsf{id}^-, \ulcorner c_L \urcorner, \ulcorner t_L \urcorner)$.*

Proof. Let us abbreviate $(\mathsf{id}^+, \mathsf{id}^-, \ulcorner c_L \urcorner, \ulcorner t_L \urcorner)$ as simply $\mathcal{R}_{ct}$. Suppose that there is a d-frame map $f : \mathcal{L} \to \mathcal{M}$ such that it provides d-complements for all elements of $L^+ \cup L^-$. We will

show that there is a unique d-frame map $\tilde{f} : \mathsf{A}(\mathcal{L}_m)/\ulcorner\mathcal{R}_{ct}\urcorner \to \mathcal{M}$ making the following triangle commute.



To show that there is such a map $\tilde{f}$, it suffices to show that the map $\overline{f} : \mathsf{A}(\mathcal{L}_m) \to \mathcal{M}$ defined as $\overline{f}^+(\nabla^+(a^+)) = f^+(a^+)$ for each $a^+ \in L^+$, and similarly for elements from L^-, respects the relation structure $\ulcorner\mathcal{R}_{ct}\urcorner$. This map respects the relation structure as desired, by its definition, and because f respects $\mathcal{R}_{ct}$. □

This also gives us another way of describing the d-frame assembly. The assembly $\mathsf{A}(\mathcal{L}_m)$ is the d-frame $\mathcal{L}_m \oplus \mathsf{Filt}(\mathcal{L}_m)$ quotiented by $(\mathsf{id}^+, \mathsf{id}^-, \mathcal{C}, \mathcal{C})$. Recall Lemma 5.1.2, and the observation made in the beginning of this section that for two d-frames $\mathcal{L}_m$ and $\mathcal{M}_m$ their coproduct still has minimal con and tot components. The d-frame $\mathcal{L}_m \oplus \mathsf{Filt}(\mathcal{L}_m)$, then, is

$$(L^+ \oplus \mathsf{Filt}(L^-), L^- \oplus \mathsf{Filt}(L^+), c_m, t_m).$$

The assembly of a general d-frame $\mathcal{L}$, then, can be described as the quotient of the d-frame $(L^+ \oplus \mathsf{Filt}(L^-), L^- \oplus \mathsf{Filt}(L^+), c_m, t_m)$ by the relation structure

$$(\mathsf{id}^+, \mathsf{id}^-, \ulcorner c_L \urcorner \cup \mathcal{C}, \ulcorner t_L \urcorner \cup \mathcal{C}).$$

We call this relation structure $\mathcal{RA}_L$, for "Relation structure of the d-frame Assembly". We will refer to this result later on, let us then write it down explicitly as a proposition.

Proposition 5.2.6. *The assembly $\mathsf{A}(\mathcal{L})$ of a d-frame $\mathcal{L}$ is the quotient of the d-frame*

$$(L^+ \oplus \mathsf{Filt}(L^-), L^- \oplus \mathsf{Filt}(L^+), c_m, t_m)$$

by the relation structure $\mathcal{RA}_L = (\mathsf{id}^+, \mathsf{id}^-, \mathcal{C} \cup \ulcorner c_L \urcorner, \mathcal{C} \cup \ulcorner t_L \urcorner)$, in which

$$\begin{aligned}
c_L &= \{(\ulcorner x^+ \urcorner, \ulcorner x^- \urcorner) : (x^+, x^-) \in c_L\} \\
t_L &= \{(\ulcorner x^+ \urcorner, \ulcorner x^- \urcorner) : (x^+, x^-) \in t_L\} \\
\mathcal{C} &= \{(\ulcorner x^+ \urcorner, \ulcorner \uparrow x^+ \urcorner) : x^+ \in L^+\} \cup \{(\ulcorner x^- \urcorner, \ulcorner \uparrow x^- \urcorner) : x^- \in L^-\}.
\end{aligned}$$

5.2.1 An alternative presentation of the d-frame assembly

We now analyze an alternative way of presenting the assembly of a d-frame. For a d-frame $\mathcal{L}$, we define $\mathcal{L}^+$ as the d-frame $(L^+, 2, c_m, t_m)$, and $\mathcal{L}^-$ as the d-frame $(2, L^-, c_m, t_m)$. Let us now compute the coproduct the assemblies of these two particular d-frames. The filter completion of 2 is just 2 itself. Additionally, 2 is the neutral element of frame coproduct: to see this, we notice that - as 2 is the initial object of **Frm** - the pairs of maps $f_1 : L \to M$ and $f_2 : 2 \to M$ are in bijective correspondence with the maps $f : L \to M$ for any two frames L and M.

$$\begin{cases} \mathsf{A}(\mathcal{L}^+) = (L^+, \mathsf{Filt}(L^+), c_m, t_m)/(\mathsf{id}^+, \mathsf{id}^-, \mathcal{C}^+, \mathcal{C}^+) \\ \mathsf{A}(\mathcal{L}^-) = (\mathsf{Filt}(L^-), L^-, c_m, t_m)/(\mathsf{id}^+, \mathsf{id}^-, \mathcal{C}^-, \mathcal{C}^-). \end{cases}$$

Here we have used the definition $\mathcal{C}^+ = \{(a^+, \uparrow a^+) : a^+ \in L^+\}$, and the set $\mathcal{C}^-$ is defined analogously. Now, let us compute the d-frame coproduct $\mathsf{A}(\mathcal{L}^+) \oplus \mathsf{A}(\mathcal{L}^-)$. For this, we will finally apply Lemma 5.2.1.

Lemma 5.2.7. *For any d-frame $\mathcal{L}$ we have that*

$$\mathsf{A}(\mathcal{L}^+) \oplus \mathsf{A}(\mathcal{L}^-) = \mathsf{A}(\mathcal{L}_m).$$

Proof. By Lemma 5.2.1, the coproduct $\mathsf{A}(\mathcal{L}^+) \oplus \mathsf{A}(\mathcal{L}^+)$ is the d-frame $(L^+ \oplus \mathsf{Filt}(L^-), L^- \oplus \mathsf{Filt}(L^+), c_m, t_m)$ quotiented by the relation structure $(\mathsf{id}^+, \mathsf{id}^-, \ulcorner \mathcal{C}^+ \urcorner \vee \ulcorner \mathcal{C}^- \urcorner, \ulcorner \mathcal{C}^+ \urcorner \vee \ulcorner \mathcal{C}^- \urcorner)$. Explicitly, this is the relation structure adding the pairs in $\{(\ulcorner x^+ \urcorner, \ulcorner \uparrow x^+ \urcorner) : x^+ \in L^+\} \cup \{(\ulcorner x^- \urcorner, \ulcorner \uparrow x^- \urcorner) : x^- \in L^-\}$ both to the con and to the tot component. By definition of the d-frame assembly, the desired claim is proven. □

Let us observe that this lemma also means that we have found a very specific kind of d-frames for which the assembly construction commutes with coproducts.

Corollary 5.2.8. *For any d-frame $\mathcal{L}$ we have that*

$$\mathsf{A}(\mathcal{L}^+) \oplus \mathsf{A}(\mathcal{L}^-) = \mathsf{A}(\mathcal{L}^+ \oplus \mathcal{L}^-).$$

Proof. This follows from Lemma 5.2.7, and from the observation that $\mathcal{L}_m = \mathcal{L}^+ \oplus \mathcal{L}^-$. □

We have also obtained the following fact, which we will use to find the new presentation of the d-frame assembly that we have mentioned.

Lemma 5.2.9. *The assembly of a d-frame $\mathcal{L}$ is the quotient of the d-frame $\mathsf{A}(\mathcal{L}^+) \oplus \mathsf{A}(\mathcal{L}^-)$ by the relation structure $(\mathsf{id}^+, \mathsf{id}^-, \ulcorner c_L \urcorner, \ulcorner t_L \urcorner)$.*

Proof. This follows from Lemma 5.2.5 and from Lemma 5.2.7. □

Finally, let us show that the assembly of a d-frame can be described in a way which is, in a sense, orthogonal to the presentation $(\mathcal{L} \oplus \mathsf{Filt}(\mathcal{L}))/(\mathsf{id}^+, \mathsf{id}^-, \mathcal{C}, \mathcal{C})$. We will show, in particular, that for a d-frame $\mathcal{L}$ its assembly can be described as the quotient of $\mathsf{A}(\mathcal{L}^+) \oplus \mathsf{A}(\mathcal{L}^-)$ by an extremal relation structure.

Proposition 5.2.10. *For any d-frame $\mathcal{L}$, we define*

$$R^+_{\mathcal{C}} = \{(\ulcorner a^+ \urcorner, \ulcorner \uparrow a^- \urcorner) : (a^+, a^-) \in c_L\} \cup \{(\ulcorner \uparrow a^- \urcorner, \ulcorner a^+ \urcorner) : (a^+, a^-) \in t_L\}.$$

Define similarly the relation $R^-_{\mathcal{C}}$ on the frame $L^- \oplus \mathsf{Filt}(L^+)$. The assembly $\mathsf{A}(\mathcal{L})$ is

$$(\mathsf{A}(\mathcal{L}^+) \oplus \mathsf{A}(\mathcal{L}^-))/(R^+_{\mathcal{C}}, R^-_{\mathcal{C}}, c_m, t_m).$$

Proof. Suppose that $\mathcal{L}$ is a d-frame. Recall that its assembly $\mathsf{A}(\mathcal{L})$ is given by quotienting $\mathsf{A}(\mathcal{L}^+) \oplus \mathsf{A}(\mathcal{L}^-)$ by the relation structure $(\mathsf{id}^+, \mathsf{id}^-, \ulcorner c_L \urcorner, \ulcorner t_L \urcorner)$. To prove our claim, then, it suffices to show that quotienting by this relation structure is equivalent to quotienting by $(R^+_{\mathcal{C}}, R^-_{\mathcal{C}}, c_m, t_m)$. This means that we need to show that a frame map $f : \mathsf{A}(\mathcal{L}^+) \oplus \mathsf{A}(\mathcal{L}^-) \to \mathcal{M}$ respects the first relation structure if and only if it respects the second. Then, let $f : \mathsf{A}(\mathcal{L}^+) \oplus \mathsf{A}(\mathcal{L}^-) \to \mathcal{M}$ be an arbitrary d-frame map. We have to show that this is such that if it satisfies the set of conditions

$$\begin{cases} (f^+(\ulcorner a^+ \urcorner), f^-(\ulcorner a^- \urcorner)) \in c_M & \text{whenever } (a^+, a^-) \in c_L \\ (f^+(\ulcorner a^+ \urcorner), f^-(\ulcorner a^- \urcorner)) \in t_M & \text{whenever } (a^+, a^-) \in t_L \end{cases}$$

then it also satisfies the relations

$$\begin{cases} f^+(\ulcorner a^+ \urcorner) \leq f^+(\ulcorner \uparrow a^- \urcorner) & \text{whenever } (a^+, a^-) \in c_L \\ f^-(\ulcorner \uparrow a^- \urcorner) \leq f^-(\ulcorner a^+ \urcorner) & \text{whenever } (a^+, a^-) \in t_L \end{cases}$$

and vice versa. If the map satisfies the first set of conditions, it must also satisfy the second, because in the structure $\mathsf{A}(\mathcal{L}^+) \oplus \mathsf{A}(\mathcal{L}^-)$ the pairs $(\ulcorner a^+ \urcorner, \ulcorner \uparrow a^+ \urcorner)$ and $(\ulcorner \uparrow a^- \urcorner, \ulcorner a^- \urcorner)$ are pairs of mutual d-complements. For the converse, suppose that the second set of relations holds, and suppose that $(a^+, a^-) \in c_L$. Then, by assumption $f^+(\ulcorner a^+ \urcorner) \leq f^+(\ulcorner \uparrow a^- \urcorner)$. Again, because in the structure $\mathsf{A}(\mathcal{L}^+) \oplus \mathsf{A}(\mathcal{L}^-)$ the pairs $(\ulcorner a^+ \urcorner, \ulcorner \uparrow a^+ \urcorner)$ and $(\ulcorner \uparrow a^- \urcorner, \ulcorner a^- \urcorner)$ are pairs of mutual d-complements, we must have that $(f^+(\ulcorner \uparrow a^- \urcorner), f^-(\ulcorner a^- \urcorner)) \in c_M$. Since consistency is a downset, we must also have $(f^+(\ulcorner a^+ \urcorner), f^-(\ulcorner a^- \urcorner)) \in c_M$. We use a similar argument using d-complements to show that if the second set of relations holds we also have that $(a^+, a^-) \in t_L$ implies that $(f^+(\ulcorner a^+ \urcorner), f^-(\ulcorner a^- \urcorner)) \in t_M$. □

Finally, let us prove that the assembly of a d-frame satisfies two properties which are the d-frame analogues of well known properties of the assemblies of a frame.

Proposition 5.2.11. *For a d-frame $\mathcal{L}$ we have the following.*

- *Its assembly $\mathsf{A}(\mathcal{L})$ is zero dimensional.*
- *The d-frame $\mathcal{L}$ is Boolean if and only if it is isomorphic to $\mathsf{A}(\mathcal{L})$.*

Proof. The first frame component of the assembly of a d-frame $\mathcal{L}$ is generated by elements of the form $\ulcorner a^+ \urcorner$ and $\ulcorner \uparrow a^- \urcorner$, and these are d-complemented by the definition of the quotient $\mathcal{C}$ on $\mathcal{L} \oplus \mathsf{Filt}(\mathcal{L})$. The argument for the other frame component is analogous. The second part of the claim is immediate from the universal property of the assembly of a frame. □

In the following chapter we will analyze the patch of the assembly of a d-frame and show that it is a partial assembly.

5.3 The assembly functor on $\mathbf{dFrm}$

Let us work towards making the assignment $\mathcal{L} \mapsto \mathsf{A}(\mathcal{L})$ into an endofunctor on the category of d-frames. The assembly $\mathsf{A}(\mathcal{L})$ of a d-frame $\mathcal{L}$ is the quotient of the d-frame

$$(L^+ \oplus \mathsf{Filt}(L^-), L^- \oplus \mathsf{Filt}(L^+), c_m, t_m)$$

by the relation structure $\mathcal{RA}_L = (\mathsf{id}^-, \mathsf{id}^-, \mathcal{C}_L \cup \ulcorner c_L \urcorner, \mathcal{C}_L \cup \ulcorner t_L \urcorner)$, where as usual we have the definitions

- $\mathcal{C}_L = \{(\ulcorner x^+ \urcorner, \ulcorner \uparrow x^+ \urcorner) : x^+ \in L^+\} \cup \{(\ulcorner \uparrow x^- \urcorner, \ulcorner x^- \urcorner) : x^- \in L^-\}$,
- $\ulcorner c_L \urcorner = \{(\ulcorner x^+ \urcorner, \ulcorner x^- \urcorner) : (x^+, x^-) \in c_L\}$,
- $\ulcorner t_L \urcorner = \{(\ulcorner x^+ \urcorner, \ulcorner x^- \urcorner) : (x^+, x^-) \in t_L\}$.

We introduce a notational convention. Let us point out that in the two frame components of $\mathsf{A}(\mathcal{L})$ all the generators $a^+ \in L^+$ are going to appear as syntactic expressions. The d-frame $\mathsf{A}(\mathcal{L})$ is obtained by quotienting the d-frame

$$(L^+ \oplus \mathsf{Filt}(L^-), L^- \oplus \mathsf{Filt}(L^+), c_m, t_m).$$

In this, as usual we denote as $\ulcorner a^+ \urcorner$ and $\ulcorner \uparrow a^- \urcorner$ the generators of $L^+ \oplus \mathsf{Filt}(L^-)$, and we denote similarly the generators of the negative frame components. To avoid confusion, then, we will use the double bracket notation $\ulcorner\ulcorner a^+ \urcorner\urcorner$ for elements of the frame components of $\mathsf{A}(\mathcal{L})$.

For any frame map $f : L \to M$ we know that this can be seen as a lattice map $f : L^{op} \to M^{op}$ between distributive lattices. Because of this, functoriality of the ideal completion of distributive lattices tells us that there is a frame map $\mathsf{Filt}(f) : \mathsf{Filt}(L) \to \mathsf{Filt}(M)$ defined on generators as $\mathsf{Filt}(f)(\uparrow a) = \uparrow f(a)$ for every $a \in L$. This means that whenever we have a d-frame map $f : \mathcal{L} \to \mathcal{M}$, if we set $\mathsf{A}(f)_m = (f^+ \oplus \mathsf{Filt}(f^-), f^- \oplus \mathsf{Filt}(f^+))$ then we may see this as a d-frame map

$$\mathsf{A}(f)_m : (L^+ \oplus \mathsf{Filt}(L^-), L^- \oplus \mathsf{Filt}(L^+), c_m, t_m) \to (M^+ \oplus \mathsf{Filt}(M^-), M^- \oplus \mathsf{Filt}(M^+), c_m, t_m).$$

In order to obtain a d-frame map $\mathsf{A}(f) : \mathsf{A}(\mathcal{L}) \to \mathsf{A}(\mathcal{M})$ we would like to be able to shift this d-frame map along the quotients of these two d-frames by $\mathcal{RA}_L$ and by $\mathcal{RA}_M$. By Lemma 5.1.5, we know that to deduce that this is possible we only need to check that $\mathsf{A}(f)_m[\mathcal{RA}_L] \leq \mathcal{RA}_M$.

Lemma 5.3.1. *If $f : \mathcal{L} \to \mathcal{M}$ is a d-frame map, we have that $\mathsf{A}(f)_m[\mathcal{RA}_L] \leq \mathcal{RA}_M$.*

Proof. Let us show this in three parts.

- First, we show that $\mathsf{A}(f)_m[\ulcorner c_L \urcorner] \subseteq \ulcorner c_M \urcorner$. Suppose that we have a pair $(\ulcorner x^+ \urcorner, \ulcorner x^- \urcorner) \in \ulcorner c_L \urcorner$. This means that $(x^+, x^-) \in c_L$. Since f is assumed to be a d-frame map, in particular it respects the con component, and so $(f^+(x^+), f^-(x^-)) \in c_M$. By definition of our map $\mathsf{A}(f)_m$, we have that the forward image of our pair $(\ulcorner x^+ \urcorner, \ulcorner x^- \urcorner)$ under this is $(\ulcorner f^+(x^+) \urcorner, \ulcorner f^-(x^-) \urcorner)$. This is in $\ulcorner c_M \urcorner$ since, as deduced before, $(f^+(x^+), f^-(x^-)) \in c_M$.
- The argument to show that $\mathsf{A}(f)_m[\ulcorner t_L \urcorner] \subseteq \ulcorner t_M \urcorner$ is analogous.
- Let us show that we also have the inclusion $\mathsf{A}(f)_m[\mathcal{C}_L] \subseteq \mathcal{C}_M$. For $a^+ \in L^+$, we have that $\mathsf{A}(f)^+_m(\ulcorner a^+ \urcorner) = \ulcorner f^+(a^+) \urcorner$ and, similarly, $\mathsf{A}(f)^-_m(\ulcorner \uparrow a^+ \urcorner) = \ulcorner \mathsf{Filt}(f^+)(a^+) \urcorner = \ulcorner \uparrow f^+(a^+) \urcorner$. This means that the forward image of a pair $(\ulcorner a^+ \urcorner, \ulcorner \uparrow a^+ \urcorner))$ is still of the form $(\ulcorner m^+ \urcorner, \ulcorner \uparrow m^+ \urcorner)$ for some element $m^+ \in M^+$, and so it must be in $\mathcal{C}_M$. □

Combining Lemmas 5.1.5 and 5.3.1, we learn that for any d-frame map $f : \mathcal{L} \to \mathcal{M}$ we can define a d-frame map

$$\begin{aligned} &\mathsf{A}(f) : \mathsf{A}(\mathcal{L}) \to \mathsf{A}(\mathcal{M}), \\ &\ulcorner\ulcorner a^+ \urcorner\urcorner \mapsto \ulcorner\ulcorner f^+(a^+) \urcorner\urcorner, \\ &\ulcorner\ulcorner \uparrow a^- \urcorner\urcorner \mapsto \ulcorner\ulcorner \uparrow f^-(a^-) \urcorner\urcorner, \end{aligned}$$

and defined similarly on generators from the second frame component. From this definition it is immediate that the assignment respects identities and compositions. Then, we may define

a functor $\mathsf{A} : \mathbf{dFrm} \to \mathbf{dFrm}$ on objects as assigning to each d-frame its assembly, and on morphisms as $f \mapsto \mathsf{A}(f)$, as described above. We have obtained the following. We will see that the spectrum of the d-frame assembly is bihomeomorphic to the *Skula bispace* of $\mathsf{dpt}(\mathcal{L})$. This bispace is still such that all d-sober subspaces of $\mathsf{dpt}(\mathcal{L})$ are patch-closed in its bitopology. For a bispace X we define its *Skula bispace*, which we will denote as $Sk(X)$, to be the bispace such that

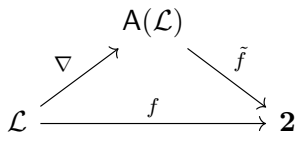
- its underlying set of points is the same as that of X,
- its positive opens are the topology generated by the positive open sets of X together with the negative closed sets of X,
- its negative opens are the topology generated by the negative open sets of X together with the positive closed sets of X.

Let us now define a functor $Sk : \mathbf{BiTop} \to \mathbf{BiTop}$. For a bispace X, we define this as $X \mapsto Sk(X)$. For a bicontinuous map $f : X \to Y$ we define $Sk(f) : Sk(X) \to Sk(Y)$ to be the function f. The fact that this is bicontinuous follows from the fact that f by our assumption is such that preimages of open (resp. closed) sets of Y are open (resp. closed) in X, and the same holds for the negative open and closed sets. The fact that the assignment respects identities and compositions follows from the fact that it simply maps each continuous function to itself. The next step is to connect the assembly and the Skula functors for d-frames and bispaces to show that they are in a relation similar to that of the frame assembly and the Skula functor for spaces. The fact that there is a bijection between the underlying sets of the relevant bispaces holds because of the universal property of the d-frame assembly.

Lemma 5.3.2. *For any d-frame $\mathcal{L}$, there is a bijection*

$$\alpha_{\mathcal{L}} : \mathsf{dpt}(\mathcal{L}) \to \mathsf{dpt}(\mathsf{A}(\mathcal{L}))$$

Proof. We recall that the assembly of a d-frame has the universal property such that every d-frame map $f : \mathcal{L} \to \mathcal{M}$ that provides d-pseudocomplements to all elements of $L^+ \cup L^-$ factors through it uniquely. Every d-point $f : \mathcal{L} \to \mathbf{2}$ is such that it provides d-complements to all elements of $L^+ \cup L^-$. Thus, the assignment $f \mapsto \tilde{f}$ mapping each point of $\mathcal{L}$ to the universal map $\tilde{f} : \mathsf{A}(\mathcal{L})$ such that this triangle commutes



is the desired bijection. □

Let us now refine this and show that this bijection is a bihomeomorphism between the Skula space of $\mathsf{dpt}(\mathcal{L})$ and the d-spectrum of the assembly of $\mathcal{L}$.

Lemma 5.3.3. *For any d-frame $\mathcal{L}$ the following holds.*

- $\alpha_{\mathcal{L}}[\varphi^+_{\mathcal{L}}(a^+)] = \varphi^+_{\mathsf{A}(\mathcal{L})}(\ulcorner\urcorner a^+ \urcorner\urcorner)$.
- $\alpha_{\mathcal{L}}[\varphi^-_{\mathcal{L}}(a^-)^c] = \varphi^+_{\mathsf{A}(\mathcal{L})}(\ulcorner\urcorner \uparrow a^- \urcorner\urcorner)$.
- $\alpha_{\mathcal{L}}[\varphi^-_{\mathcal{L}}(a^-)] = \varphi^-_{\mathsf{A}(\mathcal{L})}(\ulcorner\urcorner a^- \urcorner\urcorner)$.
- $\alpha_{\mathcal{L}}[\varphi^+_{\mathcal{L}}(a^+)^c] = \varphi^-_{\mathsf{A}(\mathcal{L})}(\ulcorner\urcorner \uparrow a^+ \urcorner\urcorner)$.

Proof. Recall that by definition of the quotient $\mathcal{RA}_L$ which we use to define the d-frame assembly each of the pairs of syntactic expressions $(\ulcorner\urcorner a^+ \urcorner\urcorner, \ulcorner\urcorner \uparrow a^+ \urcorner\urcorner)$ is a pair of mutual d-complements. Let us also recall that from Lemma 5.3.2 we know that every d-point of $\mathsf{A}(\mathcal{L})$ is of the form $\tilde{f}$ for some d-frame map $f : \mathcal{L} \to \mathbf{2}$. Let us also recall that by the definition of the map $\tilde{f}$ for every $f \in \mathsf{dpt}(\mathcal{L})$ we have that $\tilde{f}(\ulcorner\urcorner a^+ \urcorner\urcorner) = f(a^+)$ for every $a^+ \in L^+$. Let us now prove the first two parts of the claim.

- For every $a^+ \in L^+$, we have the following chain of equalities.

$$\begin{aligned}
&\alpha_{\mathcal{L}}[\varphi^+_{\mathcal{L}}(a^+)] = \\
&= \{\tilde{f} \in \mathsf{dpt}(\mathsf{A}(\mathcal{L})) : f^+(a^+) = 1\} = \\
&= \{\tilde{f} \in \mathsf{dpt}(\mathsf{A}(\mathcal{L})) : \tilde{f}(\ulcorner\urcorner a^+ \urcorner\urcorner) = 1\} = \\
&= \varphi^+_{\mathsf{A}(\mathcal{L})}(\ulcorner\urcorner a^+ \urcorner\urcorner).
\end{aligned}$$

- For every $a^- \in L^-$, we have the following chain of equalities.

$$\begin{aligned}
&\alpha_{\mathcal{L}}[\varphi^-_{\mathcal{L}}(a^-)^c] = \\
&= \{\tilde{f} \in \mathsf{dpt}(\mathsf{A}(\mathcal{L})) : f^-(a^-) = 0\} = \\
&= \{\tilde{f} \in \mathsf{dpt}(\mathsf{A}(\mathcal{L})) : \tilde{f}(\ulcorner\urcorner \uparrow a^- \urcorner\urcorner) = 1\} = \\
&= \varphi^+_{\mathsf{A}(\mathcal{L})}(\ulcorner\urcorner \uparrow a^- \urcorner\urcorner).
\end{aligned}$$

□

Proposition 5.3.4. *For any d-frame $\mathcal{L}$, the bijection $\alpha_{\mathcal{L}} : Sk(\mathsf{dpt}(\mathcal{L})) \cong \mathsf{dpt}(\mathsf{A}(\mathcal{L})))$ is a bihomeomorphism.*

Proof. To show that this bijection is a bihomeomorphism it suffices to show that in the positive topology of $\mathsf{dpt}(\mathsf{A}(\mathcal{L}))$ the subbasic opens are exactly the forward images of the subbasic opens of $Sk(\mathsf{dpt}(\mathcal{L}))$, and similarly for the negative topology. To find a subbasis for the positive topology of $\mathsf{dpt}(\mathsf{A}(\mathcal{L}))$, we reason as follows. In the coproduct $L^+ \oplus \mathsf{Filt}(L^-)$ we have a subbasis $\{\ulcorner a^+ \urcorner : a^+ \in L^+\} \cup \{\ulcorner \uparrow a^- \urcorner : a^- \in L^-\}$. Because frame quotient maps preserve the frame operations, the first frame component of $\mathsf{A}(\mathcal{L})$, too, is generated by the syntactic expressions $\{\ulcorner\ulcorner a^+ \urcorner\urcorner : a^+ \in L^+\} \cup \{\ulcorner\ulcorner \uparrow a^- \urcorner\urcorner : a^- \in L^-\}$. Finally, the positive component of the d-spatialization map $\varphi_{\mathsf{A}(\mathcal{L})}$, too, turns arbitrary joins into arbitrary unions and finite meets into finite intersections. A subbasis of the positive topology of $\mathsf{dpt}(\mathsf{A}(\mathcal{L}))$, then, is given by the collection

$$\{\varphi^+_{\mathsf{A}(\mathcal{L})}(\ulcorner\ulcorner a^+ \urcorner\urcorner) : a^+ \in L^+\} \cup \{\varphi^+_{\mathsf{A}(\mathcal{L})}(\ulcorner\ulcorner \uparrow a^- \urcorner\urcorner) : a^- \in L^-\}.$$

On the other hand, by definition of Skula bispace we have that a subbasis for the positive topology of $Sk(\mathsf{dpt}(\mathcal{L}))$ is given by

$$\{\varphi^+_{\mathcal{L}}(a^+) : a^+ \in L^+\} \cup \{\varphi^-_{\mathcal{L}}(a^-)^c : a^- \in L^-\}.$$

By Lemma 5.3.3, then, the forward images of the subbasic opens of $Sk(\mathsf{dpt}(\mathcal{L}))$ coincide with the subbasic opens of $\mathsf{dpt}(\mathsf{A}(\mathcal{L}))$. Then, $\alpha_{\mathcal{L}}$ is a homeomorphism according to the positive topology. □

We are ready to prove the main theorem of this subsection.

Theorem 5.3.5. *The following square commutes up to natural isomorphism.*

$$\begin{array}{ccc} \mathbf{dFrm} & \xrightarrow{\mathsf{dpt}} & \mathbf{BiTop} \\ \Big\downarrow{\scriptstyle \mathsf{A}} & & \Big\downarrow{\scriptstyle Sk} \\ \mathbf{dFrm} & \xrightarrow{\mathsf{dpt}} & \mathbf{BiTop} \end{array}$$

Proof. We already know that for every d-frame $\mathcal{L}$ we have a bihomeomorphism $\alpha_{\mathcal{L}} : Sk_{cf}(\mathsf{dpt}(\mathcal{L})) \cong \mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L})))$. It suffices to show that this assignment is a natural transformation. Suppose that $f : \mathcal{L} \to \mathcal{M}$ is a d-frame map. The naturality square we need to check is the following.

$$\begin{array}{ccc} Sk(\mathsf{dpt}(\mathcal{M})) & \xrightarrow{\alpha_{\mathcal{M}}} & \mathsf{dpt}(\mathsf{A}(\mathcal{M})) \\ \Big\downarrow{\scriptstyle \mathsf{dpt}(f)} & & \Big\downarrow{\scriptstyle \mathsf{dpt}(\mathsf{A}(f))} \\ Sk(\mathsf{dpt}(\mathcal{L})) & \xrightarrow{\alpha_{\mathcal{L}}} & \mathsf{dpt}(\mathsf{A}(\mathcal{L})) \end{array}$$

Since we only need to check equalities of functions, we may instead consider the following simpler square in **Set**. We have also simplified using the fact that dpt sends morphisms to their precomposition map.

$$\begin{array}{ccc} \mathsf{dpt}(\mathcal{M}) & \xrightarrow{\alpha_{\mathcal{M}}} & \mathsf{dpt}(\mathsf{A}(\mathcal{M})) \\ {\scriptstyle -\circ f}\downarrow & & \downarrow{\scriptstyle -\circ \mathsf{A}_{cf}(f)} \\ \mathsf{dpt}(\mathcal{L}) & \xrightarrow{\alpha_{\mathcal{L}}} & \mathsf{dpt}(\mathsf{A}(\mathcal{L})) \end{array}$$

For $g \in \mathsf{dpt}(\mathcal{M})$, we have to check that $\alpha_{\mathcal{M}} \circ \mathsf{A}(f) = \alpha_{\mathcal{L}}(g \circ f)$. These are both maps $\mathsf{A}(\mathcal{L}) \to \mathbf{2}$. Since elements of the form $\ulcorner\urcorner \uparrow a^{-} \urcorner\urcorner$ are d-complements of those of the form $\ulcorner\urcorner a^{-} \urcorner\urcorner$, any point of the d-frame assembly is completely determined by its action on generators of the form $\ulcorner\urcorner a^{+} \urcorner\urcorner$ and $\ulcorner\urcorner a^{-} \urcorner\urcorner$. To check that these are the same, then, it suffices to show they agree on these generators. Suppose, then, that $a^+ \in L^+$. We have the following chain of equalities.

$$\begin{aligned} &\alpha_{\mathcal{M}}(g)^+(\mathsf{A}(f)^+(\ulcorner\urcorner a^+ \urcorner\urcorner)) = \\ &= \alpha_{\mathcal{M}}(g)^+(\ulcorner\urcorner f^+(a^+) \urcorner\urcorner) = \\ &= \tilde{g}^+(\ulcorner\urcorner f^+(a^+) \urcorner\urcorner) = \\ &= g^+(f^+(a^+)) = \\ &= \alpha_{\mathcal{L}}(g \circ f)^+(\ulcorner\urcorner a^+ \urcorner\urcorner). \end{aligned}$$

□

Remark 5.3.6. Let us now build some visual intuition on what the spectrum of the d-frame assembly looks like. Recall that in Chapter 1 we showed that in case the frames L^+ and L^- are linear, for a d-frame $\mathcal{L} = (L^+, L^-, c_L, t_L)$ we can think of $\mathsf{dpt}(\mathcal{L})$ as a rectangle $\mathsf{pt}(L^+) \times \mathsf{pt}(L^-)$, in which some points are taken away because of con and tot. We have also seen that in our pictures we think of opens of $\mathsf{pt}(L^+)$ as being left segments and those of $\mathsf{pt}(L^-)$ as being downsets. The subset c_L is telling us that certain pairs of opens are disjoint, and so it takes away left lower portions of $\mathsf{pt}(L^+) \times \mathsf{pt}(L^-)$. The t_L subset, instead, takes away right upper portions.

We now keep these conventions, and we consider a d-frame $\mathcal{L}$ with L^+ and L^- linear. We identify the spectrum of the d-frame assembly $\mathsf{A}(\mathcal{L})$ with the Skula space $Sk(\mathsf{dpt}(\mathcal{L}))$. Let us look at the positive topology first. We know that this is generated by the sets of the form $\varphi^+_{\mathcal{L}}(a^+)$ together with those of the form $\varphi^-_{\mathcal{L}}(a^-)^c$.

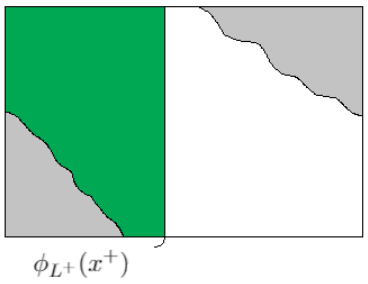


Figure 5.1: A typical subbasic open of the positive topology of $Sk(\mathsf{dpt}(\mathcal{L}))$ of the first kind, i.e. a set of the form $\varphi_L^+(x^+)$.

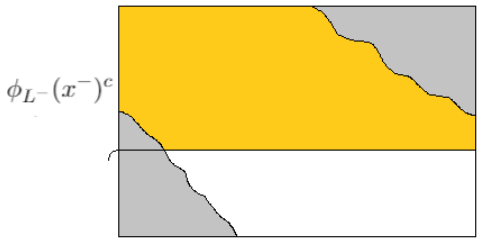


Figure 5.2: A typical subbasic open of the positive topology of $Sk(\mathsf{dpt}(\mathcal{L}))$ of the second kind, i.e. a set of the form $\varphi_L^-(x^-)^c$.

If we take intersections of sets of this form, we clearly get upper left rectangles. And, finally, taking arbitrary unions of these gives us upper left portions of our partial rectangles. The shape of a generic positive open of $Sk(\mathsf{dpt}(\mathcal{L}))$ is then an upper left portion. For the negative opens, we reason similarly. The negative topology, too, is generated by a subbasis of rectangles of two different kinds.

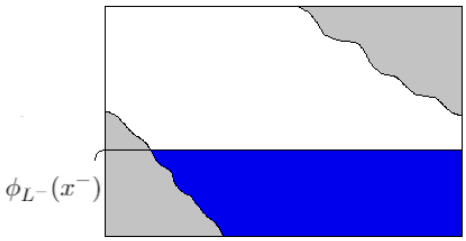


Figure 5.3: A typical subbasic open of the negative topology $Sk(\mathsf{dpt}(\mathcal{L}))$ of the first kind, i.e. a set of the form $\varphi_L^-(x^-)$.

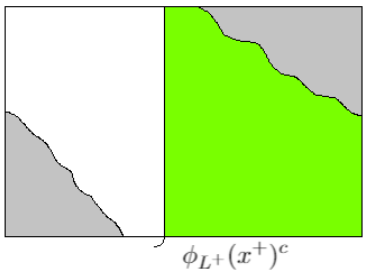
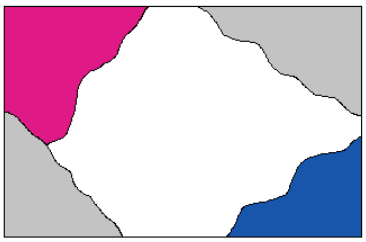


Figure 5.4: A typical subbasic open of the negative topology $Sk(\mathsf{dpt}(\mathcal{L}))$ of the second kind, i.e. a set of the form $\varphi_L^+(x^+)^c$.

Taking intersections of shapes of this form yields a right lower rectangle. We deduce that a typical negative open has the shape of a lower right portion. Then, the bitopology of $Sk(\mathsf{dpt}(\mathcal{L}))$ is such that the positive and the negative opens are portions at the opposite corners of our partial rectangle $\mathsf{dpt}(\mathcal{L})$. Below, we depict a typical positive open together with a typical negative open of the bitopology $Sk(\mathsf{dpt}(\mathcal{L}))$.



In Chapter 7 we will encounter variations of the Skula bispace construction. In particular, we will encounter two new constructions giving bispaces $Sk_{cf}(\mathsf{dpt}(\mathcal{L}))$ and $Sk_{\pm}(\mathsf{dpt}(\mathcal{L}))$, both with $|\mathsf{dpt}(\mathcal{L})|$ as their underlying set of points. Each of these bitopologies is such that each of its topologies is the one generated by the rectangles of two of the four main kinds seen above.

Chapter 6

The d-frame assembly as a frame

Introduction

We begin Section 6.1 with a revisitation of our work on partial assemblies. In [9] a theorem is proven where Noetherian frames are characterized in several ways. In particular, Noetherian frames are exactly those whose assembly is compact as well as those whose assembly is coherent. We prove a generalization of this result, which gives necessary and sufficient conditions on a frame L and a subset $S \subseteq L$ for the partial assembly $\mathsf{A}_S(L)$ to be compact.

With Section 6.2 we move on to proving several technical results on *finitary assemblies*. For a biframe $\mathcal{L}$, the finitary assembly of L is the partial assembly $\mathsf{A}_{\mathsf{fin}}(L)$ with the universal property that it provides complements freely for all finitary elements of L. As we have seen in Chapter 4 on partial assemblies, this is the same as the subframe of $\mathsf{A}(L)$ generated by the closed congruences together with the open congruences of the finitary elements of L. In this chapter we show that the bitopological nature of the frame L is inherited by its finitary assembly, in the sense that we have

$$\mathsf{A}_{\mathsf{fin}}(L) \cong (\mathsf{A}(L^+) \oplus \mathsf{A}(L^-))/C_L$$

for some congruence C_L which takes care that the inequalities of L also are inherited by its finitary assembly. We will refer to [48], where the assembly of a biframe was introduced. By *assembly* of a biframe $\mathcal{L}$ we mean a biframe $\mathsf{A}(\mathcal{L})$ with a canonical biframe embedding $\nabla : \mathcal{L} \to \mathsf{A}(\mathcal{L})$ such that it provides bicomplements freely to all elements of $L^+ \cup L^-$. In [48] the construction $\mathsf{A}(\mathcal{L})$ is called the *congruence biframe* of the biframe $\mathcal{L}$. We will show that the main component of the assembly of a biframe $\mathcal{L}$ is the finitary assembly of L. We then use our generalization to generalize to biframes the notion of a Noetherian frame. In this chapter

we prove bitopological versions of the main theorem in [9], in particular we will introduce the notion of *Noetherian* biframe and show that a biframe $\mathcal{L}$ is Noetherian if and only if its assembly $\mathsf{A}(\mathcal{L})$ is compact. We conclude Section 6.2 with the result that the finitary assembly $\mathsf{A}_{\mathsf{fin}}(L)$ of the main component L of a biframe $\mathcal{L}$ is bitopological in more than one way. For any biframe $\mathcal{L}$ the following three structures are all biframes.

- Its assembly $\mathsf{A}(\mathcal{L}) = (\mathsf{A}_{\nabla^+\Delta^-}(L), \mathsf{A}_{\nabla^-\Delta^+}(L), \mathsf{A}_{\mathsf{fin}}(L))$. Here $\mathsf{A}_{\nabla^+\Delta^-}(L)$ is the subframe of $\mathsf{A}(L)$ generated by the positive closed congruences and the negative open ones. The frame $\mathsf{A}_{\nabla^-\Delta^+}(L)$ is defined similarly.
- The structure $(\mathsf{Cl}(L), \mathsf{Fitt}_{\mathsf{fin}}(L), \mathsf{A}_{\mathsf{fin}}(L))$, where $\mathsf{Cl}(L)$ is the ordered collection of closed congruences of L, and $\mathsf{Fitt}_{\mathsf{fin}}(L)$ that of its finitary fitted congruences. We call this the *closed-fitted* assembly of $\mathcal{L}$.
- The structure $(\mathsf{A}^+(L), \mathsf{A}^-(L), \mathsf{A}_{\mathsf{fin}}(L))$. Here $\mathsf{A}^+(L)$ is the image of the assembly $\mathsf{A}(L^+)$ under the map $\mathsf{A}(i^+) : \mathsf{A}(L^+) \to \mathsf{A}(L)$, where $i^+ : L^+ \hookrightarrow L$ is subset inclusion (recall that $\mathsf{A} : \mathbf{Frm} \to \mathbf{Frm}$ is a functor). The frame $\mathsf{A}^-(L)$ is defined similarly. We call this biframe the *positive-negative* assembly of $\mathcal{L}$.

Section 6.3 is devoted to constructing variations of the assembly of a d-frame which are analogues of the closed-fitted and the positive-negative assemblies of a biframe. We will show that there are two structures $\mathsf{A}_{cf}(\mathcal{L})$ and $\mathsf{A}_{\pm}(\mathcal{L})$, for every d-frame $\mathcal{L}$, such that they have the same patch as $\mathsf{A}(\mathcal{L})$. The structure analogous to the closed-fitted assembly will be the result of quotienting

$$(\mathsf{con}(L^+ \times L^-), \mathsf{tot}(L^+ \times L^-), c_m, t_m)$$

by a certain relation structure, while the structure analogous to the positive-negative assembly will be the quotient by a certain relation structure of the d-frame

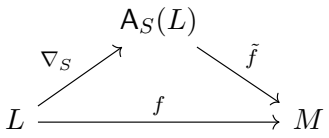
$$(\mathsf{A}(L^+), \mathsf{A}(L^+), c_m, t_m).$$

We will introduce the two structures and prove some technical results about these. In particular, we will show that the patch of both these variations of the assembly of a $\mathcal{L}$ is the finitary assembly of the patch of $\mathcal{L}$. In Chapter 7 we will look at how one should interpret these structures (in what sense, for instance, they are analogues of their biframe versions).

Convention. Throughout this chapter, to ease our notation we will assume for all our biframes $\mathcal{L}$ that $L^+, L^- \subseteq L$. All the result and proofs we present in this chapters are valid for generic biframes $\mathcal{L}$, and to obtain the generalizations of these results and proofs we simply substitute for every generator $a^+ \in L^+$ its image $\ulcorner a^+ \urcorner$, and similarly for generators on the negative component.

6.1 Partial assemblies locale theoretically

Recall that a partial assembly of a frame L is a subframe of the assembly $\mathsf{A}(L)$, and that every subset $S \subseteq L$ determines the partial assembly $\mathsf{A}_S(L)$. The frame embedding $\nabla_S : L \to \mathsf{A}_S(L)$, sending each element of L to the corresponding closed congruence, is universal with the property that it provides complements to all elements of S. This means that, whenever we have a frame map $f : L \to M$ such that $f(s) \vee \neg f(s) = 1$, there is a unique frame map $\tilde{f}$ such that the following triangle commutes.



The elements of the assembly of a frame have an interpretation as sublocales of the frame L; if we did not stop to analyze what partial assemblies are from this point of view, we would miss a big part of the picture. Let us then adapt our notation: for a partial assembly $\mathsf{A}_S(L)$ we call $\mathsf{S}_S(L)^{op}$ its image under the canonical isomorphism $\mathsf{A}(L) \to \mathsf{S}(L)^{op}$ mapping each congruence to the corresponding sublocale. Recall that in Chapter 4 we proved (Lemma 4.1.7) that for any frame L and for any subset $S \subseteq L$ the partial assembly $\mathsf{A}_S(L)$ is the subframe of $\mathsf{A}(L)$ generated by the collection

$$\{\nabla(x) : x \in L\} \cup \{\Delta(s) : s \in S\}$$

Expressing this in terms of sublocales yields the following.

Lemma 6.1.1. *For any frame L, the frame $\mathsf{S}_S(L)^{op}$ is the subframe of $\mathsf{S}(L)^{op}$ generated by all the closed sublocales of L together with the open ones of the form $\mathfrak{o}(s)$ for $s \in S$.*

Remark 6.1.2. We also remind the reader of our d-frame presentation of a general partial assembly, which we introduced in Chapter 4. For a frame L and a subset $S \subseteq L$, we present the partial assembly $\mathsf{A}_S(L)$ as a d-frame of the form

$$\mathcal{A}_S(L) = (L, \mathsf{Filt}_S(L), \mathsf{Con}_S, \mathsf{Tot}_S).$$

Here $\mathsf{Filt}_S(L)$ is the subframe of $\mathsf{Filt}(L)$ generated by principal filters of the form $\uparrow s$ for some $s \in S$. The consistency Con_S and the totality Tot_S are the smallest ones such that the pairs of the form $(s, \uparrow s)$ are pairs of mutual d-complements for every $s \in S$. In Chapter 4 we also proved a more concrete, equivalent description of the two subsets Con_S and Tot_S. We have that $(x, F) \in \mathsf{Con}_S$ if and only if $x \leq \bigwedge F$, and that $(x, F) \in \mathsf{Tot}_S$ if and only if

$x \in F$, for $x \in L$ and $F \in \mathsf{Filt}_S(L)$. Recall, now, that $\Pi : \mathbf{dFrm} \to \mathbf{Frm}$ denotes the patch functor for d-frames. Since L is the first frame component of $\mathcal{A}_S(L)$, there is a canonical map $\ulcorner - \urcorner : L \to \Pi(\mathcal{A}_S(L))$. This is the canonical embedding $\nabla_S : L \to \mathsf{A}_S(L)$ mapping each element to its closed congruence.

6.1.1 A refinement of the classical results on Noetherian frames

We begin by stating the theorem that we wish to generalize, first proven in [9]. Recall that a frame is *Noetherian* if all its elements are compact.

Theorem 6.1.3. *The following are equivalent for a frame L.*

1. *The frame $\mathsf{S}(L)^{op}$ is compact.*
2. *The frame L is Noetherian.*
3. *The complemented elements of $\mathsf{S}(L)^{op}$ are exactly the compact ones.*
4. *The frame $\mathsf{S}(L)^{op}$ is coherent.*

We would like to generalize this result and examine compactness of partial assemblies: the main reason why we are interested in this is that it will be the first step in proving a biframe version of this theorem. Assemblies are in general zero dimensional, and this readily gives us the equivalence of conditions (1), (3), and (4) above.

Lemma 6.1.4. *If L is a zero dimensional frame, the following are equivalent.*

1. *The frame L is compact.*
2. *The compact elements of L coincide with the complemented ones.*
3. *The frame L is coherent.*

Proof. Suppose that L is a zero dimensional frame.

[1 implies 2]. Suppose that L is compact. If $x \in L$ is a complemented element, we have that $x \leq \bigvee_i x_i$ implies that $\neg x \vee \bigvee_i x_i = 1$; by compactness this implies $\neg x \vee \bigvee_{f \in F} x_f$ for some finite $f \in F$, and this implies $x \leq \bigvee_{f \in F}$. If x is compact, by zero dimensionality it is the finite union of complemented elements. Finite unions of complemented elements are still complemented, since if $y_1, ..., y_n$ are complemented, we have $y_1 \vee ... \vee y_n \vee \neg(y_1 \vee ... \vee y_n) = y_1 \vee ... \vee y_n \vee (\neg y_1 \wedge ... \wedge \neg y_n) \geq (y_1 \vee \neg y_1) \wedge ... \wedge (y_n \vee \neg y_n) = 1$.

[2 implies 3]. If the complemented elements are compact, this means that the compact elements are closed under meets and, by zero dimensionality, they also generate all of L.

[3 implies 1]. Coherence implies compactness for any frame. □

However, partial assemblies are not in general zero dimensional: in a partial assembly $\mathsf{S}_S(L)^{op}$ all closed sublocales $\mathfrak{c}(x)$ with $x \in L$ appear, but not all open ones; there are not enough complemented elements to generate all of our frame. Then, we cannot so easily establish the equivalence of conditions (1), (3), and (4). In fact, the analogues of these three conditions are inequivalent for general partial assemblies. Before we look at this issue let us prove the part of the theorem that we can indeed refine to all general partial assemblies.

Lemma 6.1.5. *Let L be a frame and $S \subseteq L$ a subset such that all finite meets of elements from S are compact. If $T \subseteq S$ and $x \in L$, we have that $\bigcap_{t\in T} \mathfrak{o}(t) \subseteq \mathfrak{o}(x)$ implies $\mathfrak{o}(t_1 \wedge ... \wedge t_n) \subseteq \mathfrak{o}(x)$ for some collection $\{t_1, ..., t_n\} \in T$.*

Proof. Suppose that L is a frame and $S \subseteq L$ a subset such that all finite meets of elements from S are compact. Whenever we have $T \subseteq S$, the fitted sublocale $\bigcap_{t\in T} \mathfrak{o}(t)$ is the same as the fitted sublocale $\bigcap_{t\in F(T)} \mathfrak{o}(t)$ induced by the smallest filter $F(T)$ containing T. The filter $F(T)$ must be Scott-open. By Lemma 2.2.7, then, it must be strongly exact. By Lemma 2.2.6, we have that $\bigcap_{t\in T} \mathfrak{o}(t) \subseteq \mathfrak{o}(x)$ implies that $x \in F(T)$. This means that there must be a finite collection $\{t_1, ..., t_n\} \subseteq T$ with $t_1 \wedge ... \wedge t_n \leq x$. □

Without any assumptions on the frame L or the set S, one can prove the following.

Proposition 6.1.6. *For a frame L and any subset S we have that finite meets of elements from S are compact if and only if $\mathsf{S}_S(L)^{op}$ is a compact frame.*

Proof. Suppose that all finite meets of elements from S are compact. The sublocales of the form $\mathfrak{c}(x)$ for $x \in L$ together with those of the form $\mathfrak{o}(s)$ for $s \in S$ are a subbasis for $\mathsf{S}_S(L)^{op}$. Alexander's Lemma, then, tells us that to show that $\mathsf{S}_S(L)^{op}$ is compact it suffices to show that $\bigcap_i \mathfrak{c}(x_i) \cap \mathfrak{o}(s_i) \subseteq \{0\}$ in the coframe $\mathsf{S}_S(L)$ implies that $\bigcap_{f\in F} \mathfrak{c}(x_f) \cap \mathfrak{o}(s_f) \subseteq \{1\}$ for some finite family $F \subseteq I$ whenever $x_i \in L$ and $s_i \in S$. Let us then prove the statement working within the coframe $\mathsf{S}_S(L)$. If $\bigcap_i \mathfrak{c}(x_i) \cap \bigcap_i \mathfrak{o}(s_j) \subseteq \{0\}$, then, since $\mathfrak{c} : L \to \mathsf{S}(L)$ reverses all joins, we have that $\mathfrak{c}(\bigvee_i x_i) \cap \bigcap_i \mathfrak{o}(s_i) \subseteq \{1\}$. Since the open and the closed sublocale from the same element are complements of one another, this means $\bigcap_i \mathfrak{o}(s_i) \subseteq \mathfrak{o}(\bigvee_i x_i)$. Then, by Lemma 6.1.5, since all finite meets of elements from S are compact in L, this implies that there is some finite $F_1 \subseteq I$ such that $\bigcap_{f\in F_1} \mathfrak{o}(s_f) \subseteq \mathfrak{o}(\bigvee_i x_i)$. As F_1 is finite, and since $\mathfrak{o} : L \to \mathsf{S}(L)$ preserves all finite meets, this implies that $\bigwedge_{f\in F_1} s_f \leq \bigvee_i x_i$. Since all finite

intersections of elements from S are assumed to be compact, this implies that there is a finite family $F_2 \subseteq I$ such that $\bigwedge_{f\in F_1} s_f \leq \bigvee_{f\in F_2} x_f$. This implies that $\mathfrak{o}(\bigwedge_{f\in F_1} s_f) \subseteq \mathfrak{o}(\bigvee_{f\in F_2} x_f)$ and, using again the properties of complemented elements, we see that this is equivalent to $\mathfrak{o}(\bigwedge_{f\in F_1} s_f)\cap\mathfrak{c}(\bigvee_{f\in F_2} x_f) \subseteq \{1\}$. By the order reversing property of $\mathfrak{c}$ and the order preserving one of $\mathfrak{o}$ this implies that $\bigcap_{f\in F_1\cup F_2} \mathfrak{c}(x_f) \cap \mathfrak{o}(s_f) \subseteq \{0\}$.

Conversely, suppose that the frame $\mathsf{S}_S(L)^{op}$ is compact. Suppose that we have a finite subcollection $s_1, ..., s_n \in S$, and that $s_1 \wedge ... \wedge s_n \leq \bigvee_i x_i$. This means that $\mathfrak{o}(s_1 \wedge ... \wedge s_n) \subseteq \mathfrak{o}(\bigvee_i x_i)$, that is, $\mathfrak{o}(s_1 \wedge ... \wedge s_n) \cap \bigcap_i \mathfrak{c}(x_i) \subseteq \{1\}$. Recall that $\mathsf{S}_S(L)$ is a subcoframe of $\mathsf{S}(L)$. We can then use compactness of $\mathsf{S}_S(L)^{op}$ to deduce that there is a finite subset $F \subseteq I$ such that $\mathfrak{o}(s_1 \wedge ... \wedge s_n) \cap \bigcap_{f\in F} \mathfrak{c}(x_f) = \mathfrak{o}(s_1 \wedge ... \wedge s_n) \cap \mathfrak{c}(\bigvee_{f\in F} x_f) \subseteq \{1\}$. Then, $\mathfrak{o}(s_1 \wedge ... \wedge s_n) \subseteq \mathfrak{o}(\bigvee_{f\in F} x_f)$, that is, $s_1 \wedge ... \wedge s_n \leq \bigvee_{f\in F} x_f$. □

In particular, we notice that if a frame L is not compact it cannot possibly have a subset $S \subseteq L$ such that all its finite meets are compact, since these include the empty meet, namely 1. Then, frames which are not compact have no compact partial assemblies. Let us state this as a corollary.

Corollary 6.1.7. *If a frame L is not compact, there is no $S \subseteq L$ such that the partial assembly $\mathsf{A}_S(L)$ is compact.*

We have a bitopological version of Proposition 6.1.6, too.

Proposition 6.1.8. *For any frame L and any subset $S \subseteq L$, we have that all finite meets from S are compact if and only if $(L, \mathsf{Filt}_S(L), \mathsf{Con}_S, \mathsf{Tot}_S)$ is a compact d-frame.*

Proof. Suppose that all finite meets of elements in S are compact. Suppose that we have $(\bigvee_i x_i, \bigvee_i F_i) \in \mathsf{Tot}_S$. By definition of this totality component, this means that $\bigvee_i x_i \in \bigvee_i F_i$. The frame $\mathsf{Filt}_S(L)$ is a subframe $\mathsf{Filt}(L)$, and so joins in these two frames are computed in the same way. Since each F_i is generated by a subset of S, we can deduce that there is a finite subcollection $\{s_1, ..., s_n\} \subseteq S$ with $\{s_1, ..., s_n\} \subseteq \bigcup_i F_i$, say $s_m \in F_{i_m}$, such that $s_1 \wedge ... \wedge s_n \leq \bigvee_i x_i$. Since, by assumption, this finite meet is compact, there also is some finite $F \subseteq I$ with $s_1 \wedge ... \wedge s_n \leq \bigvee_{f\in F} x_f$. This means that $\bigvee_f x_f \in \bigvee_{m\leq n} F_{i_m}$. By definition of the totality subset of our d-frame, this means $(\bigvee_{f\in F} x_f, \bigvee_{m\leq n} F_{i_m}) \in \mathsf{Tot}_S$. Then, the d-frame must be compact.

Conversely, suppose that $(L, \mathsf{Filt}_S(L), \mathsf{Con}_S, \mathsf{Tot}_S)$ is compact, and that $\{s_1, ..., s_n\} \subseteq S$. These elements are all d-complemented as elements of the first frame component L. Since d-complemented elements form a sublattice in each of the two frame components of any d-frame (see Lemma 4.2.1), the element $s_1 \wedge ... \wedge s_n$ is d-complemented. Suppose that $s_1 \wedge ... \wedge s_n \leq \bigvee_i x_i$.

Then we have $(\bigvee_i x_i, \sim(s_1 \wedge ... \wedge s_n)) \in t_L$ and, by compactness, this implies that $(\bigvee_f x_f, \sim (s_1 \wedge ... \wedge s_n)) \in t_L$ for some finite $F \subseteq I$. We also have $(s_1 \wedge ... \wedge s_n, \sim(s_1 \wedge ... \wedge s_n)) \in c_L$, and so - by balance of $\mathcal{L}$ - this implies that $s_1 \wedge ... \wedge s_n \leq \bigvee_f x_f$. □

Restricting ourselves to zero dimensional partial assemblies amounts to adding the condition that the subset $S \subseteq L$ generates the frame L, in the sense that every element of the frame L is a join of finite meets of elements from S.

Lemma 6.1.9. *If L is a frame and $S \subseteq L$ generates L, the frame $\mathsf{S}_S(L)^{op}$ is zero dimensional.*

Proof. The partial assembly $\mathsf{S}_S(L)^{op}$, by Lemma 6.1.1, is the subframe of $\mathsf{S}(L)^{op}$ generated by the collection of sublocales $\{\mathfrak{c}(x) : x \in L\} \cup \{\mathfrak{o}(s) : s \in S\}$. Since S is a generating subset of L, every closed sublocale $\mathfrak{c}(x)$ in this collection is $\mathfrak{c}(\bigvee_i s_1 \wedge ... \wedge s_{n_i})$ for each $s_{m_j} \in S$. Since $\mathfrak{c} : L \to \mathsf{S}(L)^{op}$ preserves all joins, every closed sublocale $\mathfrak{c}(x)$ is the join (in $\mathsf{S}(L)^{op}$) of closed sublocales of the form $\mathfrak{c}(s_1 \wedge ... \wedge s_n)$ with each $s_m \in S$. Then, the collection $\{\mathfrak{c}(s_1 \wedge ... \wedge s_n) : s_1, ..., s_n \in S\} \cup \{\mathfrak{o}(s) : s \in S\}$ generates all of $\mathsf{S}_S(L)^{op}$. The elements in this collection are all complemented, because elements of the form $\mathfrak{c}(s)$ and $\mathfrak{o}(s)$ for $s \in S$ are mutual complements, and each of the elements of the form $\mathfrak{c}(s_1 \wedge ... \wedge s_n)$ is a finite meet of complemented elements. Closing this collection under finite meets yields another collection of complemented elements, which is a basis for $\mathsf{S}_S(L)^{op}$. □

We are now in a position to state the following generalization of Theorem 6.1.3. The following follows directly from combining Lemma 6.1.9, Proposition 6.1.6, and Lemma 6.1.4.

Theorem 6.1.10. *If L is a frame and $S \subseteq L$ is a generating subset, the following are equivalent.*

1. *The frame $\mathsf{S}_S(L)^{op}$ is compact.*
2. *All finite meets of elements from S are compact.*
3. *The complemented elements of $\mathsf{S}_S(L)^{op}$ are exactly the compact ones.*
4. *The frame $\mathsf{S}_S(L)^{op}$ is coherent.*

Adding among our assumptions that S generates the frame L solves the issue of zero dimensionality for d-frames, too. Let us see how.

Lemma 6.1.11. *If L is a frame and $S \subseteq L$ generates L, the d-frame $(L, \mathsf{Filt}_S(L), \mathsf{Con}_S, \mathsf{Tot}_S)$ is zero dimensional.*

Proof. All elements in the first frame component of the form $s \in S$ are d-complemented. By Lemma 4.2.1, so are their finite meets. Then, by assumption on S, in the frame L all elements are joins of complemented elements. The frame $\mathsf{Filt}_S(L)$ is the subframe of $\mathsf{Filt}(L)$ generated by principal filters of the form $\uparrow s$ for $s \in S$, and so each of its elements is a join of principal filters of the form $\uparrow s_1 \cap ... \cap \uparrow s_n$, with $\{s_1, ..., s_n\} \subseteq S$. All the principal filters in this meet are d-complemented, by definition of the con and the tot component of our d-frame. Then, by Lemma 4.2.1, they must be d-complemented, too. □

Let us now prove the d-frame version of Lemma 6.1.4. To prove this, it suffices to use some of the results which we had already proven in Section 4.2, where we had analyzed in detail the coherence property for d-frames.

Lemma 6.1.12. *If $\mathcal{L}$ is a zero dimensional d-frame, the following are equivalent.*

1. *The d-frame $\mathcal{L}$ is coherent.*
2. *The d-frame $\mathcal{L}$ is compact.*
3. *The d-complemented elements of $\mathcal{L}$ coincide with the compact ones.*

Proof. Suppose that $\mathcal{L}$ is a zero dimensional d-frame. Coherent d-frames are defined to be the compact zero dimensional ones, so the equivalence between (1) and (2) is clear. Lemma 4.2.2 states that (2) implies (3), and Lemma 4.2.3 states that (3) implies (1). □

The lemma above together with Proposition 6.1.12 yields the following.

Theorem 6.1.13. *For any frame L and any generating subset $S \subseteq L$ we have that the following are equivalent.*

1. *All finite meets of elements in S are compact.*
2. *The d-frame $(L, \mathsf{Filt}_S(L), \mathsf{Con}_S, \mathsf{Tot}_S)$ is compact.*
3. *The d-frame $(L, \mathsf{Filt}_S(L), \mathsf{Con}_S, \mathsf{Tot}_S)$ is coherent.*
4. *In the d-frame $(L, \mathsf{Filt}_S(L), \mathsf{Con}_S, \mathsf{Tot}_S)$ the compact elements coincide with the d-complemented ones.*

We now wish to relate this discussion to the way filters relate to fitted sublocales. Recall that in Subsection 2.2.1 we explored a contravariant adjunction

$$\begin{aligned} &\mathit{fit} : \mathsf{Filt}(L) \leftrightarrows \mathsf{S}(L) : \mathit{top} \\ &F \mapsto \bigcap_{f \in F} \mathfrak{o}(f) \\ &\{x \in L : S \subseteq \mathfrak{o}(x)\} \hookleftarrow S \end{aligned}$$

We observed the fact that different filters may induce the same fitted sublocale, and so the correspondence between filters and fitted sublocales is not perfect. We had also asked ourselves what are those filters such that they are fixpoints of this adjunction, and reached the conclusion that these are the *strongly exact* filters. We now work towards answering a different but related question. We want to explore the property of the map *fit* being injective, as well as the weaker property of it being injective when restricted to $\mathsf{Filt}_S(L)$ for some subset $S \subseteq L$.

Proposition 6.1.14. *For a frame L and a subset $S \subseteq L$ we have that $\mathit{fit} : \mathsf{Filt}_S(L) \to \mathsf{Fitt}_S(L)$ is injective if and only if every $\mathfrak{o}(x) \in \mathsf{Fitt}_S^{op}(L)$ is a compact element.*

Proof. If *fit* is injective, then it is an isomorphism. Principal filters are compact in $\mathsf{Filt}_S(L)$ as this is a subframe of $\mathsf{Filt}(L)$, in which principal filters are all compact. Because *fit* is an isomorphism, then, the images of principal filters are compact, too. These are exactly the open sublocales in $\mathsf{Fitt}_S(L)$. For the other direction it suffices to show that if $\mathsf{Fitt}_S(L)$ is compact, then the map $\mathit{fit} : \mathsf{Filt}_S(L) \to \mathsf{Fitt}_S(L)$ is injective. First, we notice that the elements of $\mathsf{Fitt}_S(L)$ are exactly the meets of open sublocales of the form $\mathfrak{o}(s_1 \wedge ... \wedge s_n)$ for some finite family $\{s_1, ..., s_n\} \subseteq S$. Suppose that every open $\mathfrak{o}(x) \in \mathsf{Fitt}_S(L)^{op}$ is compact. This means that in the coframe $\mathsf{Fitt}_S(L)$ we have that $\bigcap_i \mathfrak{o}(x_i) \subseteq \mathfrak{o}(x)$ implies that $\bigcap_f \mathfrak{o}(x_f) \subseteq \mathfrak{o}(x)$ for some finite $F \subseteq I$. Finiteness of F means that $\bigcap_f \mathfrak{o}(x_f) \subseteq \mathfrak{o}(x)$ implies that $\bigwedge_f x_f \leq x$. To summarize, for every $\mathfrak{o}(x) \in \mathsf{Fitt}_S(L)$ we have that whenever for a collection $\{x_i : i \in I\} \subseteq S$ we have that $\bigcap_i \mathfrak{o}(x_i) \subseteq \mathfrak{o}(x)$ then we also have that x is in the filter generated by $\{x_i : i \in I\}$. Now, suppose that $F_1 \nsubseteq F_2$ for $F_1, F_2 \in \mathsf{Filt}_S(L)$. In particular, suppose that we have $s \in S$ with $s \in F_1 \backslash F_2$. We then clearly have that $\bigcap_{f \in F_1} \mathfrak{o}(f) \subseteq \mathfrak{o}(s)$. But by the argument above we cannot have $\bigcap_{f \in F_2} \mathfrak{o}(f) \subseteq \mathfrak{o}(s)$, or we would have that $s \in F_2$, contradicting our assumption. Then, we must have $\mathit{fit}(F_2) \nsubseteq \mathit{fit}(F_1)$, and the map $\mathit{fit} : \mathsf{Filt}_S(L) \to \mathsf{Fitt}_S(L)$ must be injective. □

If we recall the fact that fit frames are those for which all sublocales are fitted, it is easy to show the following.

Proposition 6.1.15. *A fit frame is such that all filters are strongly exact if and only if it is Noetherian.*

Proof. Recall that a fit frame is such that all its sublocales are fitted, and so here $\mathsf{S}(L) = \mathsf{Fitt}(L)$. If L is Noetherian, by Lemma 6.1.5 we must have that all elements of the form $\mathfrak{o}(x)$ are compact elements of $\mathsf{S}(L)^{op}$. Then, by Proposition 6.1.8, the assignment $\mathit{fit} : \mathsf{Filt}(L) \to \mathsf{S}(L)^{op}$ is injective. Conversely, if L is fit and all its filters are strongly exact, by Proposition 6.1.14 each open sublocale $\mathfrak{o}(x) \in \mathsf{S}(L)^{op}$ is a compact element. In particular its top element $\{1\}$ must be a compact element, and so $\mathsf{S}(L)^{op}$ itself must be compact. Then, L is Noetherian by Theorem 6.1.3. □

This result means that under the assumption of fitness the case where all the filters are strongly exact collapses to the case in which all filters are Scott-open.

6.2 The patch of the biframe assembly

So far, we have not focused much of our attention on biframes. However, when it comes to assemblies and their patches, it turns out that looking at the biframe case is a useful step towards understanding the situation for d-frames. Let us then begin by looking at the assembly of a biframe, introduced in [48] under the name of *congruence biframe* of a biframe. Before we do so, we need to establish a few technical results on the main component of the assembly of a biframe $\mathcal{L}$. Recall that in a frame coproduct $L^+ \oplus L^-$ the finitary elements are defined to be those that are finite joins of elements of the form $\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner$. We may extend this definition to the case of any frame L generated by two subframes. For a biframe (L^+, L^-, L), we define an element of L to be *finitary* if it is a finite join of elements of the form $x^+ \wedge x^-$. We call $\mathsf{fin}(L)$ the ordered collection of finitary elements of L. For a biframe $\mathcal{L}$, we may consider the partial assembly $\mathsf{S}_{\mathsf{fin}}(L)^{op}$, with the embedding $\mathfrak{c} : L \to \mathsf{S}_{\mathsf{fin}}(L)^{op}$ universal such that it provides complements freely to the finitary elements of L. We refer to $\mathsf{S}_{\mathsf{fin}}(L)^{op}$ simply as the *finitary* assembly of L. For a biframe $\mathcal{L}$, we denote as $i : L^+ \hookrightarrow L$ the set inclusion map. We know, by functoriality of the assembly construction, that there is a map $\mathsf{S}(i)^{op} : \mathsf{S}(L^+)^{op} \to \mathsf{S}(L)^{op}$ acting as $\mathfrak{c}_{L^+}(x^+) \mapsto \mathfrak{c}_L(x^+)$ for every $x^+ \in L^+$. Since the open and closed sublocales from the same element are complements of each other in the assembly, and since every frame map respects complements, we must also have that $\mathsf{S}(i)^{op}(\mathfrak{o}_{L^+}(x^+)) = \mathfrak{o}_L(x^+)$ for every $x^+ \in L^+$.

Since for a biframe $\mathcal{L}$ we will be interested in mapping $\mathsf{S}(L^+)^{op}$ and $\mathsf{S}(L^-)^{op}$ into the *finitary* assembly of L, for simplicity we will slightly abuse notation and simply call $\mathsf{S}(i^+)^{op}$ the map $\mathsf{S}(L^+) \to \mathsf{S}_{\mathsf{fin}}(L)^{op}$ mapping each closed sublocale $\mathfrak{c}_{L^+}(a^+)$ to $\mathfrak{c}_L(a^+)$. The next results

will show that the finitary assembly is bitopological in nature in more than one natural way. The first bitopological description of the finitary assembly of the main component of a biframe is familiar. We know, after our work in Chapter 4, that for a biframe $\mathcal{L}$ there is a d-frame

$$(L, \mathsf{Filt}_{\mathsf{fin}}(L), \mathsf{Con}_{\mathsf{fin}}, \mathsf{Tot}_{\mathsf{fin}})$$

presenting the finitary assembly $\mathsf{A}_{\mathsf{fin}}(L)$. There is another way in which we may see the structure $\mathsf{A}_{\mathsf{fin}}(L)$ as a bitopological structure. Before we do so, we need a couple of lemmas on biframes.

Lemma 6.2.1. *For a biframe $\mathcal{L}$, we have that any frame map $f : L \to M$ such that it provides bicomplements to all elements in $L^+ \cup L^-$ also provides bicomplements to all elements of $\mathsf{fin}(L)$.*

Proof. We know that the ordered collection of complemented elements of a frame is a sublattice. Suppose that we have a biframe $\mathcal{L}$ and that the frame map $f : L \to M$ provides complements to all elements in $L^+ \cup L^-$. Since f preserves all the lattice operations, we have that $f(x^+ \wedge x^-) = f(x^+) \wedge f(x^-)$ is complemented in M, since it is the finite meet of complemented elements, and similarly $f(\bigvee_{m \leq n} x_m^+ \wedge x_m^-)$ is complemented in M too, since it is the finite join of complemented elements. □

Lemma 6.2.2. *If $\mathcal{L}$ is a biframe and $f^+ : L^+ \to M$ and $f^- : L^- \to M$ are frame maps such that*

$$\bigvee_i f^+(x_i^+) \wedge f^-(x_i^-) \leq \bigvee_j f^+(x_j^+) \wedge f^-(x_j^-)$$

whenever $\bigvee_i x_i^+ \wedge x_i^- \leq \bigvee_j x_j^+ \wedge x_j^-$ in L, there is a unique frame map $f : L \to M$ such that it restricts to L^+ as f^+ and to L^- as f^-.

Proof. Suppose that there is a biframe $\mathcal{L}$ and two frame maps $f^+ : L^+ \to M$ and $f^- : L^- \to M$ satisfying the hypothesis in our claim. We define a map $f : L \to M$ as $f_L(\bigvee_i x_i^+ \wedge x_i^-) = \bigvee_i f^+(x_i^+) \wedge f^-(x_i^-)$. We need to show that this is a well defined map, in the sense that $x \leq y$ in L implies that $f(x) \leq f(y)$. Well definedness of this map is then equivalent to our assumption on f^+ and f^-. □

We are ready to prove our second bitopological description of the finitary assembly. In the following proof, when $\mathcal{L}$ is a biframe, we will abbreviate a canonical element $\bigvee_i x_i^+ \wedge x_i^-$ of its main component as x_i. As usual, since we will have closed sublocales from different frames, for any frame L we will denote as $\mathfrak{c}_L(a)$ the closed sublocale of L relative to the element $a \in L$.

Proposition 6.2.3. *For a biframe $\mathcal{L}$, we have an isomorphism of frames*

$$\mathsf{S}_{\mathsf{fin}}(L)^{op} \cong (\mathsf{S}(L^+)^{op} \oplus \mathsf{S}(L^+)^{op})/{\ulcorner R_L \urcorner},$$

where the relation $\ulcorner R_L \urcorner$ is defined as

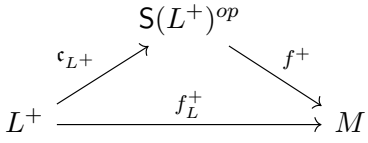
$$\{(\bigvee_i \ulcorner \mathfrak{c}(x_i^+)\urcorner \wedge \ulcorner \mathfrak{c}(x_i^-)\urcorner, \bigvee \ulcorner \mathfrak{c}(x_j^+)\urcorner \wedge \ulcorner \mathfrak{c}(x_j^-)\urcorner) : \bigvee_i x_i^+ \wedge x_i^- \leq \bigvee_j x_j^+ \wedge x_j^- \text{ in } L\}.$$

The canonical mappings of generators are the map $\mathsf{S}(i^+)^{op} : \mathsf{S}(L^+)^{op} \to \mathsf{S}_{\mathsf{fin}}(L)^{op}$ and the analogous one on the negative component.

Proof. We will show that the finitary assembly of L has the same universal property as the desired quotient of the coproduct, and in particular we show that the canonical mappings of generators are the map $\mathsf{S}(i^+)^{op} : \mathsf{S}(L^+)^{op} \to \mathsf{S}_{\mathsf{fin}}(L)^{op}$ mapping each closed sublocale $\mathfrak{c}_{L^+}(a^+)$ to the corresponding closed sublocale $\mathfrak{c}_L(a^+)$, together with the analogous map on the negative component. To show the desired claim, we need to show that whenever we have two frame maps $f^+ : \mathsf{S}(L^+)^{op} \to M$ and $f^- : \mathsf{S}(L^-)^{op} \to M$ such that their pairing $f^+ \oplus f^- : \mathsf{S}(L^+)^{op} \oplus \mathsf{S}(L^+)^{op} \to M$ respects the relation $\ulcorner R_L \urcorner$, there is a unique frame map $\tilde{f} : \mathsf{S}_{\mathsf{fin}}(L)^{op} \to M$ such that the obvious triangle commutes for both assemblies, that is, such that $\tilde{f}(\mathfrak{c}_L(a^+)) = f^+(\mathfrak{c}_{L^+}(a^+))$ for every $a^+ \in L^+$, and similarly for the elements in the frame L^-. The condition on the pairing of $f^+ \oplus f^-$ is equivalent, by definition of this pairing, to the maps f^+ and f^- being such that

$$\bigvee_i f^+(\mathfrak{c}_{L^+}(x_i^+)) \wedge f^-(\mathfrak{c}({}_{L^+}x_i^-)) \leq \bigvee_j f^+(\mathfrak{c}_{L^+}(x_j^+)) \wedge f^-(\mathfrak{c}_{L^+}(x_j^-)) \qquad (*)$$

whenever the inequality $x_i \leq x_j$ holds in L. Suppose, then, that we have two maps f^+ and f^- satisfying (*). By the universal property of the assembly of a frame, there must be a unique frame map $f_L^+ : L^+ \to M$ such that the following commutes.



Furthermore, this map must provide complements to all elements of L^+. We also have an analogous frame map $f_L^- : L^- \to M$. The maps f_L^+ and f_L^-, then, by assumption (*), we have

$$\bigvee_i f_L^+(x_i^+) \wedge f_L^-(x_i^-) \leq \bigvee_j f_L^+(x_j^+) \wedge f_L^-(x_j^-)$$

whenever $x_i \leq x_j$ in L. By Lemma 6.2.2, then, there is a unique frame map $f_L : L \to M$ which restricts as f^+ on L^+ and as f^- on L^-. This map, then, provides complements for all elements of L^+ and L^- and by Lemma 6.2.1 this implies the it provides complements to all elements of $\mathsf{fin}(L)$. Then, by the universal property of the finitary assembly, there must be a unique map $\tilde{f} : \mathsf{S}_{\mathsf{fin}}(L)^{op} \to M$ such that $\tilde{f} \circ \mathfrak{c}_L = f_L$. In particular, for each $a^+ \in L^+$, we have that $\tilde{f}(\mathfrak{c}_L(a^+)) = f_L(a^+) = f_L^+(a^+) = f^+(\mathfrak{c}_{L^+}(a^+))$. Then, the map $\tilde{f}$ is as desired and, furthermore it must be unique because any map $\mathsf{S}_{\mathsf{fin}}(L)^{op} \to M$ is completely determined by its action on closed sublocales. □

Forgetting for a moment about universal properties, and working concretely, we have several ways of describing the finitary assembly of a frame as being generated by certain collections. And, in fact, these descriptions enable us to see multiple bitopological descriptions of the finitary assembly without appealing to abstract reasoning. The fact that we can describe it in so many ways will be the starting point of subsection 6.2.1, in which we will see that there are three biframes which present the finitary assembly of the main component of some biframe. In the proof below, in light of Proposition 6.2.3, we refer to $\mathsf{S}(i^+)^{op}[\mathsf{S}(L^+)^{op}] \subseteq \mathsf{S}_{\mathsf{fin}}(L)^{op}$ simply as $\ulcorner \mathsf{S}(L^+)^{op} \urcorner$, and we use a similar notation for $\mathsf{S}(i^-)^{op}[\mathsf{S}(L^-)^{op}]$. Before we prove the desired result, we need a simple observation on generating subframes.

Observation 6.2.4. *Suppose that L is a frame. The map $frm : \mathcal{P}(L) \to \mathcal{P}(L)$, assigning to each subset of L the subframe generated by it, is a closure operator.*

Lemma 6.2.5. *For any biframe $\mathcal{L}$, the following are all descriptions of the finitary assembly of a frame L.*

1. *The subframe of $\mathsf{S}(L)^{op}$ generated by all closed sublocales of L and the open ones of the form $\mathfrak{o}(x^+ \vee x^-)$.*

2. *The subframe of $\mathsf{S}(L)^{op}$ generated by the collection*

$$\{\mathfrak{c}(x^+) : x^+ \in L^+\} \cup \{\mathfrak{o}(x^-) : x^- \in L^-\} \cup \{\mathfrak{c}(x^-) : x^- \in L^-\} \cup \{\mathfrak{o}(x^+) : x^+ \in L^+\}.$$

3. *The subframe of $\mathsf{S}(L)^{op}$ generated by $\ulcorner \mathsf{S}(L^+)^{op} \urcorner \cup \ulcorner \mathsf{S}(L^-)^{op} \urcorner$.*

4. *The ordered collection of sublocales of L of the form $\bigcap_i \mathfrak{c}(x_1^+ \wedge x_i^-) \vee \mathfrak{o}(y_1^+ \vee y_i^-)$.*

Proof. Let us prove each item separately. For a collection $C \subseteq L$, let us denote as $frm(C)$ the subframe of $\mathsf{S}(L)^{op}$ generated by C.

1. By Lemma 6.1.1, the finitary assembly of L is the subframe of $\mathsf{S}(L)^{op}$ generated by the closed sublocales of L together with the finitary open ones. The finitary open elements are those of the form $\mathfrak{o}(\bigwedge_{n\leq m} x_n^+ \vee x_n^-) = \bigvee_{n\leq m} \mathfrak{o}(x_n^+ \vee x_n^-)$ for some natural number m. Then, indeed the collection in item (1) generates the finitary assembly.

2. We show that the collection of item (2) generates the collection of item (1). Since $\mathfrak{c}$ preserves the frame operations (when seen as a map to $\mathsf{S}(L)^{op}$), the collection of closed sublocales is generated by that of closed sublocales of the form $\mathfrak{c}(x^+)$ and $\mathfrak{c}(x^-)$. The collection of open sublocales of the form $\mathfrak{o}(x^+)$ and $\mathfrak{o}(x^-)$ generates those of the form $\mathfrak{o}(x^+ \vee x^-)$ when closed under finite meets. So, we are done.

3. Since the map $\mathsf{S}(i^+)^{op} : \mathsf{S}(L^+)^{op} \to \mathsf{S}_{\mathsf{fin}}(L)^{op}$ is a frame map, and since we know that open and closed sublocales generate any assembly as a frame, we have that

$$\ulcorner \mathsf{S}(L^+)^{op} \urcorner = frm(\{\mathfrak{o}(x^+) : x^+ \in L^+\} \cup \{\mathfrak{c}(x^+) : x^+ \in L^+\}) = frm(S^+).$$

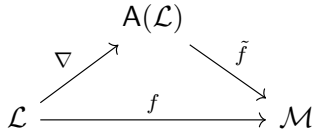
   We have a similar result for the assembly of the negative frame L^-. Substituting this into the equality we wish to claim, we obtain that this is equivalent to

$$\mathsf{S}_{\mathsf{fin}}(L)^{op} = frm(frm(S^+) \cup frm(S^-)).$$

   We use the fact that *frm* is a closure operator, which is Observation 6.2.4, to deduce that this equality amounts to $\mathsf{S}_{\mathsf{fin}}(L)^{op} = frm(S^+ \cup S^-)$. This is item (2).

4. Closing the collection in item (2) under finite meets yields the collection of sublocales of the form $\mathfrak{c}(x^+ \wedge x^-) \vee \mathfrak{o}(y^+ \vee y^-)$. Closing this under arbitrary joins yields the collection in item (4). □

In [48] it is shown (in Subsection 1.4.1) that, for every biframe $\mathcal{L}$, there exists a biframe $\mathsf{A}(\mathcal{L})$ with a biframe embedding $\nabla : \mathcal{L} \to \mathsf{A}(\mathcal{L})$ such that, whenever there is a biframe map $f : \mathcal{L} \to \mathcal{M}$ such that $f^+(x^+)$ is bicomplemented for every $x^+ \in L^+$, and similarly for each $x^- \in L^-$, there is a unique biframe map $\tilde{f} : \mathcal{L} \to \mathcal{M}$ making the following triangle commute.



In [48] this construction is called the *congruence biframe* of the biframe $\mathcal{L}$, but here we will refer to it as the assembly of a biframe. For a biframe $\mathcal{L}$, define as $\mathsf{A}_{\nabla^+\Delta^-}(L)$ the subframe

of $\mathsf{A}(L)$ generated by $\{\nabla(x^+) : x^+ \in L^+\} \cup \{\Delta(x^-) : x^- \in L^-\}$, and define as $\mathsf{A}_{\nabla^-\Delta^+}(L)$ the subframe of $\mathsf{A}(L)$ generated by $\{\nabla(x^-) : x^- \in L^-\} \cup \{\Delta(x^+) : x^+ \in L^+\}$. The assembly of a biframe $\mathcal{L}$ is the triple

$$(\mathsf{A}_{\nabla^+\Delta^-}(L), \mathsf{A}_{\nabla^-\Delta^+}(L), \mathsf{A}_{\mathsf{fin}}(L)).$$

In [48] this definition is not given in terms of the finitary assembly, instead, the third component was defined to be the subframe of $\mathsf{A}(L)$ generated by $\mathsf{A}_{\nabla^+\Delta^-}(L)$ and $\mathsf{A}_{\nabla^-\Delta^+}(L)$. So, we now have the task to prove that our definition coincides with that in [48].

Lemma 6.2.6. *For any biframe $\mathcal{L}$, the main component of $\mathsf{A}(\mathcal{L})$ is isomorphic to $\mathsf{S}_{\mathsf{fin}}(L)^{op}$.*

Proof. In terms of sublocales, the third component of the assembly of a biframe is the subframe of $\mathsf{S}(L)^{op}$ generated by the collection $\{\mathfrak{c}(x) : x \in L^+ \cup L^-\} \cup \{\mathfrak{o}(x) : x \in L^+ \cup L^-\}$. Then, by item (2) of Lemma 6.2.5, it is the finitary assembly of L. □

We now would like to prove the biframe version of the results on Noetherian frames that we have studied in Subsection 6.1.1. Since compactness of a biframe amounts to compactness of the main component, compactness of the assembly of a biframe $\mathcal{L}$ amounts to the compactness of the *finitary* assembly of the main component of $\mathcal{L}$. Compactness of $\mathsf{S}_{\mathsf{fin}}(L)^{op}$ is equivalent, by Theorem 6.1.10, to compactness of all elements of $\mathsf{fin}(L)$. This is why we define a biframe $\mathcal{L}$ to be *Noetherian* if every element of L of the form $x^+ \wedge x^-$ is compact in L. This condition is equivalent to all finitary elements of L being compact. With these definitions we obtain a biframe version of Theorem 6.1.3. Before we prove this theorem, we will prove some results on biframe coherence. Recall that a biframe is *coherent* if it is compact and zero dimensional. In particular, we will show the biframe versions of Lemmas 6.1.4 and 6.1.12 on zero dimensionality.

Lemma 6.2.7. *For a biframe $\mathcal{L}$, the bicomplemented elements of each frame L^+ and L^- form a sublattice.*

Proof. It is clear that if an element of a biframe $\mathcal{L}$ is bicomplemented then it is a complemented element of L. Complemented elements of a frame form a sublattice, and in particular complementation reverses the lattice operations for complemented elements. Thus, if $a_1^+, ..., a_n^+ \in L^+$ is a finite family of bicomplemented elements, the complement of $a_1^+ \wedge ... a_n^+$ in L is $\sim a^+ \vee ... \vee \sim a_n^+$. Since this is an element of L^-, our finite meet is bicomplemented. By an analogous argument we show that the collection of bicomplemented elements of L^+ is closed under finite joins. □

Theorem 6.2.8. *The following are equivalent for a biframe $\mathcal{L}$.*

1. *The biframe* $\mathsf{A}(\mathcal{L})$ *is compact.*

2. *The biframe* $\mathcal{L}$ *is Noetherian.*

3. *The biframe* $\mathsf{A}(\mathcal{L})$ *is coherent.*

4. *The main component of* $\mathcal{L}$ *is isomorphic to* $\mathsf{Idl}(\mathsf{fin}(L))$.

Proof. Since the main component of $\mathsf{A}(\mathcal{L})$ is the finitary assembly $\mathsf{A}_{\mathsf{fin}}(L)$ of L, its compactness is equivalent, by Theorem 6.1.10, to every finitary element of L being compact. We then have the equivalence of (1) and (2). The equivalence of (1) and (3) follows from the definition of coherence. Finally, if we have (4) all finitary elements of L must be compact, and so we have item (2). For the converse, suppose that $\mathcal{L}$ is Noetherian. Then, the main component of $\mathcal{L}$ has a basis which consists of a lattice of compact elements, namely $\mathsf{fin}(L)$. It must then be isomorphic to its ideal completion. □

6.2.1 Three presentations of the patch of the biframe assembly

Finally, we want to use Lemma 6.2.5 to show that there are three natural ways of seeing the finitary assembly of a frame as a biframe. This is a step towards showing what we have called the "main message" of the thesis, which argues that, pointfreely, there are several ways of seeing the collection of all bisubspaces of a bispace as a bispace, and two of them are particularly natural. We argue that the following two bitopologies on the finitary assembly are somewhat more natural than the one given by $\mathsf{A}(\mathcal{L})$. For any biframe $\mathcal{L}$, we now introduce the two following constructions. For a biframe $\mathcal{L}$, let us denote as $\mathsf{Fitt}_{\mathsf{fin}}(L)$ the frame fitted congruences of L coming from finitary elements, i.e. the fitted congruences of the form $\bigvee_{f\in F}\Delta(f)$, with $F \subseteq \mathsf{fin}(L)$.

- Define as $\mathsf{A}_{cf}(\mathcal{L})$ (for "closed-fitted") the biframe $(\mathsf{Cl}(L), \mathsf{Fitt}_{\mathsf{fin}}(L), \mathsf{A}_{\mathsf{fin}}(L))$. We know that this is a biframe by Lemma 6.1.1.

- Define as $\mathsf{A}_{\pm}(\mathcal{L})$ the biframe $(\mathsf{A}^+(L), \mathsf{A}^-(L), \mathsf{A}_{\mathsf{fin}}(L))$. Here $\mathsf{A}^+(L)$ is the image of $\mathsf{A}(L^-)$ under the map $\mathsf{A}(i^+) : \mathsf{A}(L^+) \to \mathsf{A}(L)$. The frame $\mathsf{A}^-(L)$ is defined similarly. The triple that we have defined is a biframe, by item (3) of Lemma 6.2.5.

We argue that these two bitopologies on $\mathsf{A}_{\mathsf{fin}}(L)$ are more natural because of the following: (a) the division into patch-closed and patch-fitted congruences (sublocales) re-traces the classical bitopology on the assembly $\mathsf{A}(L)$ of a frame; (b) the division into positive and negative congruences (sublocales) re-traces the division of L into a bitopology given to us by $\mathcal{L}$. In the next section, we will construct analogues of $\mathsf{A}_{cf}(\mathcal{L})$ and $\mathsf{A}_{\pm}(\mathcal{L})$ for d-frames.

6.3 The patch of the d-frame assembly

In the previous section we have seen that taking the main component of a biframe assembly yields the finitary assembly of its main component. The same holds for d-frames, and in this case the notion of "main component" is replaced by that of "patch". Firstly, let us remind the reader of how the d-frame assembly is constructed. The assembly $\mathsf{A}(\mathcal{L})$ of a d-frame $\mathcal{L}$ is the d-frame

$$(L^+ \oplus \mathsf{Filt}(L^-), L^- \oplus \mathsf{Filt}(L^+), c_m, t_m)$$

quotiented by the relation structure

$$(\mathsf{id}^+, \mathsf{id}^-, \mathcal{C} \cup \ulcorner c_L \urcorner, \mathcal{C} \cup \ulcorner t_L \urcorner)$$

Here we have

$$\begin{aligned}
c_L &= \{(\ulcorner x^+ \urcorner, \ulcorner x^- \urcorner) : (x^+, x^-) \in c_L\} \\
t_L &= \{(\ulcorner x^+ \urcorner, \ulcorner x^- \urcorner) : (x^+, x^-) \in t_L\} \\
\mathcal{C} &= \{(\ulcorner x^+ \urcorner, \ulcorner \uparrow x^+ \urcorner) : x^+ \in L^+\} \cup \{(\ulcorner x^- \urcorner, \ulcorner \uparrow x^- \urcorner) : x^- \in L^-\}.
\end{aligned}$$

We also remind the reader of Lemma 1.3.11, which has the key function of turning the patch of a d-frame quotient $\mathcal{L}/\mathcal{R}$ into a frame quotient of the patch $\Pi(\mathcal{L})$. Recall that this lemma essentially is telling us that, under the patch construction, d-frame quotients turn into frame quotients exactly as we would expect. Spelled out precisely, this means the following. For any d-frame $\mathcal{L}$ and any relation structure $\mathcal{R}$ on it, we have that $\Pi(\mathcal{L}/\mathcal{R})$ is the patch $\Pi(\mathcal{L})$ quotiented by the relation

$$\begin{aligned}
&\{(\ulcorner x^+ \urcorner, \ulcorner y^+ \urcorner) : (x^+, y^+) \in R^+\} \cup \{(\ulcorner x^- \urcorner, \ulcorner y^- \urcorner) : (x^-, y^- \in R^-)\} \cup \\
&\cup \{(\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner, 0) : (x^+, x^-) \in c_R\} \cup \{(1, \ulcorner x^+ \urcorner \vee \ulcorner x^- \urcorner) : (x^+, x^-) \in t_R\}.
\end{aligned}$$

Let us see how we can combine this lemma with the description of the d-frame assembly that we started this section with. We directly obtain the following useful corollary.

Corollary 6.3.1. *For a d-frame $\mathcal{L}$, the patch $\Pi(\mathsf{A}(\mathcal{L}))$ of its assembly is the coproduct $L^+ \oplus \mathsf{Filt}(L^-) \oplus L^- \oplus \mathsf{Filt}(L^+)$ quotiented by the relation*

$$\begin{aligned}
&\{(\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner, 0) : (x^+, x^-) \in c_L\} \cup \{(1, \ulcorner x^+ \urcorner \vee \ulcorner x^- \urcorner) : (x^+, x^-) \in t_L\} \cup \\
&\cup \{(\ulcorner x^+ \urcorner \wedge \ulcorner \uparrow x^+ \urcorner, 0), (1, \ulcorner x^+ \urcorner \vee \ulcorner x^+ \urcorner) : x^+ \in L^+\} \cup \\
&\cup \{(\ulcorner x^- \urcorner \wedge \ulcorner x^- \urcorner, 0), (1, \ulcorner x^- \urcorner \vee \ulcorner x^- \urcorner) : x^- \in L^-\}.
\end{aligned}$$

Now that we have this presentation of the patch of the d-frame assembly, we are ready to show that it is the finitary assembly. Before we do so, let us prove a lemma to make the task easier.

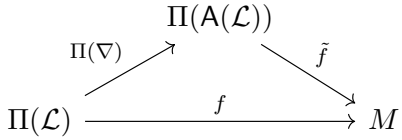
Lemma 6.3.2. *If $f : L \to M$ is a frame map providing complements for all elements of L, the map $\mathsf{Filt}_\neg(f) : \mathsf{Filt}(L) \to M$ defined on generators as $\uparrow x \mapsto \neg f(x)$ is well defined.*

Proof. First, we claim that the map $\neg f : L^{op} \to M$ is is a lattice map. This is because complemented elements of a frame form a sublattice, and in particular negation when restricted to complemented elements turns finite meets into finite joins and vice-versa. Since $\neg f : L^{op} \to M$ is a lattice map, there is a unique frame map $\mathsf{Filt}_\neg(f) : \mathsf{Filt}(L) \to M$ such that it acts as $\uparrow x \mapsto \neg f(x)$ on generators. □

We are finally ready to show the desired result.

Proposition 6.3.3. *For any d-frame $\mathcal{L}$, we have $\Pi\mathsf{A}(\mathcal{L}) \cong \mathsf{A}_{\mathsf{fin}}(\Pi\mathcal{L})$.*

Proof. We will show that the patch of the d-frame assembly $\mathsf{A}(\mathcal{L})$ satisfies the universal property of the finitary assembly of $\Pi(L)$. There is a d-frame map $\nabla : \mathcal{L} \to \mathsf{A}(\mathcal{L})$ and, by functoriality of the patch construction, this means that there is a frame map $\Pi(\nabla) : \Pi(\mathcal{L}) \to \Pi(\mathsf{A}(\mathcal{L}))$. By definition of this functor (see subsection 1.2.2) the map $\Pi(\nabla)$ acts on positive generators as $\ulcorner a^+ \urcorner \mapsto \ulcorner \nabla(a^+) \urcorner$, and similarly on negative generators. We claim that the map $\Pi(\nabla)$ is universal with the property that it provides complements to all finitary elements of $\Pi(\mathcal{L})$. For this, by Lemma 6.2.1, it suffices to show that it is universal with the property that it provides complements to elements in $\ulcorner L^+ \urcorner \cup \ulcorner L^- \urcorner$. Suppose, then, that there is a frame map $f : \Pi(\mathcal{L})$ such that $f(\ulcorner x^+ \urcorner)$ and $f(\ulcorner x^- \urcorner)$ are complemented for every $x^+ \in L^+$ and every $x^- \in L^-$. We show that there is a unique map $\tilde{f} : \Pi(\mathsf{A}(\mathcal{L})) \to M$ making the following triangle commute.



By Corollary 6.3.1, for this it suffices to define four frame maps

$$\tilde{f}^+_L : L^+ \to M$$
$$\tilde{f}^+_F : \mathsf{Filt}(L^+) \to M$$
$$\tilde{f}^-_L : L^- \to M$$
$$\tilde{f}^-_F : \mathsf{Filt}(L^-) \to M$$

and to show that they respect the relations described in Corollary 6.3.1. The elements of L^+ and L^- appear as syntactic expressions both in $\Pi(\mathcal{L})$ and in $\Pi(\mathsf{A}(\mathcal{L}))$. To distinguish them, we use the double syntactic brackets $\ulcorner\ulcorner - \urcorner\urcorner$ to denote the second ones. Define $\tilde{f}^+_L(\ulcorner\ulcorner x^+ \urcorner\urcorner) = f(\ulcorner x^+ \urcorner)$ and $\tilde{f}^+_F(\ulcorner\uparrow x^+ \urcorner) = \neg f(\ulcorner x^+ \urcorner)$, and define similarly the maps for the negative elements. By Lemma 6.3.2 the frame maps from the filter frames are well defined. Preservation of the relation structure amounts to these equalities, for each $x^+ \in L^+$

$$\begin{cases} \tilde{f}(\ulcorner\ulcorner x^+ \urcorner\urcorner) \wedge \tilde{f}(\ulcorner\uparrow x^+ \urcorner) = 0, \\ \tilde{f}(\ulcorner\ulcorner x^+ \urcorner\urcorner) \vee \tilde{f}(\ulcorner\uparrow x^+ \urcorner) = 1, \\ \tilde{f}(\ulcorner\ulcorner x^+ \urcorner\urcorner) \wedge \tilde{f}(\ulcorner x^- \urcorner) = 0 \text{ whenever } (x^+, x^-) \in c_L, \\ \tilde{f}(\ulcorner\ulcorner x^+ \urcorner\urcorner) \vee \tilde{f}(\ulcorner x^- \urcorner) = 1 \text{ whenever } (x^+, x^-) \in t_L. \end{cases}$$

as well as the analogous equalities for elements of L^-. Substituting the definition of $\tilde{f}$ into the equalities above, we obtain that these four conditions hold by assumption on f. □

6.3.1 The closed-fitted d-frame assembly $\mathsf{A}_{cf}(\mathcal{L})$

Now that we know what the patch of a d-frame assembly is, we would like to do as we did in Subsection 6.2.1 and find two other d-frames with the same patch. We start from the d-frame analogue of the "closed-fitted" biframe $\mathsf{A}_{cf}(\mathcal{L})$. With the frame assembly, we have defined the structure $\mathsf{A}(L)$ as the concrete ordered collection of all frame congruences on L, and - at a later stage - we proved that this is isomorphic to a certain free construction of the form $(L \oplus \mathsf{Filt}(L))/C$. For the d-frame $\mathsf{A}_{cf}(\mathcal{L})$, we will do the opposite. We will define $\mathsf{A}_{cf}(\mathcal{L})$ as a free construction, and later we will find a more concrete d-frame isomorphic to it. Let us recall that, at the end of Chapter 2, we defined for every pair of frames a d-frame of the form

$$(L^+ \oplus L^-, \mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-), \mathsf{Con}, \mathsf{Tot})$$

where Con and Tot are the smallest consistency and totality components making the pairs $(\ulcorner x^+ \urcorner, \ulcorner\uparrow x^+ \urcorner)$ and $(\ulcorner x^- \urcorner, \ulcorner\uparrow x^- \urcorner)$ into pairs of mutual d-complements. In other words, this d-frame is the d-frame that we obtain when we quotient

$$(L^+ \oplus L^-, \mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-), c_m, t_m)$$

by the relation structure $(\mathsf{id}^+, \mathsf{id}^-, \mathcal{C}, \mathcal{C})$, where

$$\mathcal{C} = \{(\ulcorner x^+ \urcorner, \ulcorner\uparrow x^+ \urcorner) : x^+ \in L^+\} \cup \{(\ulcorner x^- \urcorner, \ulcorner\uparrow x^- \urcorner) : x^- \in L^-\}.$$

For an arbitrary d-frame $\mathcal{L}$, we want to start from this structure, and somehow quotient it to account for the con and the tot component of $\mathcal{L}$. For an arbitrary d-frame $\mathcal{L}$, let us denote as $\mathsf{A}_{cf}(\mathcal{L})$ the d-frame $(L^+ \oplus L^-, \mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-), \mathsf{Con}, \mathsf{Tot})$ quotiented by the relation structure $(\nabla(\ulcorner c_L \urcorner), \nabla(\ulcorner t_L \urcorner), c_m, t_m)$, where we have used the definitions

$$\nabla(\ulcorner c_L \urcorner) = \nabla(\bigvee\{\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner : (x^+, x^-) \in c_L\}),$$
$$\nabla(\ulcorner t_L \urcorner) = \nabla(\bigvee\{\ulcorner \uparrow x^+ \urcorner \wedge \ulcorner \uparrow x^- \urcorner : (x^+, x^-) \in t_L\}).$$

This rather abstract construction is easily shown to have the same patch as $\mathsf{A}(\mathcal{L})$ for any d-frame $\mathcal{L}$.

Lemma 6.3.4. *For any d-frame $\mathcal{L}$, the patch $\Pi(\mathsf{A}_{cf}(\mathcal{L}))$ is the coproduct $L^+ \oplus \mathsf{Filt}(L^-) \oplus L^- \oplus \mathsf{Filt}(L^+)$ quotiented by the relation*

$$\{(\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner, 0) : (x^+, x^-) \in c_L\} \cup \{(1, \ulcorner x^+ \urcorner \vee \ulcorner x^- \urcorner) : (x^+, x^-) \in t_L\} \cup$$
$$\cup \{(\ulcorner x^+ \urcorner \wedge \ulcorner \uparrow x^+ \urcorner, 0), (1, \ulcorner x^+ \urcorner \vee \ulcorner \uparrow x^+ \urcorner) : x^+ \in L^+\} \cup$$
$$\cup \{(\ulcorner x^- \urcorner \wedge \ulcorner \uparrow x^- \urcorner, 0), (1, \ulcorner x^- \urcorner \vee \ulcorner \uparrow x^- \urcorner) : x^- \in L^-\}.$$

It follows that this patch is isomorphic to $\Pi(\mathsf{A}(\mathcal{L}))$.

Proof. By Lemma 1.3.11, the patch of $\mathsf{A}_{cf}(\mathcal{L})$ is the coproduct $L^+ \oplus \mathsf{Filt}(L^-) \oplus L^- \oplus \mathsf{Filt}(L^+)$ quotiented by

$$\{(\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner, 0) : (x^+, x^-) \in c_L\} \cup \{(\ulcorner \uparrow x^+ \urcorner \wedge \ulcorner \uparrow x^- \urcorner, 0) : (x^+, x^-) \in t_L\} \cup$$
$$\cup \{(\ulcorner x^+ \urcorner \wedge \ulcorner \uparrow x^+ \urcorner, 0), (1, \ulcorner x^+ \urcorner \vee \ulcorner \uparrow x^+ \urcorner) : x^+ \in L^+\} \cup$$
$$\cup \{(\ulcorner x^- \urcorner \wedge \ulcorner \uparrow x^- \urcorner, 0), (1, \ulcorner x^- \urcorner \vee \ulcorner \uparrow x^- \urcorner) : x^- \in L^-\}.$$

Then, to show our claim it suffices to show this quotient is unaltered if we replace the relation $\{(\ulcorner \uparrow x^+ \urcorner \wedge \ulcorner \uparrow x^- \urcorner, 0) : (x^+, x^-) \in t_L\}$ with the relation $\{(\ulcorner x^+ \urcorner \vee \ulcorner x^- \urcorner, 1) : (x^+, x^-) \in t_L\}$. But this holds because our quotients forces elements of the form $\ulcorner x^+ \urcorner$ to be d-complements of those of the form $\ulcorner \uparrow x^+ \urcorner$, and the same holds for the negative frame. The second part follows from Corollary 6.3.1. □

We now have structure $\mathsf{A}_{cf}(\mathcal{L})$, whose patch is isomorphic to the patch of the d-frame assembly of $\mathcal{L}$. The final step of this subsection will be to find an isomorphic but more concrete counterpart, corresponding to $\mathsf{A}_{cf}(\mathcal{L})$ roughly in the same way in which the concrete

assembly $\mathsf{A}(L)$ of a frame corresponds to the free construction $(L \oplus \mathsf{Filt}(L))/C$. Recall that in Chapter 2 we established two isomorphisms

$$\mathsf{con}(L^+ \times L^-) \cong L^+ \oplus L^- \qquad \mathsf{tot}(L^+ \times L^-) \cong \mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-)$$
$$c \mapsto \bigvee\{\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner : (x^+, x^-) \in c_L\} \qquad t \mapsto \bigvee\{\ulcorner \uparrow x^+ \urcorner \wedge \ulcorner \uparrow x^- \urcorner : (x^+, x^-) \in t_L\}$$

Given that these isomorphisms exist, we know that for any d-frame $\mathcal{L}$ the d-frame $(L^+ \oplus L^-, \mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-), \mathsf{Con}, \mathsf{Tot})$ is isomorphic to a d-frame of the form

$$(\mathsf{con}(L^+ \times L^-), \mathsf{tot}(L^+ \times L^-), \mathsf{Con}', \mathsf{Tot}').$$

We may obtain a concrete description of the con and the tot components of this d-frame. We will not need this in our analysis but, for completeness, let us remind the reader that we have proven in Chapter 2 that Con' and Tot' are the smallest consistency and totality making the pairs $(c_m \cup \downarrow(x^+, x^-), \uparrow(x^+, 0^-) \cup \uparrow(0^+, x^-))$ into mutual d-complements. Using the definition of these two isomorphisms, we see that the closed congruence $\nabla(\ulcorner c_L \urcorner)$ on $L^+ \oplus L^-$ corresponds, in fact, to the closed congruence $\nabla(c_L)$ on $\mathsf{con}(L^+ \times L^-)$. Similarly, on the second frame component, we have that the closed congruence $\nabla(\ulcorner t_L \urcorner)$ corresponds to the closed congruence $\nabla(t_L)$ on the ordered collection of all tot subsets of $L^+ \times L^-$. For each d-frame $\mathcal{L}$ we define the d-frame $\mathsf{A}^{\mathsf{c}}_{cf}(\mathcal{L}))$ ("c" for "concrete") to be the quotient of $(\mathsf{con}(L^+ \times L^-), \mathsf{tot}(L^+ \times L^-), \mathsf{Con}', \mathsf{Tot}')$ by the relation structure

$$(\nabla(c_L), \nabla(t_L), c_m, t_m).$$

By the reasoning that we have illustrated, this d-frame is isomorphic to the d-frame $\mathsf{A}_{cf}(\mathcal{L})$.

6.3.2 The positive-negative d-frame assembly $\mathsf{A}_{\pm}(\mathcal{L})$

To construct the d-frame $\mathsf{A}_{\pm}(\mathcal{L})$ we will follow a similar strategy. First, we will introduce it as a rather abstract construction, and the fact that it is defined as a free construction will make it easy to show that its patch is isomorphic to the patch of the d-frame assembly $\mathsf{A}(\mathcal{L})$. After this, we will introduce a more concrete construction $\mathsf{A}^{\mathsf{c}}_{\pm}(\mathcal{L})$ isomorphic to this. For a d-frame $\mathcal{L}$, let us define as C^+ the congruence on $L^+ \oplus \mathsf{Filt}(L^+)$ making the elements $\ulcorner x^+ \urcorner$ and $\ulcorner \uparrow x^+ \urcorner$ into mutual complements for every $x^+ \in L^+$. A fact which by now we have mentioned several times is that $\mathsf{A}(L^+)$ is isomorphic to $(L^+ \oplus \mathsf{Filt}(L^+))/C^+$. Define as C^- the analogous congruence on the coproduct $L^- \oplus \mathsf{Filt}(L^-)$. For a d-frame $\mathcal{L}$, we now define the d-frame $\mathsf{A}_{\pm}(\mathcal{L})$ to be the quotient of the d-frame

$$(L^+ \oplus \mathsf{Filt}(L^+), L^- \oplus \mathsf{Filt}(L^-), c_m, t_m)$$

by the relation structure

$$(C^+, C^-, \ulcorner c_L \urcorner, \ulcorner t_L \urcorner),$$

where, as usual, $\ulcorner c_L \urcorner = \{(\ulcorner x^+ \urcorner, \ulcorner x^- \urcorner) : (x^+, x^-) \in c_L\}$ and $\ulcorner t_L \urcorner$ is defined similarly. Once again, this very abstract definition has the advantage of making it easy to show that the patch of this d-frame is the same as the patch of the d-frame assembly $\mathsf{A}(\mathcal{L})$.

Lemma 6.3.5. *For any d-frame $\mathcal{L}$, the patch $\Pi(\mathsf{A}_\pm(\mathcal{L}))$ is isomorphic to the coproduct $L^+ \oplus \mathsf{Filt}(L^-) \oplus L^- \oplus \mathsf{Filt}(L^+)$ quotiented by the relation*

$$\begin{aligned}
&\{(\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner, 0) : (x^+, x^-) \in c_L\} \cup \{(1, \ulcorner x^+ \urcorner \vee \ulcorner x^- \urcorner) : (x^+, x^-) \in t_L\} \cup \\
&\cup \{(\ulcorner x^+ \urcorner \wedge \ulcorner \uparrow x^+ \urcorner, 0), (1, \ulcorner x^+ \urcorner \vee \ulcorner \uparrow x^+ \urcorner) : x^+ \in L^+\} \cup \\
&\cup \{(\ulcorner x^- \urcorner \wedge \ulcorner \uparrow x^- \urcorner, 0), (1, \ulcorner x^- \urcorner \vee \ulcorner \uparrow x^- \urcorner) : x^- \in L^-\}.
\end{aligned}$$

This implies that this patch is isomorphic to $\Pi(\mathsf{A}(\mathcal{L}))$.

Proof. The first part of the claim follows by applying Lemma 1.3.11 to our definition of the d-frame $\mathsf{A}_\pm(\mathcal{L})$. The second part of the claim follows from Corollary 6.3.1. □

Now that we have used the abstract definition for the result that we needed, let us find the concrete counterpart of the d-frame $\mathsf{A}_\pm(\mathcal{L})$. For a d-frame $\mathcal{L}$, we simply define the d-frame $\mathsf{A}^{\mathsf{c}}_\pm$ to be the d-frame $(\mathsf{A}(L^+), \mathsf{A}(L^-), c_m, t_m)$ quotiented by the relation structure

$$(\mathsf{id}^+, \mathsf{id}^-, \ulcorner c_L \urcorner, \ulcorner t_L \urcorner).$$

It is clear, given the observation that $(L^+ \oplus \mathsf{Filt}(L^+))/C^+ \cong \mathsf{A}(L^+)$ and the similar one on the negative component, that this d-frame is isomorphic to $\mathsf{A}_\pm(\mathcal{L})$.

Chapter 7

The d-frame assembly and its two variations as encoding quotients

Introduction

For a frame L, the spectrum of its assembly $\mathsf{A}(L)$ encodes information about the sober subspaces of $\mathsf{pt}(L)$. The space $\mathsf{pt}(\mathsf{A}(L))$ is homeomorphic to the Skula space of $\mathsf{pt}(L)$. Then, we may view $\mathsf{pt}(\mathsf{A}(L))$ as a new topology imposed on the set $|\mathsf{pt}(L)|$. Every congruence $C \in \mathsf{A}(L)$ determines an open $\varphi_{\mathsf{A}(L)}(C)$ which we may see as a subset of $\mathsf{pt}(L)$. It is natural to ask how we can express this set of points in terms of the congruence C.

Actually, we can get a more refined answer to this question if we see $\mathsf{pt}(\mathsf{A}(L))$ as a bispace. The assembly $\mathsf{A}(L)$ is generated by the two subframes of its closed congruences and that of the fitted congruences. Because the spatialization map $\varphi_{\mathsf{A}(L)}$ turns finite meets into finite intersections and arbitrary joins into arbitrary unions, this means that the topology of $\mathsf{pt}(\mathsf{A}(L))$ is generated by the two topologies $\{\varphi_{\mathsf{A}(L)}(\nabla(a)) : a \in L\}$ and $\{\varphi_{\mathsf{A}(L)}(\Delta(a)) : a \in L\}$. It is known that the spectrum $\mathsf{pt}(\mathsf{A}(L))$ may be viewed as a bispace such that:

- Its underlying set of points is $|\mathsf{pt}(L)|$,
- A typical positive open is a set of the form $|\mathsf{pt}(L/\nabla(a))|^c$. These sets coincide with the open sets of $\mathsf{pt}(L)$.
- A typical negative open is a set of the form $\bigcup_i |\mathsf{pt}(L/\Delta(a_i))|^c$. These sets coincide with the complements of the fitted subsets of $\mathsf{pt}(L)$.

We argue that for pointfree bispaces there exists a very similar phenomenon. The d-sober subspaces of a d-frame spectrum $\mathsf{dpt}(\mathcal{L})$ can still be seen as patch-closed sets in some bitopology on the set $|\mathsf{dpt}(\mathcal{L})|$. There is more than one natural bitopology on $|\mathsf{dpt}(\mathcal{L})|$ in which all d-sober subspaces of $\mathsf{dpt}(\mathcal{L})$ are patch-closed. We will see that there are three natural bitopologies on this set with the desired property, and that these correspond to the constructions $\mathsf{A}(\mathcal{L})$, $\mathsf{A}_{cf}(\mathcal{L})$, and $\mathsf{A}_{\pm}(\mathcal{L})$. We refer to $\mathsf{A}_{cf}(\mathcal{L})$ and to $\mathsf{A}_{\pm}(\mathcal{L})$ as two variations of the d-frame assembly $\mathsf{A}(\mathcal{L})$.

First variation: the closed-fitted assembly. For $\mathsf{A}_{cf}(\mathcal{L})$, we will see that there is a bijection between points of this d-frame and points of $\mathsf{dpt}(\mathcal{L})$, and that under this bijection this bispace is the one such that

- Its underlying set of points is $|\mathsf{dpt}(\mathcal{L})|$
- A typical positive open set corresponds to the set of points of the form $|\mathsf{dpt}(\mathcal{L}/c)|^c$ for some con update c. A set is open in this topology if and only if it is a patch-open sets of $\mathsf{dpt}(\mathcal{L})$.
- A typical negative open set corresponds to the set of points $|\mathsf{dpt}(\mathcal{L}/t)|^c$, where t is a tot update. A set is open in this topology if and only if it is the complement of a finitary fitted subset of $\mathsf{dpt}(\mathcal{L})$.

Furthermore, all of the spectra of quotients of $\mathcal{L}$ are generated by this bitopology, in the sense that all subspaces $\mathsf{dpt}(\mathcal{L}/\mathcal{R})$ are patch-closed sets of $\mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L}))$. The patch of this bitopology –in fact– is the coarsest topology on $\mathsf{dpt}(\mathcal{L})$ in which the underlying sets of d-sober subspaces of $\mathsf{dpt}(\mathcal{L})$ are patch-closed.

Second variation: the positive-negative assembly. Recall that a spectrum $\mathsf{dpt}(\mathcal{L})$ is a bispace whose underlying set of points is a subspace of $\mathsf{pt}(L^+) \times \mathsf{pt}(L^-)$. For a spectrum $\mathsf{dpt}(\mathcal{L})$, we define a subset to be *positive* if it is of the form $|(Y^+ \times \mathsf{pt}(L^-)) \cap \mathsf{dpt}(\mathcal{L})|$ for some subset $Y^+ \subseteq \mathsf{pt}(L^+)$, and we define similarly *negative* subsets of $\mathsf{dpt}(\mathcal{L})$. We say that a bisubspace is *positive* if its underlying set is, and we define *negative* bisubspaces similarly. We will see that there is a bijection between points of the d-frame $\mathsf{A}_{\pm}(\mathcal{L})$ and points of $\mathsf{dpt}(\mathcal{L})$, and that under this bijection this bispace is the one such that

- Its underlying set of points is $|\mathsf{dpt}(\mathcal{L})|$;
- A typical positive closed set corresponds to the set of points $|\mathsf{dpt}(\mathcal{L}/C^+)|$, for C^+ a positive congruence. A set is open in this topology if and only if it is a d-sober positive subset of $\mathsf{dpt}(\mathcal{L})$.

- A typical negative closed set corresponds to the set of points $|\mathsf{dpt}(\mathcal{L}/C^-)|$, with C^- a negative congruence. A set is open in this topology if and only if it is a d-sober negative subset of $\mathsf{dpt}(\mathcal{L})$.

The patch of this bitopology is the coarsest topology on $\mathsf{dpt}(\mathcal{L})$ in which the underlying sets of d-sober subspaces of $\mathsf{dpt}(\mathcal{L})$ are patch-closed.

7.1 The spectra of the d-frame assembly and its variations: Skula bispaces

Let us begin by showing what is the relation between spectra of quotients of a d-frame and the topology on the spectrum of that d-frame. In the next proof, we will use our usual convention of identifying a bispace $\mathsf{dpt}(\mathcal{L}/\mathcal{R})$ with the bisubspace $\{f \in \mathsf{dpt}(\mathcal{L}) : f \text{ respects } \mathcal{R}\} \subseteq \mathsf{dpt}(\mathcal{L})$. To our end, it suffices to re-interpret some of the result which we had got in Chapter 3 - in particular, Lemma 3.1.12. We report here the part of the lemma which we will use, for ease of consultation.

Lemma 7.1.1. *The following are true for a d-frame $\mathcal{L}$.*

1. *Whenever c is a* con *update,*

$$\mathsf{dpt}(\mathcal{L}/c) = \{f \in \mathsf{dpt}(\mathcal{L}) : f^+(x^+) = 0^+ \text{ or } f^-(x^-) = 0^- \text{ for all } (x^+, x^-) \in c\}.$$

2. *Whenever t is a* tot *update,*

$$\mathsf{dpt}(\mathcal{L}/t) = \{f \in \mathsf{dpt}(\mathcal{L}) : f^+(x^+) = 1^+ \text{ or } f^-(x^-) = 1^- \text{ for all } (x^+, x^-) \in t\}.$$

3. *Whenever R^+ is a relation on L^+,*

$$\mathsf{dpt}(\mathcal{L}/R^+) = \{f \in \mathsf{dpt}(\mathcal{L}) : f^+(a^+) = 0^+ \text{ or } f^+(b^+) = 1^+ \text{ whenever } a^+ R^+ b^+\}.$$

We obtain the following proposition relating spectra of quotients of a d-frame with the topology of its spectrum.

Proposition 7.1.2. *For every d-frame $\mathcal{L}$ we have the following.*

- $\mathsf{dpt}(\mathcal{L}/c) = \bigcap\{(\varphi_{\mathcal{L}}^+(x^+)^c \cup \varphi_{\mathcal{L}}^-(x^-)^c : (x^+, x^-) \in c\}$ *for every subset $c \subseteq L^+ \times L^-$ treated as a* con *update.*
- $\mathsf{dpt}(\mathcal{L}/t) = \bigcap\{\varphi_{\mathcal{L}}^+(x^+) \cup \varphi_{\mathcal{L}}^-(x^-) : (x^+, x^-) \in t\}$ *for every subset $t \subseteq L^+ \times L^-$ treated as a* tot *update.*

- $\mathsf{dpt}(\mathcal{L}/R^+) = \bigcap\{\varphi^+_{\mathcal{L}}(a^+)^c \cap \varphi^+_{\mathcal{L}}(b^+) : a^+ R^+ b^+\}$.
- $\mathsf{dpt}(\mathcal{L}/\Delta(a^+)) = \varphi^+_{\mathcal{L}}(a^+)$.
- $\mathsf{dpt}(\mathcal{L}/\nabla(a^+)) = \varphi^+_{\mathcal{L}}(a^+)^c$.

Proof. The first three items follow from the first three items of Lemma 3.1.12. For the fourth item, let us notice that $\Delta(a^+)$ is the congruence generated by the relation $\{(1, a^+)\}$. The maps $f : \mathcal{L} \to \mathbf{2}$ which respect the relation structure $(\Delta(a^+), \mathsf{id}^-, c_m, t_m)$ are exactly those such that f^+ respects the congruence $\Delta(a^+)$ and - by the observation we have just made - these are exactly those such that respect the relation $\{(1, a^+)\}$, that is, those such that $f^+(a^+) = 1$. But, by definition of the bitopology on $\mathsf{dpt}(\mathcal{L})$, these are exactly the elements of $\varphi^+_{\mathcal{L}}(a^+)$. The fifth item is shown in a completely analogous manner, using the fact that the congruence $\nabla(a^+)$ is the one generated by the relation $\{(a^+, 0)\}$. □

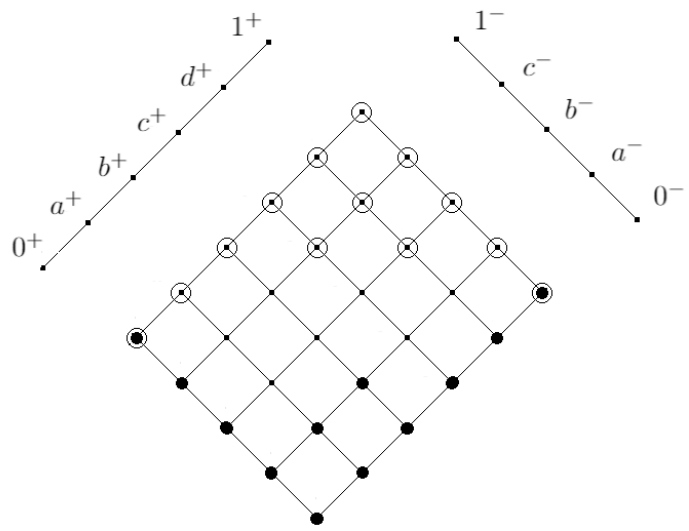
For a bispace X, we define a patch-open to be *finitary* if it is a finite union of basic patch-opens $U^+ \cap U^-$. Similarly, we define a patch-closed set to be finitary if it is the complement of a finitary patch-open set, and a patch-fitted set to be finitary if it is the intersection of finitary opens. Sometimes it will be useful to think of finitary fitted subsets as follows.

Lemma 7.1.3. *A subset of a bispace X is finitary fitted if and only if it is of the form*

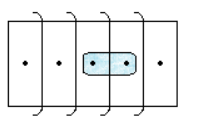
$$\bigcap_i U_i^+ \cup U_i^+.$$

We introduce a new concept to be able to describe concisely those subspaces $\mathsf{dpt}(\mathcal{L}/C^+)$ in which C^+ is an arbitrary congruence. For a spectrum $\mathsf{dpt}(\mathcal{L})$, we define a subset to be *positive* if it is of the form $|(S^+ \times \mathsf{pt}(L^-)) \cap \mathsf{dpt}(\mathcal{L})|$, where $S^+ \subseteq \mathsf{pt}(L^+)$ is an arbitrary subset. We define *negative* subsets similarly. We say that a bisubspace is *positive* if its underlying set is, and we define negative bisubspaces similarly.

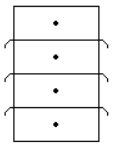
Example 7.1.4. Let us illustrate what a positive bisubspace is with an example and some pictures. We consider the following d-frame.



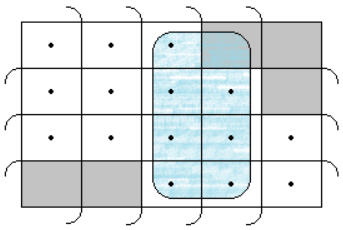
Now, we illustrate its spectrum, and we see how a subspace $Y^+ \subseteq \mathsf{pt}(L^+)$ of the positive spectrum determines a positive bisubspace of $\mathsf{dpt}(\mathcal{L})$. The situation is depicted below. As usual, we have shaded in grey the portions of $\mathsf{pt}(L^+) \times \mathsf{pt}(L^-)$ that are taken away from the con and the tot components.



$Y^+ \subseteq \mathsf{pt}(L^+)$



$\mathsf{pt}(L^-)$



$(Y^+ \times \mathsf{pt}(L^-)) \cap \mathsf{dpt}(\mathcal{L}) \subseteq \mathsf{dpt}(\mathcal{L})$

This visual intuition makes it even more immediate that the bispace $(Y^+ \times \mathsf{pt}(L^-)) \cap \mathsf{dpt}(\mathcal{L})$ is nothing but the space Y^+ stretched across all possible negative coordinates. More in general, the positive bisubspaces of $\mathsf{dpt}(\mathcal{L})$ are those subspaces such that they range across all possible negative coordinates. If we keep in mind the case where L^+ and L^- are linear for visual intuition, these are exactly subspaces of $\mathsf{dpt}(\mathcal{L})$ which are vertical rectangles. The negative subspaces are, exactly for the same reasons, horizontal rectangles.

Proposition 7.1.5. *If $\mathcal{L}$ is a d-frame and R^+ a relation on L^+, we have that*

$$(\mathsf{pt}(L^+/R^+) \times \mathsf{pt}(L^-)) \cap \mathsf{dpt}(\mathcal{L}) = \mathsf{dpt}(\mathcal{L}/R^+).$$

In particular, for every $a^+ \in L^+$ we have

- $|(\mathsf{pt}(L^+/\Delta(a^+)) \times \mathsf{pt}(L^-)) \cap \mathsf{dpt}(\mathcal{L})| = |\mathsf{dpt}(\mathcal{L}/\Delta(a^+))| = \varphi^+_{\mathcal{L}}(a^+)$,
- $|(\mathsf{pt}(L^+/\nabla(a^+)) \times \mathsf{pt}(L^-)) \cap \mathsf{dpt}(\mathcal{L})| = |\mathsf{dpt}(\mathcal{L}/\nabla(a^+))| = \varphi^+_{\mathcal{L}}(a^+)^c$.

Proof. First, we consider the first part of the claim. Both these bispaces are bisubspaces of $\mathsf{dpt}(\mathcal{L})$, and so to show that they coincide it suffices to show that the underlying set of points is the same. We have that a point $f \in \mathsf{dpt}(\mathcal{L})$ is in the underlying set of the first bispace if and only if $f^+ \in \mathsf{pt}(L^+/R^+)$, and this holds if and only if the map f^+ respects the relation R^+. But this condition, in turn, is equivalent to f respecting the relation structure $(R^+, \mathsf{id}^-, c_m, t_m)$

on $\mathcal{L}$, that is, f being in $\mathsf{pt}(\mathcal{L}/R^+)$. For the rest of the claim, we simply instantiate the first part of the claim to the case where R^+ is the relation $\{(1, a^+)\}$ and $\{(a^+, 0)\}$, and we combine this with the fourth and fifth item of Lemma 7.1.2. □

To find a d-frame analogue of the Skula space of a spectrum $\mathsf{pt}(L)$, our first question might be whether there is a topology on $|\mathsf{dpt}(\mathcal{L})|$ in which the closed sets coincide with the d-sober bisubspaces of $\mathsf{dpt}(\mathcal{L})$. This is not possible, since d-sober bisubspaces do not form a coframe (they fail to be closed under finite unions). We claim that our second best attempt is finding the coarsest topology in which the d-sober bisubspaces of $\mathsf{dpt}(\mathcal{L})$ are closed.

Theorem 7.1.6. *Suppose that $\mathcal{L}$ is a d-frame. Consider the following collection.*

$$\{\varphi^+_{\mathcal{L}}(a^+) : a^+ \in L^+\} \cup \{\varphi^+_{\mathcal{L}}(a^+)^c : a^+ \in L^+\} \cup \{\varphi^-_{\mathcal{L}}(a^-) : a^- \in L^-\} \cup \{\varphi^-_{\mathcal{L}}(a^-)^c : a^- \in L^-\}.$$

The topology generated by this collection is the coarsest one in which the underlying sets of all d-sober subspaces of $\mathsf{dpt}(\mathcal{L})$ are closed.

Proof. By Theorem 3.1.17, we know that every d-sober bisubspace of $\mathsf{dpt}(\mathcal{L})$ is an intersection of bisubspaces of the form $\mathsf{dpt}(\mathcal{L}/c) \cup \mathsf{dpt}(\mathcal{L}/t)$. By Lemma 7.1.2 this expression equals

$$\bigcap\{(\varphi^+_{\mathcal{L}}(x^+)^c \cup \varphi^-_{\mathcal{L}}(x^-)^c : (x^+, x^-) \in c\} \cup \bigcap\{\varphi^+_{\mathcal{L}}(x^+) \cup \varphi^-_{\mathcal{L}}(x^-) : (x^+, x^-) \in t\}.$$

This is an intersection of finite unions of elements from our collection. Then, all d-sober bisubspaces of $\mathsf{dpt}(\mathcal{L})$ are closed in this topology. To show that this is the coarsest topology with this property, we simply notice that any such topology must be such that it contains all the subsets of the form $\varphi^+_{\mathcal{L}}(a^+)$, $\varphi^+_{\mathcal{L}}(a^+)^c$, $\varphi^-_{\mathcal{L}}(a^-)$, and $\varphi^-_{\mathcal{L}}(a^-)^c$, as all of these are all complements of d-sober bisubspaces by Lemma 7.1.2. □

We now ask how we can divide this topology into two in a meaningful way, that is, what are some significant bitopologies on $|\mathsf{dpt}(\mathcal{L})|$ which have the topology described above as their patch. Answering this question is one of the main aims of the next three subsections. We will see that there are three reasonable ways of turning this into a bispace. These three bispaces will be called $Sk(\mathsf{dpt}(\mathcal{L}))$, $Sk_{cf}(\mathsf{dpt}(\mathcal{L}))$, and $Sk_{\pm}(\mathsf{dpt}(\mathcal{L}))$ and -as we will see- they are the spatial counterparts of the three constructions $\mathsf{A}(\mathcal{L})$, $\mathsf{A}_{cf}(\mathcal{L})$, and $\mathsf{A}_{\pm}(\mathcal{L})$.

7.1.1 The assembly functor for d-frames and its spatial counterpart

Recall that in Chapter 5 we have proven that there is an assembly functor $\mathsf{A} : \mathbf{dFrm} \to \mathbf{dFrm}$, and that its spatial counterpart is $Sk : \mathbf{BiTop} \to \mathbf{BiTop}$. The functor Sk is the spatial

counterpart of A in the sense that we have a natural isomorphism $\mathsf{dpt} \circ \mathsf{A} \cong Sk \circ \mathsf{dpt}$. The following result tells us that for a bispace $\mathsf{dpt}(\mathcal{L})$, the patch of the Skula bispace $Sk(\mathsf{dpt}(\mathcal{L}))$ is a topological space which in a certain sense encodes in it all the d-sober bisubspaces of $\mathsf{dpt}(\mathcal{L})$.

Lemma 7.1.7. *For any d-frame $\mathcal{L}$, we have that $\mathsf{patch}(Sk(\mathsf{dpt}(\mathcal{L})))$ is the coarsest topology on $|\mathsf{dpt}(\mathcal{L})|$ such that the underlying sets of all d-sober subspaces of $\mathsf{dpt}(\mathcal{L})$ are closed.*

Proof. It is clear from the way the bitopology $Sk(\mathsf{dpt}(\mathcal{L}))$ is defined that the patch is the topology generated by sets of the form $\varphi_L^+(a^+)$, $\varphi_L^+(a^+)^c$, $\varphi_L^-(a^-)$, and $\varphi_L^-(a^-)^c$. The desired results follows from Theorem 7.1.6. □

7.1.2 The A_{cf} functor and its spatial counterpart

Let us now look at our first variation of the d-frame assembly. The first step is to establish a connection between this and some suitable bitopological analogue of the Skula construction. We will then see that the bitopological space $\mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L}))$ is such that its positive opens coincide with the patch opens of $\mathsf{dpt}(\mathcal{L})$, and these also coincide with the subsets of $\mathsf{dpt}(\mathcal{L})$ of the form $|\mathsf{dpt}(\mathcal{L}/c)^c|$ for some con update c. We will also see that its negative opens are exactly the complements of the finitary fitted subsets of $\mathsf{dpt}(\mathcal{L})$, and that these also coincide with the subsets of $\mathsf{dpt}(\mathcal{L})$ of the form $|\mathsf{dpt}(\mathcal{L}/t)^c|$ for some tot update t. Thus, this will be the bispace in which we will see the interplay between the quotients obtained via increases in the con and the tot components, and where we will also see that this is just a rephrasing of the interplay between the patch-closed and the finitary patch-fitted subsets of $\mathsf{dpt}(\mathcal{L})$.

For a bispace X, let us define the *closed-fitted* Skula bispace, which we will denote as $Sk_{cf}(X)$, as the bispace such that

- its underlying set of points is that of X,
- its positive opens are the patch-opens of X,
- its negative opens are the topology generated by the positive and the negative closed sets of X.

Let us notice that for a bispace X the negative opens of $Sk_{cf}(X)$ coincide with the sets of the form $\bigcup_i U_i^{+c} \cap U_i^{-c}$. Then, the negative closed sets are exactly the sets of the form $\bigcap_i U_i^+ \cup U_i^-$, that is, the finitary fitted subsets of X. On the other hand, the positive closed sets of $Sk(X)$ coincide with the patch-closed sets of X. Now let us look at the closed-fitted Skula bispace of the spectrum of a d-frame.

Proposition 7.1.8. *For a d-frame $\mathcal{L}$, the Skula space $Sk_{cf}(\mathsf{dpt}(\mathcal{L}))$ is such that*

- *Its positive closed sets are the sets of the form $|\mathsf{dpt}(\mathcal{L}/c)|$ for some* con *update c.*
- *Its negative closed sets are the sets of the form $|\mathsf{dpt}(\mathcal{L}/t)|$ for some* tot *update t*

Additionally, the patch of this bispace is the coarsest topology on $\mathsf{dpt}(\mathcal{L})$ in which all d-sober bisubspaces are closed.

Proof. Let us show that the positive closed sets of $Sk_{cf}(\mathsf{dpt}(\mathcal{L}))$ coincide with the sets of the form $|\mathsf{dpt}(\mathcal{L}/c)|$. The positive closed sets of this topology coincide with the patch-closed sets of $\mathsf{dpt}(\mathcal{L})$. A subset of $|\mathsf{dpt}(\mathcal{L})|$ is patch-closed if and only if it is of the form

$$\bigcap\{\varphi^+_{\mathcal{L}}(a_i^+)^c \cup \varphi^-_{\mathcal{L}}(a_i^-)^c : i \in I\}.$$

If we define $c_I = \{(a_i^+, a_i^-) : i \in I\}$ we obtain that the expression above denotes the set $|\mathsf{dpt}(\mathcal{L}/c_I)|$ with c_I treated as a con update, by item (3) of Lemma 7.1.2. Conversely, if we have a set of the form $|\mathsf{dpt}(\mathcal{L}/c)|$, item (1) of Lemma 7.1.2 immediately tells us that this is patch-closed in $\mathsf{dpt}(\mathcal{L})$. Now, we look at the negative closed sets of $Sk_{cf}(\mathsf{dpt}(\mathcal{L}))$. An arbitrary negative closed set of this topology is of the form

$$\bigcap\{\varphi^+_{\mathcal{L}}(a_i^+) \cup \varphi^-_{\mathcal{L}}(a_i^-) : i \in I\}.$$

Setting $t_I = \{(a_i^+, a_i^-) : i \in I\}$, item (2) of Lemma 7.1.2 tells us that the expression above denotes the set $|\mathsf{dpt}(\mathcal{L}/t_I)|$, with t_I treated as a tot update. The converse, as before, is immediate from item (2) of Lemma 7.1.2. Finally, we notice that the patch of this bitopology is the topology generated by the positive and negative open sets of $\mathsf{dpt}(\mathcal{L})$ together with its positive and negative closed sets. This topology is the desired one, by Theorem 7.1.6. By the same proposition we also know that it is the coarsest topology in which all d-sober subspaces are closed. □

Let us now extend the definition of Sk_{cf} to morphisms. For a bicontinuous map $f : X \to Y$ we define the map $Sk_{cf}(f) : Sk_{cf}(X) \to Sk_{cf}(Y)$ to be simply the function f. Since the function f is bicontinuous, preimages of positive opens and closed sets are positive open and closed set, respectively, and the same holds for the negative topology. Then, indeed the map $Sk_{cf} : Sk_{cf}(X) \to Sk_{cf}(Y)$ is bicontinuous. The assignment also respects identities and compositions, since in the category of bitopological spaces these are just identity functions and function compositions. We have obtained the following.

Lemma 7.1.9. *The assignment $Sk_{cf} : \mathbf{BiTop} \to \mathbf{BiTop}$ is a functor.*

Let us now move to looking at the structure which we claim to be a pointfree version of the closed-fitted Skula construction. Recall that, for any d-frame $\mathcal{L}$, we defined the structure $\mathsf{A}_{cf}(\mathcal{L})$ to be the quotient of the d-frame $(L^+ \oplus L^-, \mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-), c_m, t_m)$ by the relation structure

$$(\nabla(\ulcorner c_L \urcorner), \nabla(\ulcorner t_L \urcorner), \mathcal{C}_L, \mathcal{C}_L),$$

where we have used definitions

- $\nabla(\ulcorner c_L \urcorner) = \nabla(\bigvee\{\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner : (x^+, x^-) \in c_L\})$,
- $\nabla(\ulcorner t_L \urcorner) = \nabla(\bigvee\{\ulcorner \uparrow x^+ \urcorner \wedge \ulcorner \uparrow x^- \urcorner : (x^+, x^-) \in t_L\})$,
- $\mathcal{C}_L = \{(\ulcorner x^+ \urcorner, \ulcorner \uparrow x^+ \urcorner) : x^+ \in L^+\} \cup \{(\ulcorner x^- \urcorner, \ulcorner \uparrow x^- \urcorner) : x^- \in L^-\}$.

We also introduce the following definitions.

- $R^+_{\nabla(\ulcorner c_L \urcorner)} := \{(\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner, 0) : (x^+, x^-) \in c_L\}$,
- $R^-_{\nabla(\ulcorner t_L \urcorner)} := \{(\ulcorner \uparrow x^+ \urcorner \wedge \ulcorner \uparrow x^- \urcorner, 0) : (x^+, x^-) \in t_L\}$,
- $\mathcal{CF}_L := (R^+_{\nabla(\ulcorner c_L \urcorner)}, R^-_{\nabla(\ulcorner t_L \urcorner)}, \mathcal{C}_L, \mathcal{C}_L)$.

We notice that the congruence $\nabla(\ulcorner c_L \urcorner)$ is the congruence generated by the first relation. Similarly, the congruence $\nabla(\ulcorner t_L \urcorner)$ is the congruence generated by the second one. This tells us that the closed-fitted assembly is the quotient of our usual d-frame $(L^+ \oplus L^-, \mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-), c_m, t_m)$ by the relation structure $\mathcal{CF}_L$. Let us define our endofunctor on the category of d-frames. On objects it maps each d-frame its closed-fitted assembly $\mathsf{A}_{cf}(\mathcal{L})$. To find a suitable definition of A_{cf} for morphisms, let us proceed as for the assembly. By Lemma 5.1.5, we need to consider a d-frame map $f : \mathcal{L} \to \mathcal{M}$ and compare the two relation structures $f[\mathcal{CF}_L]$ and $\mathcal{CF}_M$.

Lemma 7.1.10. *For any d-frame map $f : \mathcal{L} \to \mathcal{M}$, we have that the d-frame map $F := (f^+ \oplus f^-, \mathsf{Filt}(f^+) \oplus \mathsf{Filt}(f^-))$, seen as a d-frame map*

$$(L^+ \oplus L^-, \mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-), c_m, t_m) \to (M^+ \oplus M^-, \mathsf{Filt}(M^+) \oplus \mathsf{Filt}(M^-), c_m, t_m)$$

is such that $F[\mathcal{CF}_L] \leq \mathcal{CF}_M$.

Proof. Let us check the three subset inclusions that our claim amounts to.

- Let us check the set inclusion of the first components. Recall that the positive component of the map F is $f^+ \oplus f^-$, and that by definition of the coproduct pairing of frame maps we have $f^+ \oplus f^-(\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner) = \ulcorner f^+(x^+) \urcorner \wedge \ulcorner f^-(x^-) \urcorner$ for every $x^+ \in L^+$ and every $x^- \in L^-$. Whenever $(\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner, 0) \in R^+_{\nabla \ulcorner c_L \urcorner}$ by definition of this relation we must have $(x^+, x^-) \in c_L$. Because f is a d-frame map, it respects the con subset, and so we must also have $(F^+(\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner), F^+(0)) = (\ulcorner f^+(x^+) \urcorner \wedge \ulcorner f^-(x^-) \urcorner, 0) \in R^+_{\nabla(\ulcorner c_M \urcorner)}$ by definition of this relation.

- The proof is similar to the one above. Recall that the second component of the map F is $\mathsf{Filt}(f^+) \oplus \mathsf{Filt}(f^-)$. By definition of this map we have that $\mathsf{Filt}(f^+) \oplus \mathsf{Filt}(f^-)(\ulcorner \uparrow x^+ \urcorner \wedge \ulcorner \uparrow x^- \urcorner) = \ulcorner \uparrow f^+(x^+) \urcorner \wedge \ulcorner \uparrow f^-(x^-) \urcorner$. Then, if we have a pair $(\ulcorner \uparrow x^+ \urcorner \wedge \ulcorner \uparrow x^- \urcorner, 0) \in R^+_{\nabla(\ulcorner t_L \urcorner)}$ by definition of this relation we must have that $(x^+, x^-) \in t_L$ and so, since f is assumed to be a d-frame map, we may deduce that $(f^+(x^+), f^-(x^-)) \in t_M$, and so by definition $(\ulcorner \uparrow f^+(x^+) \urcorner \wedge \uparrow f^-(x^-), 0) \in R^+_{\nabla(\ulcorner t_L \urcorner)}$.

- Finally, we check the set inclusion $F[\mathcal{C}_L] \subseteq \mathcal{C}_M$. For this, it suffices to notice that if we have $x^+ \in L^+$, then by definition of the map F we have that $(F(\ulcorner x^+ \urcorner), F(\ulcorner \uparrow x^+ \urcorner)) = (\ulcorner f^+(x^+) \urcorner, \ulcorner \uparrow f^+(x^+) \urcorner)$). □

For the closed-fitted assembly, we adopt the same convention that we introduced for the assembly. That is, we refer to a the syntactic expression from an element $a^+ \in L^+$ as appearing in the first component of $\mathsf{A}_{cf}(\mathcal{L})$ as $\ulcorner\ulcorner a^+ \urcorner\urcorner$, and we use the same convention for elements $a^- \in L^-$. By combining Lemma 5.1.5 with Lemma 7.1.10 we obtain that whenever $f : \mathcal{L} \to \mathcal{M}$ is a d-frame map, there is a d-frame map $\mathsf{A}_{cf}(f) : \mathsf{A}_{cf}(\mathcal{L}) \to \mathsf{A}_{cf}(\mathcal{M})$ defined on positive generators as

$$\begin{aligned} &\mathsf{A}_{cf}(f) : \mathsf{A}_{cf}(\mathcal{L}) \to \mathsf{A}_{cf}(\mathcal{M}) \\ &\ulcorner\ulcorner x^+ \urcorner\urcorner \mapsto \ulcorner\ulcorner f^+(x^+) \urcorner\urcorner \\ &\ulcorner\ulcorner \uparrow x^+ \urcorner\urcorner \mapsto \ulcorner\ulcorner \uparrow f^+(x^+) \urcorner\urcorner \end{aligned}$$

and defined similarly on negative generators. It is clear that this maps an identity to an identity. The fact that it respects composition of morphisms is immediate from the definition. Then, we may take $f \mapsto \mathsf{A}_{cf}(f)$ to be the assignment for morphisms of a functor $\mathsf{A}_{cf} : \mathbf{dFrm} \to \mathbf{dFrm}$ acting on objects as $\mathsf{A}_{cf}(\mathcal{L})$. Let us state this as a lemma.

Lemma 7.1.11. *The assignment* $\mathsf{A}_{cf} : \mathbf{dFrm} \to \mathbf{dFrm}$ *is a functor.*

Let us now start working in the direction of showing that we have a bihomeomorphism $\mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L})) \cong Sk_{cf}(\mathsf{dpt}(\mathcal{L}))$. The first step is to find for every d-frame $\mathcal{L}$ a well-defined

assignment which gives a point of $\mathsf{A}_{cf}(\mathcal{L})$ for every point of $\mathcal{L}$. Before we move on to our proof, recall that if we have a point $f : L \to 2$ for a frame L we denote as $\mathsf{Filt}_{\neg}(f) : L \to 2$ the frame map defined as $\mathsf{Filt}(f)_{\neg}(\uparrow x) = \neg f(x)$ for every $x \in L$.

Lemma 7.1.12. *For any d-frame map $f : \mathcal{L} \to \mathbf{2}$, the map*

$$(f^+ \oplus f^-, \mathsf{Filt}_{\neg}(f^+) \oplus \mathsf{Filt}_{\neg}(f^-)) : (L^+ \oplus L^-, \mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-), c_m, t_m) \to \mathbf{2}$$

respects the relation structure $\mathcal{CF}_L$.

Proof. To show our claim it suffices to show that the following three conditions are satisfied. In the following, we have used the definition of $\mathsf{Filt}_{\neg}(-)$.

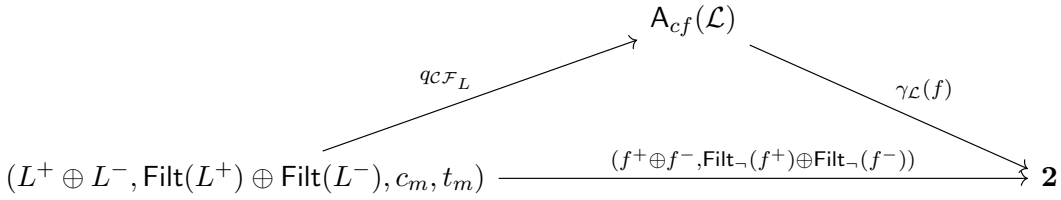
- Whenever $(x^+, x^-) \in c_L$ we must have that $\ulcorner f^+(x^+) \urcorner \wedge \ulcorner f^-(x^-) \urcorner = 0$. That is, we must have either $\ulcorner f^+(x^+) \urcorner = 0$ or $\ulcorner f^-(x^-) \urcorner = 0$.
- Whenever $(x^+, x^-) \in t_L$ we must have that $\ulcorner \mathsf{Filt}_{\neg}(f^+)(\uparrow x^+) \urcorner \wedge \ulcorner \mathsf{Filt}_{\neg}(f^-)(\uparrow x^-) \urcorner = 0$. This means that we must have either $\ulcorner f^+(x^+) \urcorner = 1$ or $\ulcorner f^-(x^-) \urcorner = 1$.
- Whenever $x^+ \in L^+$ we must have $(\ulcorner f^+(x^+) \urcorner, \ulcorner \mathsf{Filt}_{\neg}(f^+)(x^+) \urcorner) \in c_2 \cap t_2$. That is, we must have $(\ulcorner f^+(x^+) \urcorner, \ulcorner \neg f^+(x^+) \urcorner \in c_2 \cap t_2$. Similarly for $x^- \in L^-$.

The first two conditions follow from the fact that f respects the con and the tot components, while the third one follows from the fact that the elements $(1, 0)$ and $(0, 1)$ of 2×2 are both in c_2 and in t_2. □

By its definition, the d-frame $\mathsf{A}_{cf}(\mathcal{L})$ is such that its points are in bijective correspondence with the points

$$(F^+, F^-) : (f^+ \oplus f^-, \mathsf{Filt}(f^+) \oplus \mathsf{Filt}(f^-), c_m, t_m) \to \mathbf{2}$$

such that they respect the relation structure $\mathcal{CF}_L$. By Lemma 7.1.12 above, we deduce that whenever we have a point $f \in \mathsf{dpt}(\mathcal{L})$, there is a unique map $\gamma_{\mathcal{L}}(f)$, making the following commute.



For any d-frame $\mathcal{L}$ and any d-frame map $f : \mathcal{L} \to \mathbf{2}$, then, Lemma 7.1.12 above is telling us that this assignment $f \mapsto \gamma_{\mathcal{L}}(f)$ is a well-defined map $\gamma_{\mathcal{L}} : |\mathsf{dpt}(\mathcal{L})| \to |\mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L}))|$. Let us now show that this well-defined assignment is actually the bijection between the underlying sets of points $|\mathsf{dpt}(\mathcal{L})|$ and $|\mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L}))|$ that we are after.

Lemma 7.1.13. *For any d-frame $\mathcal{L}$, the map $\gamma_{\mathcal{L}} : |\mathsf{dpt}(\mathcal{L})| \to |\mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L}))|$ is a bijection.*

Proof. Injectivity of this assignment is clear: if $f^+ \neq g^+$ or $f^- \neq g^-$ we cannot have $f^+ \oplus f^- = g^+ \oplus g^-$. This means that the pairs of frame maps $(f^+ \oplus f^-, \mathsf{Filt}(f^+) \oplus \mathsf{Filt}(f^-))$ and $(g^+ \oplus g^-, \mathsf{Filt}(g^+) \oplus \mathsf{Filt}(g^-))$ must be different. Then, by the universal property of the quotient by $\mathcal{CF}_L$, we have that the two maps $\gamma_{\mathcal{L}}(f)$ and $\gamma_{\mathcal{L}}(g)$ are different. Let us show that the assignment is surjective. The points $f : L^+ \oplus L^- \to 2$ are all of the form $f^+ \oplus f^-$. So, given a point $f : \mathsf{A}_{cf}(\mathcal{L}) \to \mathbf{2}$ we may assume that it is of the form $(f^+ \oplus f^-, g^+ \oplus g^-)$. By assumption that this is a point of $\mathsf{A}_{cf}(\mathcal{L})$, we know that it respects the relation structure $\mathcal{CF}_L$. This means that

- Whenever $(x^+, x^-) \in c_L$ we have that $\ulcorner f^+(x^+) \urcorner \wedge \ulcorner f^-(x^-) \urcorner = 0$. That is, we must have either $\ulcorner f^+(x^+) \urcorner = 0$ or $\ulcorner f^-(x^-) \urcorner = 0$.
- Whenever $(x^+, x^-) \in t_L$ we have that $\ulcorner g^+(\uparrow x^+) \urcorner \wedge \ulcorner g^-(\uparrow x^-) \urcorner = 0$. That is, we must have either $\ulcorner g^+(\uparrow x^+) \urcorner = 1$ or $g^-(\ulcorner \uparrow x^- \urcorner) = 1$.
- Whenever $x^+ \in L^+$ we have $(f^+(x^+), g^+(\uparrow x^+)) \in c_2 \cap t_2$. That is, we must have either $\ulcorner f^+(x^+) \urcorner = 0$ and $\ulcorner g^+(\uparrow x^+) \urcorner = 1$ or $\ulcorner f^+(x^+) \urcorner = 1$ and $\ulcorner g^+(\uparrow x^+) \urcorner = 0$.

The last condition means that whenever $x^+ \in L^+$ we have $\ulcorner g^+(\uparrow x^+) \urcorner = \ulcorner \neg f^+(x^+) \urcorner$, and so $g^+ = \mathsf{Filt}_\neg(f^+)$. By a similar argument, also $g^- = \mathsf{Filt}_\neg(f^-)$. To complete our claim it suffices to show that $(f^+, f^-) : \mathcal{L} \to \mathbf{2}$ is a d-frame map, for which it suffices to show that it respects the con and the tot components. But this is ensured by the first two conditions above. Then, the assignment $\gamma_{\mathcal{L}}$ is surjective. □

To show that the spectrum of $\mathsf{A}_{cf}(\mathcal{L})$ is bihomeomorphic to the closed-fitted Skula space of $\mathsf{dpt}(\mathcal{L})$, we will show that the bijection above is the desired bihomeomorphism. As an intermediate step to showing this, we calculate what the open and the closed sets of $\mathsf{dpt}(\mathcal{L})$ become under the bijection $\gamma_{\mathcal{L}}$.

Lemma 7.1.14. *Suppose that $\mathcal{L}$ is a d-frame. The following holds for every $f \in \mathsf{dpt}(\mathcal{L})$.*

- *For every $a^+ \in L^+$ we have that $f^+(a^+) = 1$ if and only if $\gamma_{\mathcal{L}}(f)(\ulcorner \ulcorner a^+ \urcorner \urcorner) = 1$.*

- *For every $a^+ \in L^+$ we have that $f^+(a^+) = 1$ if and only if $\gamma_{\mathcal{L}}(f)(\ulcorner\urcorner\uparrow a^+\urcorner\urcorner) = 0$.*

Proof. Consider the triangle before Lemma 7.1.13. From this we know that for every $a^+ \in L^+$ we have that $\gamma_{\mathcal{L}}(f)(\ulcorner\urcorner a^+\urcorner\urcorner) = f^+ \oplus f^-(\ulcorner a^+\urcorner)$ and this, by definition of the coproduct pairing of two frame maps, is the same as $f^+(a^+)$. On the other hand, we have that $\gamma_{\mathcal{L}}(f)(\ulcorner\urcorner\uparrow a^+\urcorner\urcorner) = (\mathsf{Filt}_\neg(f^+) \oplus \mathsf{Filt}_\neg(f^-)(\ulcorner\uparrow a^+\urcorner)) = \neg f^+(a^+)$. □

Lemma 7.1.15. *For any d-frame $\mathcal{L}$ we have that*

- $\gamma_{\mathcal{L}}[\varphi^+_{\mathcal{L}}(a^+)] = \varphi^+_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner\urcorner a^+\urcorner\urcorner)$.
- $\gamma_{\mathcal{L}}[\varphi^+_{\mathcal{L}}(a^+)^c] = \varphi^-_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner\urcorner\uparrow a^+\urcorner\urcorner)$.
- $\gamma_{\mathcal{L}}[\varphi^-_{\mathcal{L}}(a^-)] = \varphi^-_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner\urcorner a^-\urcorner\urcorner)$.
- $\gamma_{\mathcal{L}}[\varphi^-_{\mathcal{L}}(a^-)^c] = \varphi^+_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner\urcorner\uparrow a^-\urcorner\urcorner)$.

Proof. Let us prove the first two items.

- Let us expand the left hand side. We will implicitly use the fact that $\gamma_{\mathcal{L}}$ is a bijection. We have that

$$\begin{aligned}
&\gamma_{\mathcal{L}}[\varphi^+_{\mathcal{L}}(a^+)] = \{\gamma_{\mathcal{L}}(f) : f \in \mathsf{dpt}(\mathcal{L}), f^+(a^+) = 1\} = \\
&= \{\gamma_{\mathcal{L}}(f) : f \in \mathsf{dpt}(\mathcal{L}), \gamma_{\mathcal{L}}(f)(\ulcorner\urcorner a^+\urcorner\urcorner) = 1\} = \\
&= \{F \in \mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L})) : F(\ulcorner\urcorner a^+\urcorner\urcorner) = 1\} = \\
&= \varphi^+_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner\urcorner a^+\urcorner\urcorner).
\end{aligned}$$

- Similarly, let us start from expanding the left hand side.

$$\begin{aligned}
&\gamma_{\mathcal{L}}[\varphi^+_{\mathcal{L}}(a^+)] = \{\gamma_{\mathcal{L}}(f) : f \in \mathsf{dpt}(\mathcal{L}), f^+(a^+) = 0\} = \\
&= \{\gamma_{\mathcal{L}}(f) : f \in \mathsf{dpt}(\mathcal{L}), \gamma_{\mathcal{L}}(f)(\ulcorner\urcorner\uparrow a^+\urcorner\urcorner) = 1\} = \\
&= \{F \in \mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L})) : F(\ulcorner\urcorner\uparrow a^+\urcorner\urcorner) = 1\} = \\
&= \varphi^+_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner\urcorner\uparrow a^+\urcorner\urcorner).
\end{aligned}$$

□

We are ready to prove the desired bihomeomorphism.

Proposition 7.1.16. *For any d-frame $\mathcal{L}$, the bijection $\gamma_{\mathcal{L}} : |Sk_{cf}(\mathsf{dpt}(\mathcal{L}))| \cong |\mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L}))|$ is a bihomeomorphism.*

Proof. By definition of the closed-fitted Skula construction, the positive opens of $Sk_{cf}(\mathsf{dpt}(\mathcal{L}))$ are generated by the subbasis

$$\{\varphi^+_{\mathcal{L}}(a^+) : a^+ \in L^+\} \cup \{\varphi^-_{\mathcal{L}}(a^-) : a^- \in L^-\}.$$

On the other hand, the positive opens of $\mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L}))$ are generated by the subbasis

$$\{\varphi^+_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner\ulcorner a^+ \urcorner\urcorner) : a^+ \in L^+\} \cup \{\varphi^+_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner\ulcorner a^- \urcorner\urcorner) : a^- \in L^-\}.$$

Then, by Lemma 7.1.15, we have that the collection of positive subbasic opens of $\mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L}))$ coincides with the collection of forward images of positive subbasic opens of $Sk_{cf}(\mathsf{dpt}(\mathcal{L}))$. This means that the map $\gamma_{\mathcal{L}}$ is a homeomorphism according to the positive topology. For the negative topology, we notice that the negative opens of $Sk_{cf}(\mathsf{dpt}(\mathcal{L}))$ are generated by the subbasis

$$\{\varphi^+_{\mathcal{L}}(a^+)^c : a^+ \in L^+\} \cup \{\varphi^-_{\mathcal{L}}(a^-)^c : a^- \in L^-\}.$$

On the other hand, the negative opens of $\mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L}))$ are generated by the subbasis

$$\{\varphi^-_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner\ulcorner \uparrow a^+ \urcorner\urcorner) : a^+ \in L^+\} \cup \{\varphi^-_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner\ulcorner \uparrow a^- \urcorner\urcorner) : a^- \in L^-\}.$$

Then, by Lemma 7.1.15, we have that the collection of positive subbasic opens of $\mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L}))$ coincides with the collection of forward images of positive subbasic opens of $Sk_{cf}(\mathsf{dpt}(\mathcal{L}))$. This means that the map $\gamma_{\mathcal{L}}$ is a homeomorphism according to the negative topology. □

In proving the following theorem, we use Lemma 4.3.15. Recall that this lemma states that whenever we have frame maps $f : L \to M$ and $g : M \to 2$, we have that $\mathsf{Filt}_\neg(g) \circ \mathsf{Filt}(f) = \mathsf{Filt}_\neg(g \circ f)$. We are now ready to prove that the bihomeomorphisms of the form $\gamma_{\mathcal{L}}$ combine together to form a natural transformation $\gamma : \mathsf{Obj}(\mathbf{dFrm}^{op}) \to \mathsf{Mor}(\mathbf{BiTop})$.

Theorem 7.1.17. *The following square commutes up to natural isomorphism.*

$$\begin{array}{ccc} \mathbf{dFrm} & \xrightarrow{\mathsf{dpt}} & \mathbf{BiTop} \\ {\scriptstyle \mathsf{A}_{cf}}\downarrow & & \downarrow{\scriptstyle Sk_{cf}} \\ \mathbf{dFrm} & \xrightarrow{\mathsf{dpt}} & \mathbf{BiTop} \end{array}$$

Proof. We already know that for every d-frame $\mathcal{L}$ we have a bihomeomorphism $\gamma_{\mathcal{L}} : Sk_{cf}(\mathsf{dpt}(\mathcal{L})) \cong \mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L})))$. It suffices to show that this assignment is a natural transformation. Suppose

that $f : \mathcal{L} \to \mathcal{M}$ is a d-frame map. The naturality square we need to check is the following.

$$\begin{array}{ccc} Sk_{cf}(\mathsf{dpt}(\mathcal{M})) & \xrightarrow{\gamma_{\mathcal{M}}} & \mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{M})) \\ \downarrow{\scriptstyle \mathsf{dpt}(f)} & & \downarrow{\scriptstyle \mathsf{dpt}(\mathsf{A}_{cf}(f))} \\ Sk_{cf}(\mathsf{dpt}(\mathcal{L})) & \xrightarrow{\gamma_{\mathcal{L}}} & \mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L})) \end{array}$$

Since we only need to check equalities of functions, we may instead consider the following simpler square in **Set**. We have also simplified using the fact that dpt sends morphisms to their precomposition map.

$$\begin{array}{ccc} \mathsf{dpt}(\mathcal{M}) & \xrightarrow{\gamma_{\mathcal{M}}} & \mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{M})) \\ \downarrow{\scriptstyle -\circ f} & & \downarrow{\scriptstyle -\circ \mathsf{A}_{cf}(f)} \\ \mathsf{dpt}(\mathcal{L}) & \xrightarrow{\gamma_{\mathcal{L}}} & \mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L})) \end{array}$$

For $g \in \mathsf{dpt}(\mathcal{M})$, we have the following chain of equalities.

$$\begin{aligned} &\gamma_{\mathcal{M}}(g) \circ \mathsf{A}_{cf}(f) = \\ &= (g^+ \oplus g^-, \mathsf{Filt}_\neg(g^+) \oplus \mathsf{Filt}_\neg(f^-)) \circ (f^+ \oplus f^-, \mathsf{Filt}(f^+) \oplus \mathsf{Filt}(f^-)) = \\ &= ((g^+ \circ f^+) \oplus (g^- \circ f^-), (\mathsf{Filt}_\neg(g^+) \circ \mathsf{Filt}(f^+)) \oplus (\mathsf{Filt}_\neg(g^+) \circ \mathsf{Filt}(f^+))) = \\ &= ((g^+ \circ f^+) \oplus (g^- \circ f^-), \mathsf{Filt}_\neg(g^+ \circ f^+) \oplus \mathsf{Filt}_\neg(g^- \circ f^-)) = \\ &= \gamma_{\mathcal{L}}(g \circ f). \end{aligned}$$

□

Since we have the bijection $\gamma_{\mathcal{L}}$, we may view the bispace $\mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L}))$ as a new bitopology on the set of points $|\mathsf{dpt}(\mathcal{L})|$. The positive opens of this new bispace are all determined by con subsets of $L^+ \times L^-$, and the negative ones by tot subsets of $L^+ \times L^-$. The result above tells us that the positive open corresponding to a con subset consists exactly of the points of $\mathsf{dpt}(\mathcal{L})$ that get taken away by quotienting $\mathcal{L}$ by that con update.

Remark 7.1.18. Now that we have a rather simple description of the spectrum of the closed-fitted assembly, we would like to be able to visualize this space. We consider the case where both L^+ and L^- are linear frames. Let us identify the spectrum of $\mathsf{A}_{cf}(\mathcal{L})$ with the closed-fitted Skula bispace of $\mathsf{dpt}(\mathcal{L})$. First, we focus on its positive topology. A natural subbasis for this is given by the sets of the form $\varphi^+_{\mathcal{L}}(a^+)$ together with those of the form $\varphi^-_{\mathcal{L}}(a^-)$.

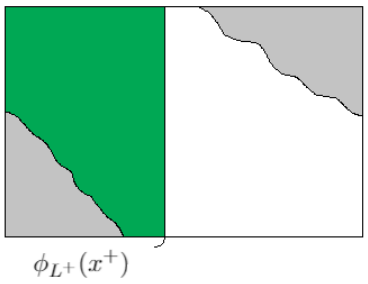


Figure 7.1: A typical subbasic open of the positive topology $Sk_{cf}(\mathsf{dpt}(\mathcal{L}))$ coming from an open of $\mathsf{pt}(L^+)$.

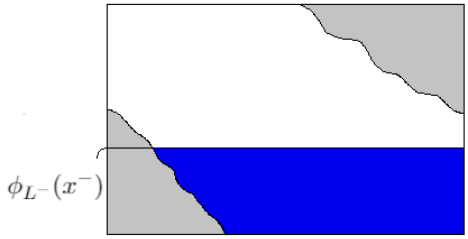


Figure 7.2: A typical subbasic open of the positive topology of $Sk_{cf}(\mathsf{dpt}(\mathcal{L}))$ coming from an open of $\mathsf{pt}(L^-)$.

A typical positive open, then, is a lower left portion. Notice that, by Proposition 7.1.2, these coincide with the complements of the spectra $|\mathsf{dpt}(\mathcal{L}/c)|$. Now, let us move to the negative topology. The negative topology of $Sk_{cf}(\mathsf{dpt}(\mathcal{L}))$ has two types of subbasic opens. These are the remaining two kinds of rectangles: the upper ones and the right ones.

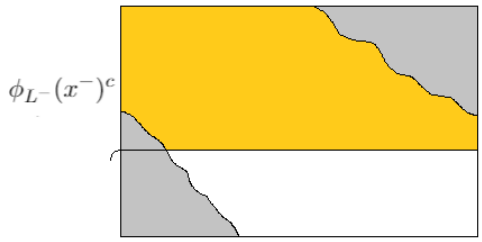


Figure 7.3: A typical subbasic open of the negative topology of $Sk_{cf}(\mathsf{dpt}(\mathcal{L}))$ coming from a closed set of $\mathsf{pt}(L^+)$.

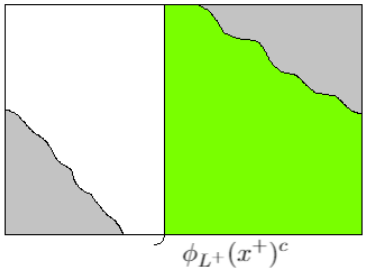
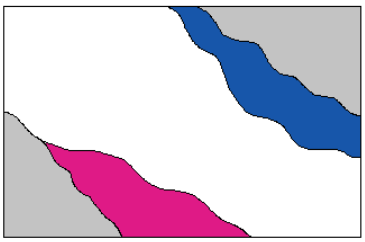


Figure 7.4: A typical subbasic open of the positive topology $Sk_{cf}(\mathsf{dpt}(\mathcal{L}))$ coming from a closed set of $\mathsf{pt}(L^-)$.

Then, a typical negative open of this topology is always an upper right portion. An alternative way to describe this set is that it is the complement of $|\mathsf{dpt}(\mathcal{L}/t)|$ for some tot update t. We now depict a typical positive open of $\mathsf{A}_{cf}(L)$ together with a typical negative one. In Chapter 1, we had built the visual intuition that the d-spectrum of a d-frame (L^+, L^-, c, t) is obtained by taking away a lower left portion and an upper right one from the d-spectrum of (L^+, L^-, c_m, t_m). This intuition can be generalized: increasing the con and the tot component of any d-frame in any way takes away lower left portions and upper left ones from its spectrum, respectively.



We now come back to the more concrete version of the closed-fitted assembly. Recall that we have isomorphisms

$$\iota_c : \mathsf{con}(L^+ \times L^-) \cong L^+ \oplus L^- \qquad \iota_t : \mathsf{tot}(L^+ \times L^-) \cong \mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-)$$
$$c \mapsto \bigvee \{\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner : (x^+, x^-) \in c\} \qquad t \mapsto \bigvee \{\ulcorner \uparrow x^+ \urcorner \wedge \ulcorner \uparrow x^- \urcorner : (x^+, x^-) \in t\}$$

This tells us that we may identify the elements of $L^+ \oplus L^-$ with the con subsets of $L^+ \times L^-$, and we may similarly identify the elements of $\mathsf{Filt}(L^+) \oplus \mathsf{Filt}(L^-)$ with the tot subsets. We already know that the opens of $\mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L}))$ from the first frame component $L^+ \oplus L^-$, i.e. those of the form $\varphi^+_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner \bigvee_i \ulcorner x_i^+ \urcorner \wedge \ulcorner x_i^- \urcorner \urcorner)$ correspond to the complements of the sets $|\mathsf{dpt}(\mathcal{L}/c)|$. But we also know that the element $\bigvee_i \ulcorner x_i^+ \urcorner \wedge \ulcorner x_i^- \urcorner \in L^+ \oplus L^-$ is $\iota(c')$ for some con subset $c' \subseteq L^+ \times L^-$, it is natural to ask what is the relation between c and c'. It turns out that for any con subset $c \subseteq L^+ \times L^-$ the open $\varphi^+_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner \iota_c(c) \urcorner)$ when seen as a subset of $\mathsf{dpt}(\mathcal{L})$ is the same as the open $|\mathsf{dpt}(\mathcal{L}/c)|^c$.

Proposition 7.1.19. *For any d-frame $\mathcal{L}$, we have the following.*

- $\varphi_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner \iota_c(c) \urcorner) = \gamma_{\mathcal{L}}[|\mathsf{dpt}(\mathcal{L}/c)|^c]$ *for every* $c \in \mathsf{con}(L^+ \times L^-)$.
- $\varphi_{\mathsf{A}_{cf}(\mathcal{L})}(\iota_t(t)) = \gamma_{\mathcal{L}}[|\mathsf{dpt}(\mathcal{L}/t)|^c]$ *for every* $t \in \mathsf{tot}(L^+ \times L^-)$.

Proof. Let us prove the two items in turn.

- For every $c \in \mathsf{con}(L^+ \times L^-)$, we may expand the left hand side as follow. In the equality between the third and the fourth line we have used Lemma 7.1.15, and for the one after that we have used that $\gamma_{\mathcal{L}}[-]$ preserves all intersections and unions since it is a bijection.

Finally, for the last step we have used item (3) of Lemma 7.1.2.

$$\begin{aligned}
&\varphi^+_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner \iota_c(c) \urcorner) = \\
&= \varphi^+_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner \bigvee \{\ulcorner x^+ \urcorner \wedge \ulcorner x^- \urcorner) \urcorner : (x^+, x^-) \in c\} = \\
&= \bigcup \{\varphi^+_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner\ulcorner x^+ \urcorner\urcorner) \cap \varphi^+_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner\ulcorner x^- \urcorner\urcorner) : (x^+, x^-) \in c\} = \\
&= \bigcup \{\gamma_{\mathcal{L}}[\varphi^+_{\mathcal{L}}(x^+)] \cap \gamma_{\mathcal{L}}[\varphi^-_{\mathcal{L}}(x^-)] : (x^+, x^-) \in c\} = \\
&= \gamma_{\mathcal{L}}[\bigcup \{\varphi^+_{\mathcal{L}}(x^+) \cap \varphi^-_{\mathcal{L}}(x^-) : (x^+, x^-) \in c\} = \\
&= \gamma_{\mathcal{L}}[|\mathsf{dpt}(\mathcal{L}/c)|^c].
\end{aligned}$$

- For the second equality, we proceed similarly. For the equality between the third and the fourth line, we have used Lemma 7.1.15, for the following one the preservation property of $\gamma_{\mathcal{L}}[-]$, and for the last one item (4) of Lemma 7.1.2.

$$\begin{aligned}
&\varphi^-_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner \iota_t(t) \urcorner) = \\
&= \varphi^-_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner \bigvee \{\ulcorner \uparrow x^+ \urcorner \wedge \ulcorner \uparrow x^- \urcorner) \urcorner : (x^+, x^-) \in t\} = \\
&= \bigcup \{\varphi^-_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner\ulcorner \uparrow x^+ \urcorner\urcorner) \cap \varphi^-_{\mathsf{A}_{cf}(\mathcal{L})}(\ulcorner\ulcorner \uparrow x^- \urcorner\urcorner) : (x^+, x^-) \in t\} = \\
&= \bigcup \{\gamma_{\mathcal{L}}[\varphi^+_{\mathcal{L}}(x^+)^c] \cap \gamma_{\mathcal{L}}[\varphi^-_{\mathcal{L}}(x^-)^c] : (x^+, x^-) \in t\} = \\
&= \gamma_{\mathcal{L}}[\bigcup \{\varphi^+_{\mathcal{L}}(x^+)^c \cap \varphi^-_{\mathcal{L}}(x^-)^c : (x^+, x^-) \in t\} = \\
&= \gamma_{\mathcal{L}}[|\mathsf{dpt}(\mathcal{L}/t)|^c].
\end{aligned}$$

□

7.1.3 The $\mathsf{A}_{\pm}$ functor and its spatial counterpart

Let us move to looking at our second variation of the d-frame assembly. We want to describe the spectrum of $\mathsf{A}_{\pm}(\mathcal{L})$ and show that it corresponds to some bitopological Skula-like construction on $\mathsf{dpt}(\mathcal{L})$. Once we have done this we will express this bitopology in terms of the d-sober subspaces of $\mathsf{dpt}(\mathcal{L})$. We will uncover encoded in the bispace $\mathsf{dpt}(\mathsf{A}_{\pm}(\mathcal{L}))$ an interplay between the d-frame quotients of $\mathcal{L}$ obtained from frame quotients of the L^+ and the L^- components. We will also see that this is just a rephrasing of the interplay between the d-sober positive and negative bisubspaces of $\mathsf{dpt}(\mathcal{L})$.

For a bispace X, let us define the topological space $Sk_{\pm}(X)$ as the bispace such that

- its underlying set of points is that of X,
- its positive opens are the topology generated by the positive opens and the positive closed sets,

- its negative opens are the topology generated by the negative open sets and the negative closed sets of X.

Alternatively, the negative opens are the Skula topology relative to $(X, \Omega^+(X))$, and the negative opens are the Skula topology relative to $(X, \Omega^-(X))$. We call the bispace $Sk_\pm(X)$ the *positive-negative* Skula bispace of X.

Proposition 7.1.20. *For any d-frame $\mathcal{L}$, the space $Sk_\pm(\mathsf{dpt}(\mathcal{L}))$ is such that*

- *Its positive closed sets are exactly the sets of the form $|\mathsf{dpt}(\mathcal{L}/R^+)|$ for R^+ a relation on L^+, and these sets coincide with the underlying sets of the d-sober positive bisubspaces of $\mathsf{dpt}(\mathcal{L})$;*
- *Its negative closed sets are exactly the sets of the form $|\mathsf{dpt}(\mathcal{L}/R^-)|$ for R^- a relation on L^-, and these sets coincide with the underlying sets of the d-sober negative bisubspaces of $\mathsf{dpt}(\mathcal{L})$.*

Additionally, its patch is the coarsest topology on $\mathsf{dpt}(\mathcal{L})$ in which all d-sober subspaces of $\mathsf{dpt}(\mathcal{L})$ are closed.

Proof. We need to show that the positive open sets of $Sk_\pm(\mathsf{dpt}(\mathcal{L}))$ are exactly the sets of of the form $|\mathcal{L}/R^+|$ for some relation R^+ on L^+. A subset of $\mathsf{dpt}(\mathcal{L})$ is positive open in $Sk_\pm(\mathsf{dpt}(\mathcal{L}))$ if and only if it is of the form

$$\bigcup\{\varphi^+_{\mathcal{L}}(a^+_i) \cap \varphi^+_{\mathcal{L}}(b^+_i)^c : i \in I\}.$$

Let us now define the relation R_I on L^+ as the set of pairs $\{(a^+_i, b^+_i) : i \in I\}$. By item (3) of Lemma 7.1.2, we know that this is the same as $\mathsf{dpt}(\mathcal{L}/R_I)$. Then, every positive open of $Sk_\pm(\mathsf{dpt}(\mathcal{L}))$ is of the form $\mathcal{L}/R^+$. The converse, i.e. the fact that any bisubspace of $\mathsf{dpt}(\mathcal{L})$ of the form $\mathsf{dpt}(\mathcal{L}/R^+)$ is a positive open of $Sk_\pm(\mathsf{dpt}(\mathcal{L}))$ follows immediately from item (3) of Lemma 7.1.2. □

Let us now extend the definition of $Sk_\pm$ to morphisms. For a bicontinuous map $f : X \to Y$ we define the map $Sk_\pm(f) : Sk_\pm(X) \to Sk_\pm(Y)$ to be simply the function f. As usual, bicontinuity of $Sk_\pm(f)$ holds because - since f is bicontinuous - its preimage map preserves the property of being an open set in one of the two topologies, as well as that of being closed.

Lemma 7.1.21. *The assignment $Sk_\pm : \mathbf{BiTop} \to \mathbf{BiTop}$ is a functor.*

The next step is to show that, for every d-frame $\mathcal{L}$, we have a bihomeomorphism between the bispace $\mathsf{dpt}(\mathsf{A}_{\pm}(\mathcal{L}))$ and $Sk_{\pm}(\mathsf{dpt}(\mathcal{L}))$. Recall that, for any d-frame $\mathcal{L}$, we had defined the structure $\mathsf{A}_{\pm}(\mathcal{L})$ to be the quotient of the d-frame

$$(L^+ \oplus \mathsf{Filt}(L^+), L^- \oplus \mathsf{Filt}(L^-), c_m, t_m)$$

by the relation structure $\mathcal{PM}_L := (C_L^+, C_L^-, \ulcorner c_L \urcorner, \ulcorner t_L \urcorner)$, for which we have used the following definitions

- C_L^+ is the relation making elements $\ulcorner a^+ \urcorner$ and $\ulcorner \uparrow a^+ \urcorner$ into mutual complements,
- C_L^- is the relation making elements $\ulcorner a^- \urcorner$ and $\ulcorner \uparrow a^- \urcorner$ into mutual complements,
- $\ulcorner c_L \urcorner = \{(\ulcorner x^+ \urcorner, \ulcorner x^- \urcorner) : (x^+, x^-) \in c_L\}$,
- $\ulcorner t_L \urcorner = \{(\ulcorner x^+ \urcorner, \ulcorner x^- \urcorner) : (x^+, x^-) \in t_L\}$.

Let us extend this to an endofunctor in **dFrm**, following the usual technique.

Lemma 7.1.22. *Whenever $f : \mathcal{L} \to \mathcal{M}$ is a d-frame map, let F be the map $(f^+ \oplus \mathsf{Filt}(f^+), f^- \oplus \mathsf{Filt}(f^-))$, seen as a map*

$$(L^+ \oplus \mathsf{Filt}(L^+), L^- \oplus \mathsf{Filt}(L^-), c_m, t_m) \to (M^+ \oplus \mathsf{Filt}(M^+), M^- \oplus \mathsf{Filt}(M^-), c_m, t_m).$$

This is such that $F[\mathcal{PM}_L] \leq \mathcal{PM}_M$.

Proof. Whenever $a^+ \in L^+$, by definition of the map F we have that the forward image of a typical pair $(\ulcorner a^+ \urcorner, \ulcorner \uparrow a^+ \urcorner)$ in C_L^+ is $(\ulcorner f^+(a^+) \urcorner, \ulcorner f^+(a^+) \urcorner)$, which by definition is a pair in C_M^+. Then, we have that $F[C_L^+] \subseteq C_M^+$. To show that $F[\ulcorner c_L \urcorner] \subseteq \ulcorner c_M \urcorner$ and the similar inclusion for the tot components, the argument is similar to that in Lemma 7.1.10. □

By Lemma 7.1.22, and by Lemma 5.1.5, we then have learned that for every d-frame map $f : \mathcal{L} \to \mathcal{M}$ there is a d-frame map

$$\begin{aligned}
&\mathsf{A}_{\pm}(f) : \mathsf{A}_{\pm}(\mathcal{L}) \to \mathsf{A}_{\pm}(\mathcal{M}),\\
&\ulcorner\ulcorner x^+ \urcorner\urcorner \mapsto \ulcorner\ulcorner f^+(x^+) \urcorner\urcorner,\\
&\ulcorner\ulcorner \uparrow x^+ \urcorner\urcorner \mapsto \ulcorner\ulcorner \uparrow f^+(x^+) \urcorner\urcorner.
\end{aligned}$$

We take this to be the definition of the assignment $\mathsf{A}_{\pm} : \mathbf{dFrm} \to \mathbf{dFrm}$ on morphisms. This assignment clearly respects identities. For compositions, the argument is similar to that showing that A_{cf} respects compositions. Then, the assignment is functorial.

Lemma 7.1.23. *The assignment* $\mathsf{A}_{\pm} : \mathbf{dFrm} \to \mathbf{dFrm}$ *is a functor.*

We now want to show that there is a bihomeomorphism $\mathsf{dpt}(\mathsf{A}_{\pm}(\mathcal{L})) \cong Sk_{\pm}(\mathsf{dpt}(\mathcal{L}))$. Let us now see how to obtain a point of $\mathsf{A}_{\pm}(\mathcal{L})$ for every point of $\mathcal{L}$.

Lemma 7.1.24. *For any d-frame* $\mathcal{L}$*, the map*

$$(f^+ \oplus \mathsf{Filt}_{\neg}(f^+), f^- \oplus \mathsf{Filt}_{\neg}(f^-)) : (L^+ \oplus \mathsf{Filt}(L^+), L^- \oplus \mathsf{Filt}(L^-), c_m, t_m) \to \mathbf{2}$$

respects the relation structure $\mathcal{PM}_L$*.*

Proof. For our claim, it suffices to show that the map satisfies the next three conditions.

- That whenever $x^+ \in L^+$ we have that the elements $f^+(\ulcorner x^+ \urcorner)$ and $\mathsf{Filt}_{\neg}(f^+)(\ulcorner \uparrow x^+ \urcorner)$ are complements in 2. This follows immediately from the definition of $\mathsf{Filt}_{\neg}(-)$.
- That whenever $(x^+, x^-) \in c_L$ we have $(\ulcorner f^+(x^+) \urcorner, \ulcorner f^-(x^-) \urcorner) \in c_2$. This follows immediately from the fact that f is a d-frame map.
- That whenever $(x^+, x^-) \in t_L$ we have $(\ulcorner f^+(x^+) \urcorner, \ulcorner f^-(x^-) \urcorner) \in t_2$. This holds by an argument analogous to the one above. □

Lemma 7.1.24 above is telling us that for every d-frame $\mathcal{L}$ and any point $f \in \mathsf{dpt}(\mathcal{L})$ determines a point

$$(f^+ \oplus \mathsf{Filt}_{\neg}(f^+), f^- \oplus \mathsf{Filt}_{\neg}(f^-)) : (L^+ \oplus \mathsf{Filt}(L^+), L^- \oplus \mathsf{Filt}(L^-), c_m, t_m) \to \mathbf{2}$$

and there is a unique d-frame map $\rho_{\mathcal{L}}(f) : \mathsf{A}_{\pm}(\mathcal{L}) \to \mathbf{2}$ such that the map above is $\rho_{\mathcal{L}}(f) \circ q_{\mathcal{PM}_L}$. In other words, Lemma 7.1.24 is telling us that the following assignment is well-defined.

$$\begin{aligned} &\rho_{\mathcal{L}} : |\mathsf{dpt}(\mathcal{L})| \to |\mathsf{dpt}(\mathsf{A}_{\pm}(\mathcal{L}))| \\ &f \mapsto \overline{(f^+ \oplus \mathsf{Filt}_{\neg}(f^+), f^- \oplus \mathsf{Filt}_{\neg}(f^-))}. \end{aligned}$$

Injectivity of this map is clear. To show that it is a bijection, it only remains to show that it is surjective.

Lemma 7.1.25. *For any d-frame* $\mathcal{L}$*, the map* $\rho_{\mathcal{L}} : |\mathsf{dpt}(\mathcal{L})| \to |\mathsf{dpt}(\mathsf{A}_{\pm}(\mathcal{L}))|$ *is a bijection.*

Proof. We notice that, by universal property of the frame coproduct, every point of $\mathsf{dpt}(\mathsf{A}_{\pm}(\mathcal{L}))$ is of the form $(f^+ \oplus f_F^+, f^- \oplus f_F^-)$. This point also respects the relation structure $\mathcal{PM}_L$, and this conditions amounts to the following three facts.

- Whenever $x^+ \in L^+$ we have that the elements $f^+(\ulcorner x \urcorner)$ and $f_F^+(\ulcorner \uparrow x^+ \urcorner)$ are complements in 2. In particular, we have that $f_F^+(\ulcorner \uparrow x^+ \urcorner) = \neg \ulcorner f^+(x^+) \urcorner$. Then, we deduce that $f^+ = \mathsf{Filt}_\neg(f^+)$. We deduce by a similar argument that $f_F^- = \mathsf{Filt}_\neg(f^-)$.
- Whenever $(x^+, x^-) \in c_L$ we have that $(\ulcorner f^+(x^+) \urcorner, \ulcorner f^-(x^-) \urcorner) \in c_2$.
- Whenever $(x^+, x^-) \in t_L$ we have that $(\ulcorner f^+(x^+) \urcorner, \ulcorner f^-(x^-) \urcorner) \in t_2$.

From the last two facts, we deduce that the pairing of frame maps (f^+, f^-) respects the con and the tot components of $\mathcal{L}$, and so it is a d-frame map. Combining this with the first fact, we deduce that our arbitrary point of the positive-negative assembly is actually the forward image under $\rho_{\mathcal{L}}$ for a d-frame map $(f^+, f^-) : \mathcal{L} \to \mathbf{2}$. □

Let us now refine this and show that this bijection is a bihomeomorphism.

Lemma 7.1.26. *Suppose that $\mathcal{L}$ is a d-frame. The following holds for every $f \in \mathsf{dpt}(\mathcal{L})$.*

- *For every $a^+ \in L^+$ we have that $f^+(a^+) = 1$ if and only if $\rho_{\mathcal{L}}(f)(\ulcorner \ulcorner a^+ \urcorner \urcorner) = 1$.*
- *For every $a^+ \in L^+$ we have that $f^+(a^+) = 1$ if and only if $\rho_{\mathcal{L}}(f)(\ulcorner \ulcorner \uparrow a^+ \urcorner \urcorner) = 0$.*

Proof. The proof is similar to that of Lemma 7.1.14. □

Lemma 7.1.27. *For every d-frame $\mathcal{L}$ we have the following.*

- $\rho_{\mathcal{L}}[\varphi^+_{\mathcal{L}}(a^+)] = \varphi^+_{\mathsf{A}_\pm(\mathcal{L})}(\ulcorner \ulcorner a^+ \urcorner \urcorner)$.
- $\rho_{\mathcal{L}}[\varphi^+_{\mathcal{L}}(a^+)^c] = \varphi^+_{\mathsf{A}_\pm(\mathcal{L})}(\ulcorner \ulcorner \uparrow a^+ \urcorner \urcorner)$.
- $\rho_{\mathcal{L}}[\varphi^-_{\mathcal{L}}(a^-)] = \varphi^-_{\mathsf{A}_\pm(\mathcal{L})}(\ulcorner \ulcorner a^- \urcorner \urcorner)$.
- $\rho_{\mathcal{L}}[\varphi^-_{\mathcal{L}}(a^-)^c] = \varphi^-_{\mathsf{A}_\pm(\mathcal{L})}(\ulcorner \ulcorner \uparrow a^- \urcorner \urcorner)$.

Proof. The proof is similar to that of Lemma 7.1.15. □

Proposition 7.1.28. *For any d-frame $\mathcal{L}$, the bijection*

$$\begin{aligned} &\rho_{\mathcal{L}} : \mathsf{dpt}(\mathcal{L}) \to \mathsf{dpt}(\mathsf{A}_\pm(\mathcal{L})) \\ &(f^+, f^-) \mapsto (f^+ \oplus \mathsf{Filt}_\neg(f^+), f^- \oplus \mathsf{Filt}_\neg(f^-)) \end{aligned}$$

is a bihomeomorphism.

Proof. We show that the map is a homeomorphism for the positive topology. It suffices to show that the subbasic opens of $\mathsf{dpt}(\mathsf{A}_\pm(\mathcal{L}))$ coincide with the forward images of the subbasic opens of $Sk_\pm(\mathcal{L})$. By definition of the positive-negative Skula bispace of $\mathsf{dpt}(\mathcal{L})$, we know that its positive topology is the same as the Skula space of $(|\mathsf{dpt}(\mathcal{L})|, \varphi^+_{\mathcal{L}}[L^+])$. Then, this is generated by the subbasis

$$\{\varphi^+_{\mathcal{L}}(a^+) : a^+ \in L^+\} \cup \{\varphi^+_{\mathcal{L}}(a^+)^c : a^+ \in L^+\}.$$

On the other hand, the collection of positive opens of $\mathsf{dpt}(\mathsf{A}_\pm(\mathcal{L}))$ is generated by the subbasis

$$\{\varphi^+_{\mathsf{A}_\pm(\mathcal{L})}(\ulcorner\urcorner a^+ \urcorner\urcorner) : a^+ \in L^+\} \cup \{\varphi^+_{\mathsf{A}_\pm(\mathcal{L})}(\ulcorner\urcorner \uparrow a^+ \urcorner\urcorner) : a^+ \in L^+\}.$$

By Lemma 7.1.27, then, indeed the positive opens of $\mathsf{dpt}(\mathsf{A}_\pm(\mathcal{L}))$ are exactly the forward images under $\rho_{\mathcal{L}}$ of the subbasic opens of $Sk_\pm(\mathsf{dpt}(\mathcal{L}))$. □

Let us now show that pasting together all the bihomeomorphisms of the form $\rho_{\mathcal{L}}$ for some d-frame $\mathcal{L}$ gives us a natural isomorphism.

Theorem 7.1.29. *The following square commutes up to natural isomorphism.*

$$\begin{array}{ccc} \mathbf{dFrm} & \xrightarrow{\mathsf{dpt}} & \mathbf{BiTop} \\ {\scriptstyle \mathsf{A}_\pm}\downarrow & & \downarrow{\scriptstyle Sk_\pm} \\ \mathbf{dFrm} & \xrightarrow{\mathsf{pt}} & \mathbf{BiTop} \end{array}$$

Proof. Since we already have an assignment $\rho : \mathsf{Obj}(\mathbf{dFrm}^{\mathbf{op}}) \to \mathsf{Mor}(\mathbf{BiTop})$ which is an isomorphism evaluated at each component, we only need to check its naturality. Let $f : \mathcal{L} \to \mathcal{M}$ be a d-frame map. The naturality square to check is the following.

$$\begin{array}{ccc} Sk_\pm(\mathsf{dpt}(\mathcal{M})) & \xrightarrow{\rho_{\mathcal{M}}} & \mathsf{dpt}(\mathsf{A}_\pm(\mathcal{M})) \\ {\scriptstyle \mathsf{dpt}(f)}\downarrow & & \downarrow{\scriptstyle \mathsf{dpt}(\mathsf{A}_\pm(f))} \\ Sk_\pm(\mathsf{dpt}(\mathcal{L})) & \xrightarrow{\rho_{\mathcal{L}}} & \mathsf{dpt}(\mathsf{A}_\pm(\mathcal{L})) \end{array}$$

We are only interested in showing an equality between functions, so we only need to check the commutativity of the following in **Set**.

$$\begin{array}{ccc} \mathsf{dpt}(\mathcal{M}) & \xrightarrow{\rho_{\mathcal{M}}} & \mathsf{dpt}(\mathsf{A}_\pm(\mathcal{M})) \\ {\scriptstyle -\circ f}\downarrow & & \downarrow{\scriptstyle -\circ \mathsf{A}_\pm(f)} \\ \mathsf{dpt}(\mathcal{L}) & \xrightarrow{\rho_{\mathcal{L}}} & \mathsf{dpt}(\mathsf{A}_\pm(\mathcal{L})) \end{array}$$

Suppose that $g \in \mathsf{dpt}(\mathcal{M})$. We have the following chain of equalities.

$$\begin{aligned}
&\rho_{\mathcal{M}}(g) \circ \mathsf{A}_{\pm}(f) = \\
&= (g^+ \oplus \mathsf{Filt}_{\neg}(g^+), g^- \oplus \mathsf{Filt}_{\neg}(g^-)) \circ (f^+ \oplus \mathsf{Filt}(f^+), f^- \oplus \mathsf{Filt}(f^-)) = \\
&= ((g^+ \circ f^+) \oplus \mathsf{Filt}_{\neg}(g^+) \circ \mathsf{Filt}(f^+), (g^- \circ f^-) \oplus (\mathsf{Filt}_{\neg}(g^-) \circ \mathsf{Filt}(f^-))) = \\
&= ((g^+ \circ f^+) \oplus \mathsf{Filt}_{\neg}(g^+ \circ f^+), (g^- \circ f^-) \oplus \mathsf{Filt}_{\neg}(g^- \circ f^-)) = \\
&= \rho_{\mathcal{L}}(g \circ f).
\end{aligned}$$

□

Since we have the bijection $\rho_{\mathcal{L}}$, we may view the bispace $\mathsf{dpt}(\mathsf{A}_{\pm}(\mathcal{L}))$ as a new bitopology on the set of points $|\mathsf{dpt}(\mathcal{L})|$. The positive opens of this new bispace are all determined by congruences C^+ on the frame L^+, and the negative ones by congruences C^- on L^-.

Remark 7.1.30. Let us now see how we can visualize the bitopology of $\mathsf{pt}(\mathsf{A}_{\pm}(\mathcal{L}))$, by identifying it with the positive-negative Skula bispace of $\mathsf{dpt}(\mathcal{L})$. For this, we select yet another way of pairing the four types of subbasic rectangles. This time, we are pairing the two vertical types of rectangles together, and the two horizontal ones. We then exhaust all the possible ways of partitioning the four subbasic rectangles into groups of two.

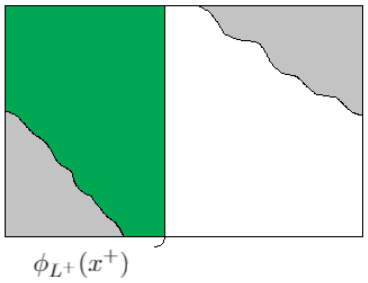


Figure 7.5: A typical subbasic open of the positive topology $Sk_{\pm}(\mathsf{dpt}(\mathcal{L}))$ coming from an open of $\mathsf{pt}(L^+)$.

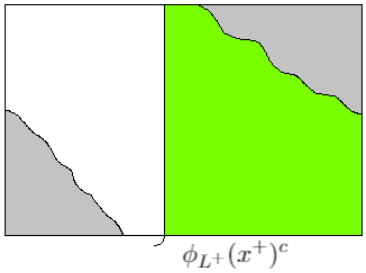


Figure 7.6: A typical subbasic open of the positive topology of $Sk_{\pm}(\mathsf{dpt}(\mathcal{L}))$ coming from closed set of $\mathsf{pt}(L^+)$.

A typical positive open of $Sk_{\pm}$, as we know, is a d-sober positive bisubspace of $\mathsf{dpt}(\mathcal{L})$. As we know, the positive bisubspaces of $\mathsf{dpt}(\mathcal{L})$ stretch across all negative coordinates that they intersect, and a typical one is thus of a vertical shape. Of course, we could have seen this also from the shape of the subbasic opens that generate this topology. For the negative topology, let us look at the two kinds of horizontal rectangles.

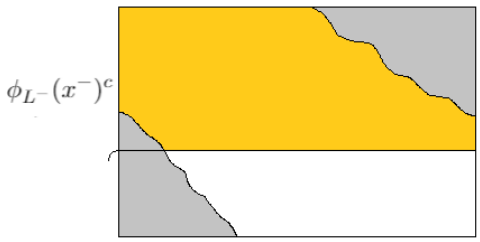


Figure 7.7: A typical subbasic open of the negative topology of $Sk_{\pm}(\mathsf{dpt}(\mathcal{L}))$ coming from an open set of $\mathsf{pt}(L^-)$.

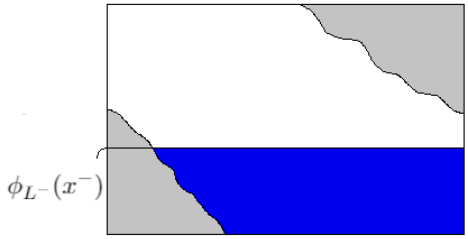
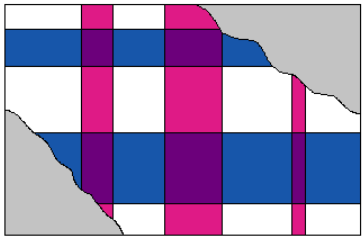


Figure 7.8: A typical subbasic open of the positive topology $Sk_{\pm}(\mathsf{dpt}(\mathcal{L}))$ coming from a closed set of $\mathsf{pt}(L^-)$.

Finally, as a summary we depict a typical positive open together with a typical negative open. We recall that the positive opens here are the same as the complements of the d-sober positive bisubspaces of $\mathsf{dpt}(\mathcal{L})$, and that the negative opens are the complements of the d-sober negative bisubspaces. In the same way in which the duality between closed and (finitary) fitted bisubspaces was represented visually by the duality "lower left-upper right", the duality between positive and negative bisubspaces is represented visually by the duality "horizontal-vertical".



Let us now look at the more concrete version of the positive-negative d-frame assembly, namely the d-frame $\mathsf{A}^{\mathsf{c}}_{\pm}(\mathcal{L})$. The d-frame $\mathsf{A}_{\pm}(\mathcal{L})$ may be defined as the one obtained quotienting the d-frame $((L^+ \oplus \mathsf{Filt}(L^+))/C^+, (L^- \oplus \mathsf{Filt}(L^-))/C^-, c_m, t_m)$ by the relation structure $(\mathsf{id}^+, \mathsf{id}^-, \ulcorner c_L \urcorner, \ulcorner t_L \urcorner)$. We now have on one side a d-frame which is $(\mathsf{A}(L^+), \mathsf{A}(L^-), c_m, t_m)$ quotiented by some relation structure, and on the other its d-spectrum, which we know being isomorphic to a bispace (that is $Sk_{\pm}(\mathsf{dpt}(\mathcal{L}))$) whose positive opens are sets $|\mathsf{dpt}(\mathcal{L}/C^+)|$ and whose negative opens are of the form $|\mathsf{dpt}(\mathcal{L}/C^-)|$.

We know that the first frame component is isomorphic to $\mathsf{A}(L^+)$ and that the isomorphism

is given by

$$\iota^+ : \mathsf{A}(L^+) \to (L^+ \oplus \mathsf{Filt}(L^+))/C^+$$
$$C^+ \mapsto \bigvee\{\ulcorner x^+ \urcorner \wedge \ulcorner \uparrow y^+ \urcorner : \nabla(x^+) \cap \Delta(y^+) \subseteq C^+\}.$$

This frame map, together with the analogous negative one for the negative component, yields an isomorphism $\mathsf{A}^{\mathsf{c}}_{\pm}(\mathcal{L}) \cong \mathsf{A}_{\pm}(\mathcal{L})$. We now ask what is the general form of a typical open $\varphi^+_{\mathsf{A}_{\pm}(\mathcal{L})}(\iota^+(C^+))$ of $\mathsf{dpt}(\mathsf{A}_{\pm}(\mathcal{L}))$, in terms of concrete frame congruences on L^+. The question is: what is the connection between the open $\varphi^+_{\mathsf{A}_{\pm}(\mathcal{L})}(\ulcorner C^+ \urcorner)$ and the open $|\mathsf{dpt}(\mathcal{L}/C^+)^c|$? Once again, the connection is the simplest possible one: they are the same. We will now see that the open coming from a congruence C^+ is exactly the set of points that gets taken away from $\mathsf{dpt}(\mathcal{L})$ when quotienting $\mathcal{L}/C^+$.

Proposition 7.1.31. *For any d-frame $\mathcal{L}$, we have the following.*

- $\rho_{\mathcal{L}}[|\mathsf{dpt}(\mathcal{L}/C^+)|^c] = \varphi^+_{\mathsf{A}_{\pm}(\mathcal{L})}(\ulcorner \iota^+(C^+) \urcorner)$.
- $\rho_{\mathcal{L}}[|\mathsf{dpt}(\mathcal{L}/C^+|)^c] = \varphi^+_{\mathsf{A}_{\pm}(\mathcal{L})}(\ulcorner \iota^+(C^-) \urcorner)$.

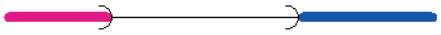
Proof. Let us prove the first claim. For the first part, let us expand the left hand side. For the equality between the second and the third line, we have used the fact that forward images from bijective functions preserve arbitrary intersections and unions. For the equality after we have used Lemma 7.1.27.

$$\begin{aligned}
&\rho_{\mathcal{L}}[|\mathsf{dpt}(\mathcal{L}/C^+)|^c] = \\
&= \rho_{\mathcal{L}}[\bigcup\{\varphi^+_L(a^+) \cap \varphi^+_L(b^+)^c : a^+ C^+ b^+\}] = \\
&= \bigcup\{\rho_{\mathcal{L}}[\varphi^+_L(a^+)] \cap \rho_{\mathcal{L}}[\varphi^+_L(b^+)^c] : a^+ C^+ b^+\} = \\
&= \bigcup\{\varphi^+_{\mathsf{A}_{\pm}(\mathcal{L})}(\ulcorner\ulcorner a^+ \urcorner\urcorner) \cap \varphi_{\mathsf{A}_{\pm}(\mathcal{L})}(\ulcorner\ulcorner \uparrow b^+ \urcorner\urcorner) : a^+ C^+ b^+\} = \\
&= \varphi^+_{\mathsf{A}_{\pm}(\mathcal{L})}(\bigvee\{\ulcorner\ulcorner a^+ \urcorner\urcorner \wedge \ulcorner\ulcorner \uparrow b^+ \urcorner\urcorner : a^+ C^+ b^+\}) = \\
&= \varphi_{\mathsf{A}_{\pm}(\mathcal{L})}(\ulcorner \iota^+(C^+) \urcorner).
\end{aligned}$$

□

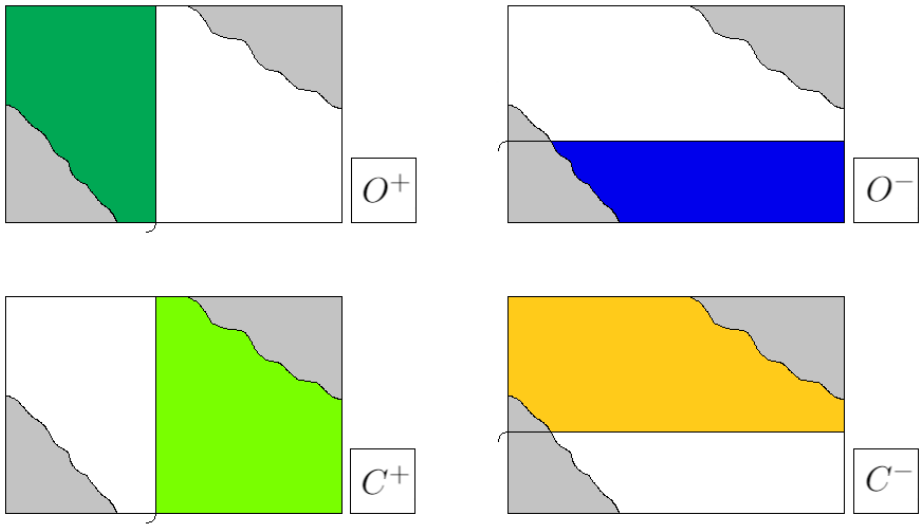
7.2 A visual summary

To conclude this chapter, we gather all of the visual intuition that we have gained through this chapter. For any frame L we have its assembly $\mathsf{A}(L)$. If we take L to be linear, then we may visualize its spectrum as a line, where opens are left closed segments. The assembly $\mathsf{A}(L)$ has a spectrum which may be viewed as a bitopological space with $|\mathsf{pt}(L)|$ as its underlying set of points. Its positive opens are just the opens of the original topology, and its negative opens are the topology generated by the closed sets of $\mathsf{pt}(L)$. Below we depict a typical positive and a typical negative open of $Sk(\mathsf{pt}(L))$, seen as a bitopology.



These two topologies "generate" all the sober subspaces of L in the sense that all sober subspaces are patch-closed in this bitopology. There is no other obvious way of dividing the topology generated by these sets (namely $Sk(\mathsf{pt}(L))$) into two topologies.

When we are dealing with a pointfree bispace $\mathcal{L}$, instead, we have to look at the coarsest topology on $\mathsf{dpt}(\mathcal{L})$ in which all d-sober bisubspaces are closed. This is the topology generated by the positive opens, together with the positive closed sets, together with the negative opens, together with the negative closed sets. Once again, to visualize this let us assume that $\mathcal{L}$ is a linear d-frame. Then, these four kinds of sets correspond to four kinds of rectangles. Suppose that our set of types of rectangles is $\{O^+, O^-, C^+, C^-\}$, for "positive opens, negative opens, positive closed sets, negative closed sets".



There are only three ways of partitioning a set of four elements $\{a, b, c, d\}$ into two sets of two, namely $\{\{a, b\}, \{c, d\}\}$, $\{\{a, c\}, \{b, d\}\}$, and $\{\{a, d\}, \{b, c\}\}$. Each such partition of $\{O^+, O^-, C^+, C^-\}$ determines a bitopological space whose patch is the topology generated by $O^+ \cup O^- \cup C^+ \cup C^-$, in the sense that a partition $\{\{x_1, x_2\}, \{x_3, x_4\}\}$ determines the bitopological space in which the positive topology is the one generated by the collection $x_1 \cup x_2$, and the negative one the one generated by the collection $x_3 \cup x_4$. The three partitions of $\{O^+, O^-, C^+, C^-\}$ into two sets of cardinality two correspond to the three Skula bispaces. Let us summarize the information that we have gathered about each of these.

- The partition $\{\{O^+, O^-\}, \{C^+, C^-\}\}$ yields the closed-fitted Skula space. This is the spectrum of the closed-fitted assembly $\mathsf{A}_{cf}(\mathcal{L})$.

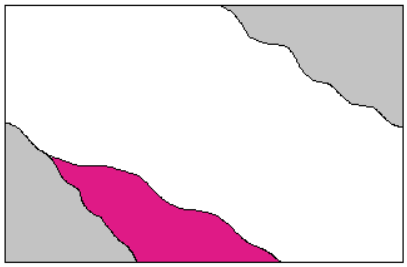


Figure 7.9: A typical positive open of $\mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L}))$ is a lower left portion of the partial rectangle $\mathsf{dpt}(\mathcal{L})$. These sets also coincide with the patch-open sets of $\mathsf{dpt}(\mathcal{L})$, and the complements $|\mathsf{dpt}(\mathcal{L}/c)|^c$ for a con update c.

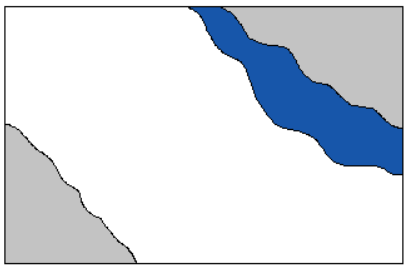


Figure 7.10: A typical negative open of $\mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L}))$ is an upper right portion of $\mathsf{dpt}(\mathcal{L})$. These sets also coincide with the complements of the finitary patch-fitted sets of $\mathsf{dpt}(\mathcal{L})$, and the complements $|\mathsf{dpt}(\mathcal{L}/t)|^c$ for a tot update t.

- The partition $\{\{O^+, C^+\}, \{O^-, C^-\}\}$ yields the positive-negative Skula space. This is the spectrum of the positive-negative assembly $\mathsf{A}_{\pm}(\mathcal{L})$.

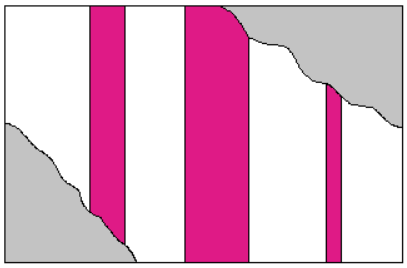


Figure 7.11: A typical positive open of $\mathsf{dpt}(\mathsf{A}_{\pm}(\mathcal{L}))$ is a collection of vertical portions of the partial rectangle $\mathsf{dpt}(\mathcal{L})$. These sets also coincide with the complements of the d-sober positive bisubspaces of $\mathsf{dpt}(\mathcal{L})$, and the complements $|\mathsf{dpt}(\mathcal{L}/C^+)|^c$ for a congruence C^+ on L^+.

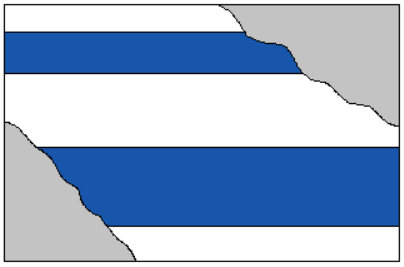


Figure 7.12: A typical negative open of $\mathsf{dpt}(\mathsf{A}_{\pm}(\mathcal{L}))$ is a collection of horizontal portions of the partial rectangle $\mathsf{dpt}(\mathcal{L})$. These sets also coincide with the complements of the d-sober negative bisubspaces of $\mathsf{dpt}(\mathcal{L})$, and the complements $|\mathsf{dpt}(\mathcal{L}/C^-)|^c$ for a congruence C^- on L^-.

- The partition $\{\{O^+, C^-\}, \{O^-, C^+\}\}$ yields the Skula bispace. Let us look at its point-free version $\mathsf{A}(\mathcal{L})$.

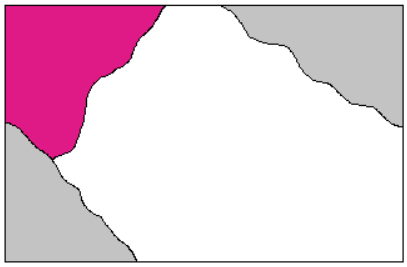


Figure 7.13: A typical positive open of $\mathsf{dpt}(\mathsf{A}(\mathcal{L}))$ is an upper left portion of the partial rectangle $\mathsf{dpt}(\mathcal{L})$. These sets also coincide with those of the topology generated by the positive opens and the negative closed sets of $\mathsf{dpt}(\mathcal{L})$.

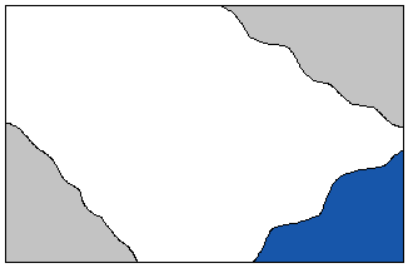


Figure 7.14: A typical positive open of $\mathsf{dpt}(\mathsf{A}(\mathcal{L}))$ is an lower right portion of the partial rectangle $\mathsf{dpt}(\mathcal{L})$. These sets also coincide with those of the topology generated by the positive opens and the negative closed sets of $\mathsf{dpt}(\mathcal{L})$.

Chapter 8

From point-set to pointfree bisubspaces and back

Introduction

One of the features distinguishing pointfree topology from point-set topology is the behaviour of subspaces. While for a space X the lattice of its subspaces is simply the powerset $\mathcal{P}(X)$ of its underlying set, the lattice $\mathsf{S}(L)$ of sublocales of a frame L is in general just a coframe. In particular, the lattice $\mathsf{S}(\Omega(X))$ of sublocales of the frame of open sets $\Omega(X)$ of a space X need not represent perfectly the subspaces of X, and might have in general many nonspatial sublocales. An account of the theory of locales and sublocales can be found in [21, 26, 37]. A question that we address in this chapter is what are those frames L such that $\mathcal{P}(\mathsf{pt}(L))$ is isomorphic to the collection of all *spatial* sublocales in $\mathsf{S}(L)$. These are the frames for which $\mathsf{pt}(L)$ is a T_D space. We provide several characterizations of these frames and, in particular, we show that the condition is equivalent to the spatialization of $\mathsf{S}(L)^{op}$ being Boolean, as well as all of the primes in L being *weakly covered*.

Our starting point is the observation that for any frame L there is an order isomorphism between the collection of *sobrifications* of the subspaces of $\mathsf{pt}(L)$ (ordered under subspace inclusion) and that of the *spatializations* of the sublocales of L. Both these collections are coframes which we denote as $\mathsf{sob}[\mathcal{P}(\mathsf{pt}(L))]$ and $\mathsf{sp}[\mathsf{S}(L)]$, respectively. We will show that these two coframes are naturally seen as part of a commuting diagram in the category of coframes of the following form:

$$\begin{array}{ccc} \mathsf{sp}[\mathsf{S}(L)] & \xrightarrow{\mathsf{pt}(\cong)} & \mathsf{sob}[\mathcal{P}(\mathsf{pt}(L))] \\ \Big\uparrow{\scriptstyle \mathsf{sp}} & & \Big\downarrow \\ \mathsf{S}(L) & \xrightarrow{\mathsf{pt}} & \mathcal{P}(\mathsf{pt}'(L)) \end{array}$$

In Section 8.2, we will go as far as we can proving analogous results for d-frames. The analysis will be centered around an axiom for bispaces which plays a role in the d-frame duality analogous to that of the T_D axiom in the frame duality. Recall that we say that a space X is T_D when for every point $x \in X$ there are opens U and V such that $\{x\} = U \backslash V$.

Proposition 8.0.1. *The following are equivalent conditions on a space X.*

1. *The space X is T_D.*
2. *If for two subspaces $Y, Z \subseteq X$ we have $Y \nsubseteq Z$, then we also have $\Omega'(Y) \nsubseteq \Omega'(Z)$.*
3. *For all points $x \in X$ we have that the dualization $\Omega(i)$ of the subspace inclusion $i : X\backslash\{x\} \hookrightarrow X$ is not an isomorphism of frames.*
4. *The Skula space $Sk(X)$ is discrete.*

We will see that there is an axiom for bispaces such that we may characterize it with a d-frame version of the proposition above. We recall that in Chapter 3 we have seen that for any d-frame $\mathcal{L}$ the following commutes in the category of complete meet-semilattices.

$$\begin{array}{ccc} \mathsf{sp}[\mathsf{S}(\mathcal{L})] & \xrightarrow{\mathsf{dpt}(\cong)} & \mathsf{sob}[\mathcal{P}(\mathsf{dpt}(\mathcal{L}))] \\ \Big\uparrow{\scriptstyle \mathsf{sp}} & & \Big\downarrow \\ \mathsf{S}(\mathcal{L}) & \xrightarrow{\mathsf{dpt}} & \mathcal{P}(\mathsf{dpt}(\mathcal{L})) \end{array}$$

We will use our d-frame version of the T_D axiom to characterize in several ways those d-frames $\mathcal{L}$ such that the right vertical arrow is an isomorphism.

8.1 The relation between subspaces and sublocales

8.1.1 The coframe of spatializations

In the same way in which in Chapter 3 we have seen spatialization as an interior operator on the complete lattice $\mathsf{S}(\mathcal{L})$ of quotients of a d-frame $\mathcal{L}$, we will now look at spatialization as an

interior operator on the coframe $\mathsf{S}(L)$ of sublocales of a frame L. Since we will work on the coframe $\mathsf{S}(L)$ of sublocales of a frame L, we will adapt the terminology from frame theory by just adding the prefix "co-" to the classical notion. For example, for a coframe C we will call "conucleus" an operator $\nu : C \to C$ which is a nucleus on C^{op}, "cosublocale" a subset $S \subseteq C$ which is a sublocale of C^{op}, and so on. We also use the following pieces of notation.

- For a frame L, we call $\mathsf{pt}'(L)$ its prime spectrum.
- For a complete lattice L and a subset $Y \subseteq L$, we call $\mathfrak{M}(Y)$ the ordered collection $\{\bigwedge M : M \subseteq Y\}$. Similarly, we call $\mathfrak{m}(Y)$ the collection $\{\bigwedge M : M \subseteq Y, M \text{ is finite}\}$.

The first step is to notice that the spatialization $\phi_L : L \twoheadrightarrow \Omega(\mathsf{pt}(L))$ of a frame L, as any frame surjection, determines a sublocale of L. We now show how to obtain this sublocale explicitly. Before doing this, we will gather some useful facts that will simplify our calculations with sublocales. First of all, let us remember that for every frame L and every element $x \in L$, there exists a sublocale $\mathfrak{b}(x)$ defined as $\bigcap\{S \in \mathsf{S}(L) : x \in S\}$. This is clearly the smallest sublocale containing the element x. The reason behind this notation is that "$\mathfrak{b}$" stands for "Boolean" - the sublocales of L that are Boolean algebras coincide with those of the form $\mathfrak{b}(x)$ for some $y \in L$. This is not a fact that we will use in our analysis, however. We provide proofs for the facts on primes and Boolean sublocales that we show, but proofs of these results are also in [37]. Before we start with basic facts about Boolean sublocales, let us look at very general properties of the Heyting implication in frames.

Lemma 8.1.1. *For any frame L we have a contravariant Galois connection*

$$- \to x : L \leftrightarrows L : - \to x$$

Proof. Let us fix an element $x \in L$. For any two elements $a, b \in L$ we have that $a \leq b \to x$ if and only if $a \wedge b \leq x$ and this holds if and only if $b \leq a \to x$. Then, indeed the pair of maps described in the statement form a Galois connection. □

From the properties of antitone Galois connections, the following facts hold.

Corollary 8.1.2. *For any frame L and every $x \in L$ we have the following.*

1. $\bigvee_i y_i \to x = \bigwedge(y_i \to x)$ *for every collection* $y_i \in L$.
2. $y \leq (y \to x) \to x$ *for every* $y \in L$
3. $y \to x = ((y \to x) \to x) \to x$ *for every* $y \in L$.

Let us see some results on Boolean sublocales. Compare the two results in the lemma below with the following two facts about open sublocales.

1. For every $x \in L$ we have that $\mathfrak{o}(x) = \{x \to y : y \in L\}$
2. For every $x \in L$ we have that $a \in \mathfrak{o}(x)$ if and only if $a = x \to a$.

Lemma 8.1.3. *For every frame L and every element $x \in L$, we have the following.*

1. *For every $x \in L$ we have $\mathfrak{b}(x) = \{y \to x : y \in L\}$.*
2. *For every $x \in L$ we have that $a \in \mathfrak{b}(x)$ if and only if $a = (a \to x) \to x$.*
3. *For every $x \in L$ the frame $\mathfrak{b}(x)$ is a Boolean algebra, where the negation of an element $a \in \mathfrak{b}(x)$ is $a \to x$.*

Proof. Let us prove each item in turn.

1. By definition of $\mathfrak{b}(x)$, it suffices to show that $\{y \to x : y \in L\}$ is a sublocale containing x, and that it is the smallest such sublocale. We have that $1 \to x = x$, and so $x \in \{y \to x : y \in L\}$. Let us show that this collection is a sublocale. For every $a \in L$, we have that $a \to (y \to x) = (a \wedge y) \to x$, and so the collection is stable under left implication. For closure under meets, we notice that by item (1) of Corollary 8.1.2 we have that $\bigwedge(y_i \to x) = \bigvee_i y_i \to x$. Finally, by the stability property of sublocales, whenever $x \in S$ for some sublocale $S \subseteq L$, we must have $\{y \to x : y \in L\} \subseteq S$.

2. If $a = (a \to x) \to x$ then $a \in \mathfrak{b}(x)$ by item (1). Suppose, now, that $a \in \mathfrak{b}(x)$. We have that $a \leq (a \to x) \to x$ by item (2) of Corollary 8.1.2. For the other inequality, let us use the assumption that $a \in \mathfrak{b}(x)$. In particular, let $y \in L$ be such that $a = y \to x$. By item (3) of Corollary 8.1.2, we have $a = y \to x = ((y \to x) \to x) \to x$.

3. The collection $\mathfrak{b}(x)$ is a frame. To show that it is a Boolean algebra, it suffices to show that each element joins to 1 with its Heyting pseudocomplement. First, let us show that the Heyting pseudocomplement of some $a \in \mathfrak{b}(x)$ is $a \to x$. The bottom element of the sublocale $\mathfrak{b}(x)$ is x itself, by item (1) and because $x \leq y \to x$ for each $y \in L$; furthermore $x \in \mathfrak{b}(x)$ as $1 \to x = x$. For every $a \in \mathfrak{b}(x)$, we have that $y \leq a \to x$ if and only if $y \wedge x \leq a$. Since meets in $\mathfrak{b}(x)$ are computed in the same way as in L, this means that the element $x \to a$ satisfies the definition of pseudocomplement of x in $\mathfrak{b}(x)$. Item (2), then, implies that every element of $\mathfrak{b}(x)$ is its own Heyting pseudocomplement, and so $\mathfrak{b}(x)$ is a Boolean algebra. □

Let us see some more generic frametheoretical facts which we will use to simplify our calculations.

Lemma 8.1.4. *For a frame L the following facts hold.*

1. *For any collection $\{S_1, ..., S_n\} \subseteq L$ of sublocales, we have that*
$$S_1 \vee ... \vee S_n = \mathfrak{m}(S_1 \cup ... \cup S_n).$$
2. *If $p \in \mathsf{pt}'(L)$, then for each $x \in L$ we have that $x \to p = p$ if and only if $x \not\leq p$.*
3. *If $p \in L$ is such that $y \not\leq p$ implies $y \to p = p$ for every $y \in L$ then p must be prime.*
4. *For every prime $p \in \mathsf{pt}'(L)$, we have that $\mathfrak{b}(p) = \{1, p\}$.*
5. *If a sublocale S has two elements, it is of the form $\mathfrak{b}(p)$ for some prime $p \in \mathsf{pt}'(L)$.*
6. *For a collection $p_i \in \mathsf{pt}'(L)$ we have $\bigvee_i \mathfrak{b}(p_i) = \mathfrak{M}(\{p_i : i \in I\})$.*

Proof. Let us prove each item in turn.

1. We will show that $\mathfrak{m}(S_1 \cup ... \cup S_n)$ is a sublocale, and that it is the smallest one which contains each S_m. Since each of the S_m's is closed under meets, the collection $\mathfrak{m}(S_1 \cup ... \cup S_n)$ is closed under all meets, too. If $y \in L$, let us consider an arbitrary element $s_1 \wedge ... \wedge s_n \in \mathfrak{m}(S_1 \cup ... \cup S_n)$; say that $s_m \in S_m$ for each $m \leq n$, with some s_m's possibly equal to 1. The element $y \to (s_1 \wedge ... \wedge s_n)$ is the same as $\bigwedge_{m \leq n}(y \to s_m)$. Since every sublocale S_m is stable under left implication, indeed we have $y \to s_m \in S_m$ and so $y \to (s_1 \wedge ... \wedge_n) \in \mathfrak{m}(S_1 \cup ... \cup S_n)$. This set is then a sublocale, and it clearly contains all of the S_m's. For $T \subseteq L$ a sublocale, we have that $S_1, ..., S_n \subseteq T$ implies that $\mathfrak{m}(S_1 \cup ... \cup S_n) \subseteq T$, by meet closure of T.

2. Suppose that $p \in L$ is a prime element and that $x \not\leq p$. We expand the definition of $x \to p$ to obtain that this is the same as $\bigvee\{y \in L : y \leq p \text{ or } x \leq p\}$, in which expression we have used primality of p. Then, $x \to p = p$. Conversely, if for $x \in L$ we have that $x \to p = p$ for a prime p, we deduce that $x \to p \neq 1$, since primes are different from 1 by definition. Then, we cannot have $x \leq p$.

3. Suppose that $p \in L$ is such that $x \not\leq p$ implies that $x \to p = p$ for every $x \in L$. To show that p is prime, we show that if we have $x \wedge y \leq p$, and $x \not\leq p$, this implies $y \leq p$. Then, let us assume the antecedent. Since $y \not\leq p$, by assumption $y \to p = p$. This means, by definition of Heyting implication, that $y \wedge z \leq p$ implies that $z \leq p$. In particular, we deduce that $x \leq p$. So, p is prime.

4. First, let us show that for every prime p we have that $\{1, p\}$ is a sublocale. This set is immediately seen to be closed under meets. For the stability condition, we notice that for $y \in L$ we have that $y \to 1 = 1 \in \{1, p\}$ for every $y \in L$; and that we also have $y \to p = 1 \in \{1, p\}$ if $y \leq p$, and $y \to p = p \in \{1, p\}$ if $y \not\leq p$, by item (2). Then, indeed, $\{1, p\}$ is a sublocale. It is clearly the smallest sublocale containing p: since sublocales are closed under arbitrary meet, in particular they must all contain the empty meet 1.

5. Suppose that for some element $x \in L$ we have that $\{1, p\}$ is a sublocale, where $1 \neq p$. If $y \not\leq p$ we must have that $y \to p \neq 1$ and that $y \to p \in \{1, p\}$, by stability under left implication. Then, $y \not\leq p$ implies that $y \to p = p$ for all $y \in L$; by item (3) this implies that p is prime.

6. First, we show that $\mathfrak{M}(\{p_i : i \in I\})$ is a sublocale for every collection of primes $\{p_i : i \in I\}$. By its definition, the collection is closed under all meets. Suppose that $\bigwedge_i p_j$ is an arbitrary element of our collection, with each p_j a prime in $\{p_i : i \in I\}$. We have that $y \to \bigwedge_j p_j = \bigwedge_j (y \to p_j)$. For every j, by item (2) we have that each $y \to p_j$ is either 1 or p_j, and so $y \to \bigwedge_j p_j = \bigwedge_k p_k$, with each p_k equal to some p_j. By the definition of this collection, this is in $\mathfrak{M}(\{p_i : i \in I\})$. □

The next results will be tools to define a spatialization operator on $\mathsf{S}(L)$.

Lemma 8.1.5. *For any frame L and any sublocale $S \subseteq L$, we have that $\mathsf{pt}'(S) = \mathsf{pt}'(L) \cap S$.*

Proof. For the inclusion $\mathsf{pt}'(L) \cap S \subseteq \mathsf{pt}'(S)$, let us assume that $p \in \mathsf{pt}'(L)$ is a prime which is in S. The element is also prime as an element of S since by definition this condition is formally stronger than $p \in S$ and p being prime. Conversely, if $p \in S$ is prime as an element of S, suppose that $x \wedge y \leq p$ for $x, y \in L$. Then, by monotonicity of nuclei we have that $\nu_S(x \wedge y) \leq \nu_S(p) = p$, and by their meet preservation property also $\nu_S(x) \wedge \nu_S(y) \leq p$. By primality of p in S, we have that either $\nu_S(x) \leq p$ or $\nu_S(y) \leq p$. Since nuclei are inflationary, if the former holds then $x \leq p$, and if the latter holds $y \leq p$. Then, p is prime as an element of L, too. □

Lemma 8.1.6. *A frame is spatial if and only if all its elements are meets of primes.*

Proof. By definition of the prime spectrum of a frame, we know that a frame is spatial if and only if the map $\phi'_L : L \twoheadrightarrow \Omega(\mathsf{pt}'(L))$ is injective, that is, if whenever $a \not\leq b$ we have some prime $p \in L$ such that $b \leq p$ and $a \not\leq p$. If all elements of L are meets of primes, each element must be the meet of all the primes above it. then whenever $a \not\leq b$, there must be a prime p with $b \leq p$ and $a \not\leq p$, otherwise we would have that $\uparrow b \cap \mathsf{pt}'(L) \subseteq \uparrow a \cap \mathsf{pt}'(L)$, from which

$\bigwedge\{p \in \mathsf{pt}'(L) : a \leq p\} \leq \bigwedge\{p \in \mathsf{pt}'(L) : b \leq p\}$, contradicting our hypothesis that $s \not\leq b$. Conversely, if the frame L is spatial, and if we have some $a \in L$ such that $a \neq \bigwedge_i p_i$ for every collection $p_i \in \mathsf{pt}'(L)$, we also have that $a < \bigwedge\{p \in \mathsf{pt}'(L) : a \leq p\} =: a'$. But since these two elements of L have the same primes above them, we deduce that $\phi'_L(a) = \phi'_L(a')$, contradicting spatiality of L. □

Lemma 8.1.7. *For any frame L, its spatialization sublocale is $\mathfrak{M}(\mathsf{pt}'(L))$.*

Proof. Translating the universal property of the frame spatialization to a condition on sublocales, we obtain that the spatialization sublocale of L is the largest sublocale of L which is spatial. The sublocale $\mathfrak{M}(\mathsf{pt}'(L))$ is spatial, since by its definition all its elements are meets of primes. Suppose, now, that $T \subseteq L$ is a spatial sublocale. Then, all its elements are meets of primes, and so we have $\mathfrak{M}(\mathsf{pt}'(T)) \subseteq T$. We need to have $\mathsf{pt}'(T) \subseteq \mathsf{pt}'(L)$. Since the operator $\mathfrak{M}$, too, is monotone, we also have $\mathfrak{M}(\mathsf{pt}'(T)) \subseteq \mathfrak{M}(\mathsf{pt}'(L))$. So, $T \subseteq \mathfrak{M}(\mathsf{pt}'(L))$. □

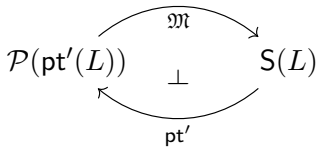
From now on, we will refer to $\mathfrak{M}(\mathsf{pt}'(L))$ as $\mathsf{sp}(L)$ for any frame L. There is an alternative way of seeing the spatialization of a sublocale.

Proposition 8.1.8. *For every sublocale $S \subseteq L$ we have that $\mathsf{sp}(S) = \bigvee\{\mathfrak{b}(p) : p \in \mathsf{pt}'(S)\}$.*

Proof. By item (5) of 8.1.4, for any sublocale $S \subseteq L$ we have that $\bigvee\{\mathfrak{b}(p) : p \in \mathsf{pt}'(L)\} = \mathfrak{M}(\mathsf{pt}'(S))$. □

Let us now show that the map $\mathsf{sp} : \mathsf{S}(L) \to \mathsf{S}(L)$ is an interior operator. In the frame theoretical case, in contrast with the d-frame case, we can prove that this interior operator preserves finite joins too. As we will see, this implies that the collection of its fixpoints is actually a cosublocale of $\mathsf{S}(L)$. Before we prove this, let us see a more general result.

Proposition 8.1.9. *For every frame L there is an adjunction of posets as below.*



Proof. It is clear from their definition that both these maps are monotone. To show that we have an adjunction between posets, it is sufficient to show that for any subspace $Y \subseteq \mathsf{pt}'(L)$ and any sublocale $S \subseteq L$ we have that $\mathfrak{M}(Y) \subseteq S$ if and only if $Y \subseteq \mathsf{pt}'(L)$. Suppose that the former holds. Then, since pt' is monotone we have $\mathsf{pt}'(\mathfrak{M}(Y)) \subseteq \mathsf{pt}'(S)$. We also have that $Y \subseteq \mathsf{pt}'(\mathfrak{M}(Y))$ since $\mathsf{pt}'(Y) = Y$ by assumption and $\mathsf{pt}'(Y) \subseteq \mathsf{pt}'(\mathfrak{M}(Y))$ by monotonicity

of pt'. Then, the first of the desired implications holds. Suppose, now, that the latter holds. Since $\mathfrak{M}$ is monotone, we deduce that $\mathfrak{M}(Y) \subseteq \mathfrak{M}(\mathsf{pt}'(S))$. Since $\mathfrak{M}(\mathsf{pt}'(S)) \subseteq S$ as the left hand side is just its spatialization sublocale, we are done. Then, indeed we have an adjunction $\mathfrak{M} \dashv \mathsf{pt}'$. □

Observation 8.1.10. *If we have a frame L and a collection of sublocales $S_1, ..., S_n$ of this frame, we have that by item (1) of Lemma 8.1.4 their join in $\mathsf{S}(L)$ is computed as $\mathfrak{m}(S_1 \cup ... \cup S_n)$. Since primes cannot be nontrivial meets, this means that $\mathsf{pt}'(S_1 \vee ... \vee S_n) = \mathsf{pt}'(S_1) \cup ... \cup \mathsf{pt}'(S_n)$.*

Proposition 8.1.11. *Spatialization $\mathsf{sp} : \mathsf{S}(L) \to \mathsf{S}(L)$ is a conucleus: it is an interior operator which preserves finite joins.*

Proof. For any sublocale $S \subseteq L$, we have that $\mathsf{pt}'(S) \subseteq S$. Since sublocales are closed under meets, this implies that $\mathsf{sp}(S) \subseteq \mathfrak{M}(S) = S$. Then, spatialization is decreasing in $\mathsf{S}(L)$. It is also idempotent: if S is a spatial sublocale, all its elements are meets of primes and so $\mathsf{sp}(S) = S$. Now, let us show that the map preserves finite joins of $\mathsf{S}(L)$. The map $\mathfrak{M}$ preserved all colimits, since it is a left adjoint. On the other hand, the map pt' preserves finite colimits by Observation 8.1.10. Then, their composition $\mathfrak{M} \circ \mathsf{pt}'$ preserves finite joins. □

Corollary 8.1.12. *For every frame L the lattice of its spatial sublocales $\mathsf{sp}[\mathsf{S}(L)]$ is a coframe. Spatialization is a coframe map $\mathsf{sp} : \mathsf{S}(L) \to \mathsf{sp}[\mathsf{S}(L)]$.*

Proof. Both parts of the claim holds because sp is a conucleus on $\mathsf{S}(L)$. □

We now claim that the cosublocale $\mathsf{sp}[\mathsf{S}(L)] \subseteq \mathsf{S}(L)$ corresponds exactly to the spatialization cosublocale of $\mathsf{S}(L)$.

Lemma 8.1.13. *The primes of $\mathsf{S}(L)^{op}$ are exactly the sublocales of the form $\mathfrak{b}(p)$ for some prime $p \in L$.*

Proof. Suppose we have a sublocale $S \subseteq L$ with more than one element, say $\{1, x, y\} \subseteq S$ with all three these elements pairwise different. Suppose without losing generality that $x \not\leq y$. This means that $y \notin \mathfrak{c}(x)$. We also have $x \to x = 1 \neq x$, and so $x \notin \mathfrak{o}(x)$. Since $\mathfrak{c}(x) \vee \mathfrak{o}(x) = L$, we have that $S \subseteq \mathfrak{o}(x) \vee \mathfrak{c}(x)$, but we also have $S \not\subseteq \mathfrak{o}(x)$ and $S \not\subseteq \mathfrak{c}(x)$, with witnesses x and y, respectively. Then, S is not join prime. Conversely, if $S = \{1, p\}$ and $S \subseteq T \vee U$, in particular $\{p\} \subseteq T \vee U = \mathfrak{m}(T \cup U)$. This implies that $\{p\} \subseteq T \cup U$ since primes cannot be nontrivial meets. We must have either $p \in T$ or $p \in U$. If the former holds then $\{1, p\} \subseteq T$ and if the latter holds then $\{1, p\} \subseteq U$. Then, $\{1, p\}$ is join prime. □

Proposition 8.1.14. *For every frame L the coframe $\mathsf{sp}[\mathsf{S}(L)]$ is the spatialization cosublocale of $\mathsf{S}(L)$.*

Proof. The spatialization sublocale of a frame is the closure under meets of the ordered collection of its prime elements. Then the spatialization cosublocale of $\mathsf{S}(L)$ is the closure under joins of the ordered collection of its join primes, which by Lemma 8.1.13 is $\{\mathfrak{b}(p) : p \in \mathsf{pt}'(L)\}$. □

As a corollary, we obtain this well-known fact.

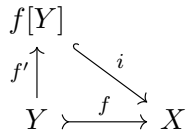
Corollary 8.1.15. *For any frame L, the frame $\mathsf{S}(L)^{op}$ is spatial if and only if all sublocales of L are spatial.*

8.1.2 The coframe of sobrifications

Suppose that we have the spectrum $\mathsf{pt}'(L)$ of a frame L. We want to see sobrification as a closure operator on $\mathcal{P}(\mathsf{pt}'(L))$ in the same way in which we have seen spatialization as an interior operator on $\mathsf{S}(L)$. In particular, we will see that there is a symmetry between spatialization on $\mathsf{S}(L)$ and sobrification on $\mathcal{P}(\mathsf{pt}'(L))$. When we have an extremal monomorphism $Y \rightarrowtail X$ in **Top** we can turn it into a concrete subspace inclusion $Y \subseteq X$. Recall that in a category $\mathcal{C}$ a morphism $f : A \to B$ is an extremal monomorphism if it is a monomorphism and if $f = ge$ with e an epimorphism, e must be an isomorphism. It is very well-known that the epimorphisms in the category of topological spaces are exactly the surjections. The following result is also quite well-known.

Lemma 8.1.16. *In the category of topological spaces, a continuous function $f : Y \to X$ is an extremal monomorphism if and only if it is an injection such that the map $Y \to f[Y]$ defined as $y \mapsto f(x)$ is a homeomorphism.*

It is also known that the subspace inclusions $Y \hookrightarrow X$ in the category of topological spaces are essentially the extremal monomorphisms, in the sense that on the one hand all subspace inclusions are extremal monomorphisms, and on the other hand every extremal monomorphism is a subspace inclusion up to homeomorphism, in the following sense: every extremal monomorphisms $f : Y \rightarrowtail X$ in **Top** is such that the map $Y \to f[Y]$ acting as $y \mapsto f(y)$ is a homeomorphism making the following diagram commute



It is clear that any subspace inclusion $Y \subseteq X$ is such that the image of Y under this map is homeomorphic to Y, since it is identical to Y. We have already seen the fact that the sobrification map $\psi_X : X \to \mathsf{pt}(\Omega(X))$ of a space X is not in general bijective. When X is a T_0 space, the map is always injective. In fact, we have more. We know that when X is T_0, this map in general is an extremal monomorphism.

Lemma 8.1.17. *If X is a T_0 space, the sobrification map $\psi_X : X \rightarrowtail \mathsf{pt}(\Omega(X))$ is an extremal monomorphism.*

Proof. Suppose that X is a T_0 space. Its sobrification map is then a continuous injection. By Lemma 8.1.16, then, to show our claim it suffices to show that for every open $U \subseteq X$ the set $\psi_X[U]$ is open in $\psi_X[X] \subseteq \mathsf{pt}(\Omega(X))$. Suppose, then, that $U \subseteq X$ is an open set. Recall that a typical open of $\mathsf{pt}(\Omega(X))$ is of the form $\{f \in \mathsf{pt}(\Omega(X)) : f(U) = 1\}$ for some open $U \subseteq X$, and so an open of the subspace $\psi_X[X]$ is a set of the form $\{N_x \in \mathsf{pt}(\Omega(X)) : N_x(U) = 1\}$ for some open $U \subseteq X$. The set $\psi_X[U]$ is $\{N_x \in \mathsf{pt}(\Omega(X)) : x \in U\} = \{N_x \in \mathsf{pt}(\Omega(X)) : N_x(U) = 1\}$, and so it is open in $\psi_X[X]$. Then, the sobrification map ψ_X is an extremal monomorphism. □

This means that whenever we have T_0 topological space, we may see it up to homeomorphism as a subspace of its sobrification.

Lemma 8.1.18. *If $f : X \to Y$ is an injective function its forward image map preserves intersections.*

Lemma 8.1.19. *The composition of extremal monomorphisms in* **Top** *is an extremal monomorphism.*

Proof. Suppose that $m_1 : X \to Y$ and $m_2 : Y \to Z$ are extremal monomorphisms in **Top**. By Lemma 8.1.16, it suffices to show that $m_2 \circ m_1$ is a continuous injection such that $m_2[m_1[X]]$ is homeomorphic to X. Since continuous functions as well as injections are closed under composition, it suffices to show that $m_2 \circ m_1$ is such that forward images of opens of X are opens of $m_2[m_1[X]]$. Suppose that $U \subseteq X$ is open. We have that $m_1[U]$ is open in $Y \cap m_1[X]$, let $V \subseteq Y$ be such that $m_1[X] = V \cap m_1[X]$. The map m_2 is an injection and so by Lemma 8.1.18 we have $m_2[m_1[U]] = m_2[V \cap m_1[X]] = m_2[V] \cap m_2[m_1[X]]$. By assumption of m_2, we have that $m_2[V]$ must be $W \cap m_2[Y]$ for some open $W \subseteq Z$. This means that $m_1[X] = W \cap m_2[Y] \cap m_2[m_1[Y]]$. Since m_2 is an injection this is also the same as $W \cap m_2[Y \cap m_1[X]] = W \cap m_2[m_1[X]]$. Then, indeed $m_2[m_1[U]]$ is an open set of the subspace $m_2[m_1[X]] \subseteq Z$. So, the map $m_2 \circ m_1$ is an extremal monomorphism. □

Lemma 8.1.20. *Suppose that X is a sober space, and that $\psi_X^{-1} : \mathsf{pt}(\Omega(X)) \to X$ is the inverse of the sobrification map. For any subspace $i_Y : Y \subseteq X$ the map $\psi_X^{-1} \circ \mathsf{pt}(\Omega(i_Y)) : \mathsf{pt}(\Omega(Y)) \rightarrowtail X$ is an extremal monomorphism.*

Proof. By Lemma 8.1.19, it suffices to show that isomorphisms are extremal epimorphisms. This follows from Lemma 8.1.16, since for a continuous open bijection $f : X \cong Y$ indeed we have that the image $f[X]$ is homeomorphic to X. □

In the lemma below we have kept the convention that for a continuous function $f : Y \to X$, we have denoted as f' the map $Y \to f[Y]$ acting as $y \mapsto f(y)$.

Lemma 8.1.21. *Suppose that X is a sober space, and that $Y \subseteq X$ is a subspace. We have a chain of subspace inclusions $Y \subseteq \mathsf{sob}(Y) \subseteq X$ such that the following diagram in* **Top** *commutes. All vertical arrows are homeomorphisms.*

$$\begin{array}{ccccc} Y & \hookrightarrow & \mathsf{sob}(Y) & \hookrightarrow & X \\ {\scriptstyle \mathsf{id}_Y}\uparrow & & {\scriptstyle (\psi_X^{-1} \circ \mathsf{pt}(\Omega(i)))'}\uparrow & & {\scriptstyle \psi_X^{-1}}\uparrow \\ Y & \xrightarrow{\psi_Y} & \mathsf{pt}(\Omega(Y)) & \xrightarrow{\mathsf{pt}(\Omega(i))} & \mathsf{pt}(\Omega(X)) \end{array}$$

Proof. The map $\psi_X^{-1} \circ \mathsf{pt}(\Omega(i))$ is an extremal monomorphism, by Lemma 8.1.20.By the observation made after Lemma 8.1.16, then, there is a subspace $\mathsf{sob}(Y)$ of X such that the subspace inclusion $\mathsf{sob}(Y) \hookrightarrow X$ makes the right square of the rectangle commute. Let us show that the left square commutes. We need to show that for every $y \in Y$ we have that $\psi_X^{-1}(\mathsf{pt}(\Omega(i))(\psi_Y(y))) = y$ for every $y \in Y$. We notice that for the subspace inclusion $i : Y \hookrightarrow X$ we have that $\Omega(i) : \Omega(X) \to \Omega(Y)$ is the map $i^{-1} : U \mapsto U \cap Y$. Then, the map $\mathsf{pt}(\Omega(i)) : \mathsf{pt}(\Omega(Y)) \to \mathsf{pt}(\Omega(X))$ is precomposition with $- \cap Y$, that is, it assigns to each frame map $f : \Omega(X) \to 2$ the map $\Omega(Y) \to 2$ defined as $U \mapsto f(U \cap Y)$. In particular, for a map of the form $N_y : \Omega(Y) \to 2$, we will have that $\mathsf{pt}(\Omega(i))(N_y)$ is a map $\Omega(X) \to 2$ such that it gives 1 whenever $y \in U \cap Y$, and it gives 0 whenever $y \notin U \cap Y$. It then coincides with the neighborhood map $N_y : \Omega(X) \to 2$.

$$\begin{aligned} &\psi_X^{-1}(\mathsf{pt}(\Omega(i))(\psi_Y(y))) = \\ &= \psi_X^{-1}(\mathsf{pt}(\Omega(i))(N_y)) = \\ &= \psi_X^{-1}(N_y) = \\ &= y. \end{aligned}$$

□

For any sober space X and any subspace $Y \subseteq X$, we may calculate $\mathsf{sob}(Y)$ as a closure operator on $\mathcal{P}(X)$. Let us remind the reader that we have already shown that the sober

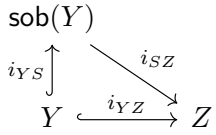
subspaces of a sober space form a closure system: for a sober space X, the sober subspaces coincide with those whose underlying set is a Skula-closed set.

Lemma 8.1.22. *If X is a sober space and $Y \subseteq X$ a subspace, we have*

$$\mathsf{sob}(Y) = \bigcap \{Z \subseteq X : Y \subseteq Z, Z \text{ is sober}\}$$

of X with sobrification map the set inclusion $Y \subseteq \mathsf{sob}(Y)$.

Proof. By Lemma 8.1.21, we know that there is a subspace $\mathsf{sob}(Y) \subseteq X$ such that it is isomorphic to the sobrification of Y, with the sobrification map given by the set inclusion $i_{YS} : Y \hookrightarrow \mathsf{sob}(Y)$. By the universal property of the sobrification of a space, whenever Z is a sober subspace of X such that we have an inclusion $i_{YZ} : Y \subseteq Z$, there must be a unique map $i_{SZ} : \mathsf{sob}(Y) \to Z$ such that the following commutes.



Because this triangle commutes, i_{SZ}, too, must be a subset inclusion. Then, every sober subspace of X which contains Y must be a superset of $\mathsf{sob}(Y)$, and so $\mathsf{sob}(Y) \subseteq \bigcap \{Z \subseteq X : Z \text{ is sober}, Y \subseteq Z\}$. For the other set inclusion, it suffices to notice that $\mathsf{sob}(Y) \in \{Z \subseteq X : Z \text{ is sober}, Y \subseteq Z\}$. □

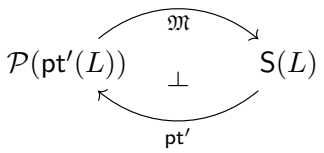
We now look at the specific case where we want to sobrify subspaces of a sober space of the form $\mathsf{pt}'(L)$ for some frame L.

Lemma 8.1.23. *For any frame L and any subspace $Y \subseteq \mathsf{pt}'(L)$ we have that its sobrification is $\mathsf{pt}'(\mathfrak{M}(Y))$, with sobrification map the set inclusion $Y \subseteq \mathsf{pt}'(\mathfrak{M}(Y))$*

Proof. For this, it suffices to show that for every subspace $Y \subseteq \mathsf{pt}'(L)$ the intersection $\mathsf{sob}(Y)$ of all sober subspaces of $\mathsf{pt}'(L)$ which contain it is $\mathsf{pt}'(\mathfrak{M}(Y))$. □

Finally, we may come back to the adjunction introduced in Proposition 8.1.9. The next proposition will make precise our opening statement for this subsection; that there is a symmetry between spatialization of sublocales and sobrification of subspaces.

Proposition 8.1.24. *For every frame L, consider the following adjunction.*



The fixpoints of $\mathfrak{M} \circ \mathsf{pt}'$ *are exactly the spatial sublocales, and the fixpoints of* $\mathsf{pt}' \circ \mathfrak{M}$ *are exactly the sober subspaces.*

Proof. This follows from Proposition 8.1.9 combined with Lemmas 8.1.7 and 8.1.23. □

8.1.3 The relation between subspaces and sublocales

We can now finally see that we can exploit the equivalence between sober spaces and spatial frames to show that the coframes of sober subspaces of $\mathsf{pt}'(L)$ and that of spatial sublocales of L are the same.

Proposition 8.1.25. *For any frame* L *we have that the map* $\mathsf{pt}' : \mathsf{sp}[\mathsf{S}(L)] \to \mathsf{sob}[\mathcal{P}(\mathsf{pt}'(L))]$ *is a coframe isomorphism. Its inverse acts as* $S \mapsto \mathfrak{M}'(S)$.

Proof. Proposition 8.1.9 tells us that we have an adjunction $\mathfrak{M} : \mathcal{P}(\mathsf{pt}'(L)) \leftrightarrows \mathsf{S}(L) : \mathsf{pt}'$ with $\mathfrak{M} \dashv \mathsf{pt}'$. This adjunction restricts to an equivalence of categories between the two full subcategories determined by the objects which are fixpoints of $\mathfrak{M} \circ \mathsf{pt}'$ and $\mathsf{pt}' \circ \mathfrak{M}$. By Proposition 8.1.24, these are the spatial sublocales and the sober subspaces, respectively. □

We can gather together this result and the results of the two previous subsections in the following intuitive picture. The following is a commutative diagram in the category of coframes. We will refer to this as the "main diagram" of this section.

$$\begin{array}{ccc} \mathsf{sp}[\mathsf{S}(L)] & \xrightarrow{\mathsf{pt}'(\cong)} & \mathsf{sob}[\mathcal{P}(\mathsf{pt}(L))] \\ \Big\uparrow{\scriptstyle \mathsf{sp}} & & \Big\downarrow \\ \mathsf{S}(L) & \xrightarrow{\mathsf{pt}'} & \mathcal{P}(\mathsf{pt}'(L)) \end{array}$$

Our analysis of the connection between subspaces and sublocales will be centred around analyzing what happens when the right vertical arrow in this diagram is an isomorphism.

In that case, the spatial sublocales of $\mathsf{S}(L)$ perfectly represent the subspaces of $\mathsf{pt}'(L)$. We will show that this is equivalent to the space $\mathsf{pt}'(L)$ being T_D and the primes of L being *weakly covered*. This is a weak version of the notion of coveredness which was introduced in [11], and then used in [39] in relation to T_D spaces. In spatial frames coveredness is equivalent to weak coveredness.

8.1.4 When are all subspaces sober?

For a frame L, we say that a prime $p \in \mathsf{pt}'(L)$ is *weakly covered* if whenever $p = \bigwedge P$ for $P \subseteq \mathsf{pt}'(L)$ we have $p \in P$. This is a weakening of the notion of *covered prime*, by which is meant a prime $p \in L$ such that whenever $p = \bigwedge A$ we have $p \in A$. The notion of covered prime is already present in relation with the T_D axiom in the literature, for instance in [39] it is observed that X being a T_D space is equivalent to all primes of $\Omega(X)$ of the form p_x for some $x \in X$ being covered. Our notion of weakly covered stems from the fact that we want a notion of covered prime generalized to nonspatial frames.

Lemma 8.1.26. *For a frame L, every prime of L is weakly covered if and only if all subspaces of $\mathsf{pt}'(L)$ are sober.*

Proof. Suppose that all primes of L are weakly covered. For every subspace $Y \subseteq \mathsf{pt}'(L)$ we have that $\mathfrak{M}(Y)$ contain the same primes as Y, since we cannot get any new prime as an meet $\bigwedge P$ of primes $P \subseteq Y$, or this would be a prime which is not weakly covered. Then, $Y = \mathsf{pt}'(\mathfrak{M}(Y))$ for every subspace $Y \subseteq \mathsf{pt}'(L)$. By Proposition 8.1.23 this means that every subspace of $\mathsf{pt}'(L)$ is sober. Conversely, suppose that there is a prime $p \in \mathsf{pt}'(L)$ which is not weakly covered. Let $P \subseteq \mathsf{pt}'(L)$ be a collection of primes with $p \notin P$ and such that $\bigwedge P = p$. We have that $p \in \mathsf{pt}'(\mathfrak{M}(P))$ but $p \notin P$, and so the subspace $P \subseteq \mathsf{pt}'(L)$ is not its own sobrification. □

The condition that all subspaces of $\mathsf{pt}'(L)$ are sober is equivalent to the right vertical arrow of the main diagram of this section being an isomorphism. This is also equivalent to having that pt' establishes an isomorphism of coframes $\mathsf{sp}[\mathsf{S}(L)] \cong \mathcal{P}(\mathsf{pt}'(L))$. Let us look at some more equivalent ways of phrasing this condition.

Lemma 8.1.27. *For a frame L, prime $p \in \mathsf{pt}'(L)$ is weakly covered if and only if the point $\mathfrak{b}(p)$ is isolated in $\mathsf{pt}'(\mathsf{S}(L)^{op})$.*

Proof. Suppose that L is a frame and that $p \in \mathsf{pt}'(L)$ be a weakly covered prime. Let us consider the singleton $\{\mathfrak{b}(p)\} \subseteq \mathsf{pt}'(\mathsf{S}(L)^{op})$. A typical open of $\mathsf{pt}'(\mathsf{S}(L)^{op})$ is a set of the form $\{\mathfrak{b}(q) : \mathfrak{b}(q) \nsubseteq T\}$, with q ranging over the primes of L. Let $\{q_i : i \in I\}$ be the set of primes $\mathsf{pt}'(L)\backslash\{p\}$. We claim that $\{\mathfrak{b}(p)\} = \{\mathfrak{b}(q) : \mathfrak{b}(q) \nsubseteq \bigvee_i \mathfrak{b}(q_i)\}$. Since $\bigvee_i \mathfrak{b}(q_i) = \mathfrak{M}(\{q_i : i \in I\})$, by weak coveredness of p we have $p \notin \bigvee_i \mathfrak{b}(q_i)$, that is, $\mathfrak{b}(p) \subseteq \bigvee_i \mathfrak{b}(q_i)$. For all other primes $q \in L$, by definition of the collection q_i we must have $\mathfrak{b}(q) \subseteq \bigvee_i \mathfrak{b}(q_i)$. Then, indeed $\{\mathfrak{b}(p)\}$ is isolated. For the converse, let us suppose that $\{\mathfrak{b}(p)\}$ is isolated. Let S be the sublocale such that $\mathfrak{b}(p) \nsubseteq S$ and such that $\mathfrak{b}(q) \subseteq S$ whenever q is a prime other than p. Suppose

that Q is a collection of primes not containing p. We claim that $\bigwedge Q \neq p$. By property of the sublocale S, we must have $\bigvee\{\mathfrak{b}(q) : q \in Q\} \subseteq S$, and by meet closure of sublocales this implies $\bigwedge Q \in S$. Then, we must have $\bigwedge Q \neq p$, or we would have $p \in S$, contradicting our hypothesis. □

Lemma 8.1.28. *For a frame L, a prime $p \in \mathsf{pt}'(L)$ is weakly covered if and only if it is an isolated point of the subspace $\mathsf{pt}'(\uparrow p) = \uparrow p \cap \mathsf{pt}'(L)$ of $\mathsf{pt}'(L)$.*

Proof. Suppose that $p \in L$ is a weakly covered prime. Consider the space $\uparrow p \cap \mathsf{pt}'(L) \subseteq \mathsf{pt}'(L)$. Let P be the collection $\{q \in \uparrow p \cap \mathsf{pt}'(L) : q \neq p\}$. By assumption that p is weakly covered, we have that $p < \bigwedge P$, and so $\bigwedge P \not\leq p$. This means that the open $\{q \in \mathsf{pt}'(\uparrow p) : \bigwedge P \leq q\} \subseteq \uparrow p \cap \mathsf{pt}'(L)$ only contains the point p. Conversely, suppose that $p \in \mathsf{pt}'(L)$ is not weakly covered, and let P be a collection of primes with $\bigwedge P = p$. Consider the subspace $\mathsf{pt}'(\uparrow p) \subseteq \mathsf{pt}'(L)$. Suppose that there is an open $\{q \in \mathsf{pt}'(\uparrow p) : a \not\leq q\}$ containing p. We show that it must contain at least one other point of $\mathsf{pt}'(\uparrow p)$. If we had $a \leq q$ for all primes $q \neq p$ with $p \leq q$, we would also have that $a \leq \bigwedge\{q \in \mathsf{pt}'(\uparrow p) : q \neq p\} \leq \bigwedge P = p$, contradicting that $a \not\leq p$. Then, the point p is not isolated in $\mathsf{pt}'(\uparrow p)$. □

Let us now recall the fact that $\mathsf{pt}'(\mathsf{S}(L)^{op})$ is the Skula space of $\mathsf{pt}'(L)$. We also remind the reader that for a frame L the Skula-closed sets of its spectrum coincide with the underlying sets of its sober subspaces.

Lemma 8.1.29. *In a complete Boolean algebra B the meet prime elements coincide with the coatoms.*

Proof. Suppose that B is a complete Boolean algebra. Suppose that $p \in \mathsf{pt}'(B)$, and suppose towards contradiction that there is $x \in B$ with $p < x < 1$. Since $x \neq 1$ and since B is a Boolean algebra we must have $\neg x \neq 0$. We have that $\neg x \leq \neg p$. We cannot have $\neg x \leq \neg p$, otherwise we would have $\neg x \leq p \wedge \neg p$, contradicting that $\neg x \neq 0$. We deduce $\neg x \not\leq \neg p$. By assumption, we also have $x \not\leq p$, and by primality of p this means that $x \wedge \neg x \not\leq p$, which is a contradiction. Conversely, suppose that $p \in B$ is a coatom. Suppose that $x \wedge y \leq p$. Using distributivity we have $(x \vee p) \wedge (y \vee p) = p$. Since p is a coatom, we have $x \vee p = p$ or $x \vee p = 1$, and similarly for $y \vee p$. We cannot have $x \vee p = 1 = y \vee p$ or we would contradict our assumption that $(x \vee p) \wedge (y \vee p) = p$. Then, we must have either $x \vee p = p$, that is $x \leq p$, or $y \vee p = p$, that is $y \leq p$. □

Lemma 8.1.30. *The prime spectrum of a Boolean algebra is the discrete space whose points are its coatoms.*

Proof. Suppose that B is a complete Boolean algebra which is also a spatial frame. Its prime spectrum $\mathsf{pt}'(B)$ is the set of its coatoms, by Lemma 8.1.29. We claim that every coatom is an isolated point of $\mathsf{pt}'(B)$. Suppose, then, that $p \in \mathsf{pt}'(B)$. Since B is a Boolean algebra and $p \neq 1$ we must also have $\neg p \neq 0$. This means that $\neq p \not\leq \neg\neg p = p$. If $q \in \mathsf{pt}'(L)$ is such that $q \neq p$, we have that $p \not\leq q$, since p and q are distinct coatoms, and that $p \wedge \neg p \leq q$. Since q is a prime, this implies that $p \leq q$. Then, we have that the open $\{q \in \mathsf{pt}'(B) : \neg p \not\leq q\} \subseteq \mathsf{pt}'(B)$ consists only of the singleton $\{p\}$. Then, p is isolated. □

Lemma 8.1.31. *If a sober space has a frame of opens which is a Boolean algebra, it must be discrete.*

Proof. If a sober space is such that its frame of opens is Boolean, it must be homeomorphic to the prime spectrum of a Boolean algebra, which we know to be discrete, by Lemma 8.1.30. □

Lemma 8.1.32. *For a frame L, we have that $\mathsf{sp}[\mathsf{S}(L)]$ is Boolean if and only if the space $\mathsf{pt}'(\mathsf{S}(L)^{op})$ is discrete.*

Proof. If $\mathsf{pt}'(\mathsf{S}(L)^{op})$ is a discrete space, its frame of opens is a Boolean algebra. By proposition 8.1.14, this frame of opens is anti-isomorphic to the coframe $\mathsf{sp}[\mathsf{S}(L)]$. For the converse, suppose that $\mathsf{sp}[\mathsf{S}(L)]$ is a Boolean algebra. Since this is anti-isomorphic to $\Omega(\mathsf{pt}'(\mathsf{S}(L)^{op}))$, this means that the frame of opens of $\mathsf{pt}'(\mathsf{S}(L)^{op})$ is Boolean. By Lemma 8.1.31, the space $\mathsf{pt}'(\mathsf{S}(L)^{op})$ is discrete. □

Theorem 8.1.33. *For a frame L, the following are equivalent.*

1. *All the primes in L are weakly covered.*
2. *All subspaces of $\mathsf{pt}'(L)$ are sober.*
3. *The map $\mathsf{pt}' : \mathsf{sp}[\mathsf{S}(\Omega(X))] \to \mathcal{P}(\mathsf{pt}'(L))$ is an isomorphism.*
4. *The space $\mathsf{pt}'(\mathsf{S}(L)^{op})$ is discrete.*
5. *The space $\mathsf{pt}'(L)$ is T_D.*
6. *The coframe $\mathsf{sp}[\mathsf{S}(L)]$ is a Boolean algebra.*

Proof. The equivalence between (1) and (2) is Lemma 8.1.26. As we have observed already, all subspaces of $\mathsf{pt}'(L)$ being sober is a condition equivalent to the right vertical arrow of the main diagram of this section being an isomorphism. Then, (2) and (3) are equivalent. By Lemma 8.1.27, a prime $p \in L$ being covered is equivalent to the point $\mathfrak{b}(p)$ of the space $\mathsf{pt}'(\mathsf{S}(L)^{op})$

being isolated, and so all primes of L being weakly covered is equivalent to $\mathsf{pt}'(\mathsf{S}(L)^{op})$ being discrete. Then, (1) and (4) are equivalent. Since $\mathsf{pt}'(\mathsf{S}(L)^{op})$ is the Skula space of $\mathsf{pt}'(L)$, by the last item of Proposition 8.0.1 the space $\mathsf{pt}'(\mathsf{S}(L)^{op})$ being discrete is equivalent to $\mathsf{pt}'(L)$ being a T_D space, and so (4) and (5) are equivalent. Lemma 8.1.32 states the equivalence of (4) and (6). □

8.2 The relation between bisubspaces and d-quotients

We would like to carry out an analysis similar to the one in the previous sections for the relation between point-set and pointfree bisubspaces, too. For d-frames, we will look only at the first question, namely "when are all bisubspaces d-sober?", by analyzing a d-frame analogue of the T_D axiom. Unfortunately, we cannot say much, yet, about when is it that all d-quotients of a d-frame are spatial, since in the d-frame case there is no relation between spatiality of the assembly $\mathsf{A}(\mathcal{L})$ of a d-frame $\mathcal{L}$ and spatiality of all the d-quotients of $\mathcal{L}$. Instead, we will explore a bitopological version of this matter in the next section, where we will analyze the primes of main components of biframes.

8.2.1 The d-T_D axiom

We say that a bispace X is *d-T_D* if and only if every singleton $\{x\}$ is such that

- either $\{x\} = U^+ \cap V^{+c}$ for positive opens U^+ and V^+,
- or $\{x\} = U^- \cap V^{-c}$,
- or $\{x\} = U^+ \cap U^-$,
- or $\{x\} = U^{+c} \cap U^{-c}$.

Let us see how the d-T_D axiom relates to other classical bitopological separation axioms. Recall that a bispace X is *pairwise* T_1 is for any two distinct points $x, y \in X$ either there is a positive open U^+ with $x \in U^+$ and $y \notin U^+$, or there is a negative open U^- with the same property.

Lemma 8.2.1. *A bispace X is pairwise T_1 if and only if every singleton is of the form $U^{+c} \cap U^{-c}$.*

Proof. Suppose that X is a bispace which is pairwise T_1. Let $x \in X$. For every point $y \in X$ distinct from x, there is either some U^+_y with $y \in U^{+c}$ and $x \notin U^{+c}$, or some U^-_y with the

same property. Let Y^+ be the set of points $y \in X$ such that the first holds, and Y^- the set of points such that the second holds. Consider the set $\bigcap\{U_y^{+c} : y \in Y^+\} \cap \bigcap\{U_y^{-c} : y \in Y^-\}$. Observe that this is the intersection of a positive and a negative closed set. This set also equals $\{x\}$ since by definition of the U_y^+'s and the U_y^-'s all of them omit x. On the other hand, still by definition every $y \in X$ distinct from x is omitted by the set, since there is either some U_y^+ with $y \in U_y^+$ or some U_y^- with the same property. For the converse, suppose that every singleton of X is of the form $U^{+c} \cap U^{-c}$. Consider two distinct points $x, y \in X$. Let $x = U^{+c} \cap U^{-c}$. We must have that $y \notin U^{+c} \cap U^{-c}$. That is, we must have that either $y \in U^+$ or $y \in U^-$. This means that we have either a positive or a negative open that contains y and omits x, and so the space X is pairwise T_1. □

Proposition 8.2.2. *All pairwise T_1 bispaces are d-T_D. All d-T_D bispaces are pairwise T_0.*

Proof. Using the characterization of pairwise T_1 spaces given by Lemma 8.2.1, it is immediate that this condition is stronger than a space being d-T_D. Suppose that a bispace X is d-T_D. Let $x, y \in X$ be two distinct points. We have four cases to consider. Suppose that $\{x\} = U^+ \cap V^{+c}$ for positive opens U^+ and V^+. Then, either we have that $y \notin U+$ or $y \notin V^{+c}$. In the first case, we have a positive open that contains x and omits y, and in the second case a negative open that contains y and omits x. The argument to show that we can find a separating positive or negative open for the three other cases is similar. □

The following property of the d-T_D axiom is a d-frame version of a classical topological result, too.

Lemma 8.2.3. *The property of being d-T_D is inherited by bisubspaces.*

Proof. Suppose that X is a d-T_D bispace and that $Y \subseteq X$ is a bisubspace. Let $y \in Y$. If $\{y\} = U^+ \cap V^{+c}$ for positive opens U^+ and V^{+c} of X, then we have that $\{y\} = (U^+ \cap Y) \cap (V^{+c} \cap Y)$, that is, the singleton $\{y\}$ is the intersection of a positive open and a positive closed set of Y. The argument for the other three cases is completely analogous. □

We have a connection between the d-T_D axioms and the d-frames of opens of subspaces of X which is a d-frame version of the classical results for the T_D axioms.

Observation 8.2.4. *When we have a topological space X and a subspace inclusion $i : Y \subseteq X$ we have that the dualization of the inclusion is $i^{-1} : \Omega(X) \to \Omega(Y)$, and this equals $U \cap Y = \{y \in Y : i(y) \in U\}$ for every open set $U \subseteq X$. For d-frames the situation is analogous. For a bispace X and a subspace inclusion $i : Y \subseteq X$, the dualization of this inclusion is the pairing*

(i^{-1}, i^{-1}) *of the preimage map with itself. For a positive open* $U^+ \subseteq X$ *we then have that* $\mathsf{d}\Omega(i)(U^+) = \{y \in Y : i(y) \in U^+\} = U^+ \cap Y$.

Lemma 8.2.5. *Suppose that* X *is a bispace and that* $x \in X$. *We have the following facts.*

- *We have that* $\{x\} = U^+ \cap V^{+c}$ *for some positive opens* U^+ *and* V^+ *if and only if the positive component of the dualization of the inclusion* $i : X\backslash\{x\} \subseteq X$ *is not a frame isomorphism.*
- *We have that* $\{x\} = U^- \cap V^{-c}$ *if and only if the negative component of the dualization of the inclusion* $i : X\backslash\{x\} \subseteq X$ *is not a frame isomorphism.*
- *We have that* $\{x\} = U^+ \cap U^-$ *if and only if the dualization of the inclusion* $i : X\backslash\{x\} \subseteq X$ *does not reflect the property of being in the* con *component.*
- *We have that* $\{x\} = U^{+c} \cap U^{-c}$ *if and only if the dualization of the inclusion* $i : X\backslash\{x\} \subseteq X$ *does not reflect the property of being in the* tot *component.*

Proof. Let us prove the four items in turn.

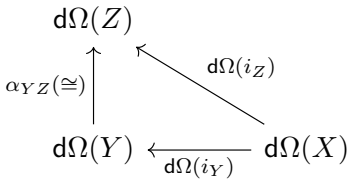
- Suppose first that $\{x\} = U^+ \cap V^{+c}$ for positive opens U^+ and V^{+c}. We then have that $(U^+ \cap X\backslash\{x\}) \subseteq (V^+ \cap X\backslash\{x\})$ but $U^+ \nsubseteq V^+$. By Observation 8.2.4 this means that $\mathsf{d}\Omega(i)^+(U^+) \subseteq \mathsf{d}\Omega(i)^+(V^+)$ but $\mathsf{d}\Omega(i_Y)^+(U^+) \nsubseteq \mathsf{d}\Omega(i_Y)^+(V^+)$. Then, the positive component of $\mathsf{d}\Omega(i)$ does not reflect the order and so it cannot be a frame isomorphism. Conversely, if $\mathsf{d}\Omega(i)$ is such that the positive component is not a frame isomorphism, in particular, since we know it is surjective, it must not be injective. Suppose that $U^+ \nsubseteq V^+$ and that $\mathsf{d}\Omega(i)^+(U^+) \subseteq \mathsf{d}\Omega(i)^+(V^+)$. This means that $U^+\backslash\{x\} \subseteq V^+\backslash\{x\}$, and so we must have that $\{x\} = U^+ \cap V^{+c}$.
- For the second item, the argument is similar: if $\{x\} = U^- \cap V^{-c}$ one shows similarly that the second frame component of the map $\mathsf{d}\Omega(i)$ does not reflect the order. The argument to show the converse is similar, too.
- For the third item, suppose that $\{x\} = U^+ \cap U^-$. We have that the two sets $U^+ \cap (X\backslash\{x\})$ and $U^- \cap (X\backslash\{x\})$ are disjoint, but the two sets U^+ and U^- are not. By Observation 8.2.4, this means that the pair $(\mathsf{d}\Omega(i)^+(U^+), \mathsf{d}\Omega(i)^-(U^-))$ is disjoint in $X\backslash\{x\}$ and so it is in the con component of $\mathsf{d}\Omega(X\backslash\{x\})$, but the pair (U^+, U^-) is not in the con component of $\mathsf{d}\Omega(X)$. Then, the map $\mathsf{d}\Omega(i)$ does not reflect the property of being in the con component. For the converse, suppose that $\mathsf{d}\Omega(i)$ does not reflect the con

component. This means that there must be $U^+, U^- \subseteq X$ such that $U^+ \cap U^-$ are not disjoint, but $U^+ \backslash \{x\}$ and $U^- \backslash \{x\}$ are. We must then have that $\{x\} = U^+ \cap U^-$.

- For the fourth item, one uses a similar argument to show that if $\{x\} = U^{+c} \cap U^{-c}$ the map $\mathsf{d}\Omega(i)$ does not reflect the property of being in the tot component. The argument for showing the converse is similar, too. □

Lemma 8.2.6. *If X is a T_D bispace, and $Y, Z \subseteq X$ are distinct bisubspaces, they induce different d-frame quotients on $\mathsf{d}\Omega(X)$.*

Proof. Suppose that we have distinct subspaces $Y, Z \subseteq X$, and let $i_Y : Y \subseteq X$ and $i_Z : Z \subseteq X$ be their subspace inclusions. We need to show that there can be no isomorphism α_{YZ} making the following diagram commute.



We will show that any map α_{YZ} making the diagram above commute cannot be a d-frame isomorphism. Suppose, then, that α_{YZ} is a d-frame map with this property. Since Y and Z are assumed to be distinct, we can assume without loss of generality that we have $Y \nsubseteq Z$. In particular, suppose that $x \in Y$ and $x \notin Z$. We have four cases to consider. Suppose that $\{x\} = U^+ \cap V^{+c}$ for positive opens U^+ and V^{+c}. We then have that $(U^+ \cap Z) \subseteq (V^+ \cap Z)$ but $(U^+ \cap Y) \nsubseteq (V^+ \cap Y)$. By Observation 8.2.4 this means that $\mathsf{d}\Omega(i_Z)^+(U^+) \subseteq \mathsf{d}\Omega(i_Z)^+(V^+)$ but $\mathsf{d}\Omega(i_Y)^+(U^+) \nsubseteq \mathsf{d}\Omega(i_Y)^+(V^+)$. Then we have $\mathsf{d}\Omega(U^+) \nsubseteq \mathsf{d}\Omega(V^+)$ and $\alpha^+_{YZ}(\mathsf{d}\Omega(U^+)) \subseteq \alpha^+_{YZ}(\mathsf{d}\Omega(V^+))$, and so α_{YZ} is not an isomorphism since its first frame component does not reflect the order. For the second case, the argument is similar: if $\{x\} = U^- \cap V^{-c}$ one shows similarly that the second frame component of the map α_{YZ} does not reflect the order. For the third case, suppose that $\{x\} = U^+ \cap U^-$. We have that the two sets $U^+ \cap Z$ and $U^- \cap Z$ are disjoint, but the two sets $U^+ \cap Y$ and $U^- \cap Y$ are not. By Observation 8.2.4, this means that $\mathsf{d}\Omega(i_Z)^+(U^+)$ and $\mathsf{d}\Omega(i_Z)^-(U^-)$ are disjoint in Z and so their pairing $(\mathsf{d}\Omega(i_Z)^+(U^+), \mathsf{d}\Omega(i_Z)^-(U^-)) = (\alpha^+_{YZ}(\mathsf{d}\Omega(i_Y)^+(U^+)), \alpha^-_{YZ}(\mathsf{d}\Omega(i_Y)^-(U^-)))$ is in the con component of $\mathsf{d}\Omega(Y)$, but the pair $(\mathsf{d}\Omega(i_Y)(U^+), \mathsf{d}\Omega(i_Y)(U^-))$ is not in the con component of $\mathsf{d}\Omega(Y)$. Then, we have that α_{YZ} does not reflect the property of being in the con component, and so it cannot be a d-frame isomorphism. For the fourth case, one uses a similar argument to show that if $\{x\} = U^{+c} \cap U^{-c}$ the map α_{YZ} does not reflect the property of being in the tot component. □

Proposition 8.2.7. *The following are equivalent for a bispace X.*

1. *The space is d-T_D.*
2. *Whenever $Y \not\subseteq Z$ are subspaces of X, they induce different d-frame quotients of $\mathsf{d}\Omega(X)$.*
3. *Whenever $x \in X$, the dualization $\mathsf{d}\Omega(i) : \mathsf{d}\Omega(X) \twoheadrightarrow \mathsf{d}\Omega(X\backslash\{x\})$ of the inclusion $i : X\backslash\{x\} \subseteq X$ is not a d-frame isomorphism.*
4. *Every singleton is isolated in one of the two topologies of $Sk_{cf}(X)$ or in one of the two topologies of $Sk_{\pm}(X)$.*

Proof. The fact that (1) implies (2) is established by Lemma 8.2.6. The fact that (2) implies (3) is immediate by setting $Y = X$ and $Z = Z\backslash\{x\}$. Let us show that (3) implies (1). Suppose that X is a bispace such that (3) holds, and suppose that $x \in X$. By our assumption, the dualization of the subspace inclusion $i : X\backslash\{x\} \subseteq X$ is not a d-frame isomorphism. There are four cases: either the positive frame component of the map is not an isomorphism, or the negative one, or the con component is not reflected, or the tot component is not reflected. By Lemma 8.2.5, any of these four conditions implies that the space X is T_D. Finally, let us show that (1) and (4) are equivalent. By definition of the topologies $Sk_{cf}(X)$ and $Sk_{\pm}(X)$, one direction is immediate. For the other, we notice that a singleton being open in one of these two topologies is equivalent to the singleton being either of the form $U^+ \cap U^-$, or of the form $U^{+c} \cap U^{-c}$, or of the form $U^+ \cap V^{+c}$, or of the form $U^- \cap V^{-c}$. □

Let us now consider the case where X is also a d-sober space. In particular, we will explore what the T_D property implies, for the spectrum $\mathsf{dpt}(\mathcal{L})$ of a d-frame $\mathcal{L}$. We will see characterizations of this property in terms of the relation between quotients of $\mathcal{L}$ and subspaces of $\mathsf{dpt}(\mathcal{L})$. Recall that in Chapter 3 we have seen that for every d-frame $\mathcal{L}$ we have the following commutative square in the category of complete inf-lattices.

$$\begin{array}{ccc} \mathsf{sp}[\mathsf{S}(\mathcal{L})] & \xrightarrow{\mathsf{dpt}(\cong)} & \mathsf{sob}[\mathcal{P}(\mathsf{dpt}(\mathcal{L}))] \\ \Big\uparrow{\scriptstyle \mathsf{sp}} & & \Big\downarrow \\ \mathsf{S}(\mathcal{L}) & \xrightarrow{\mathsf{dpt}} & \mathcal{P}(\mathsf{dpt}(\mathcal{L})) \end{array}$$

As in the frame case, then, all bisubspaces of $\mathsf{dpt}(\mathcal{L})$ being d-sober is a condition on the frame $\mathcal{L}$ equivalent to the spatial d-frame quotients of $\mathcal{L}$ perfectly representing the bisubspaces of $\mathsf{dpt}(\mathcal{L})$. Let us explore the condition of all bisubspaces of a d-frame spectrum $\mathsf{dpt}(\mathcal{L})$ being sober.

Lemma 8.2.8. *If $\mathcal{C}$ and $\mathcal{D}$ are categories and $F : \mathcal{C} \to \mathcal{D}$ is part of an equivalence of categories, we have that whenever $F(f) : F(A) \to F(B)$ is an isomorphism in $\mathcal{D}$, the morphism $f : A \to B$ is an isomorphism in $\mathcal{C}$.*

Proposition 8.2.9. *For a d-frame $\mathcal{L}$ the following are equivalent.*

1. *All bisubspaces of $\mathsf{dpt}(\mathcal{L})$ are d-sober.*
2. *The collection of d-sober subspaces of $\mathsf{dpt}(\mathcal{L})$ is closed under arbitrary unions.*
3. *The collection of d-sober subspaces of $\mathsf{dpt}(\mathcal{L})$ is closed under complementation.*
4. *The bisubspaces of the form $\mathsf{dpt}(\mathcal{L})\backslash\{f\}$ are d-sober.*
5. *The space $\mathsf{dpt}(\mathcal{L})$ is d-T_D.*

Proof. It is immediate that (1) implies (2), (3), and (4). Let us show the three reverse directions. Item (2) implies item (1) because singletons are all d-sober, and every bisubspace is a union of singletons. To see that (3) implies (1), we observe that an arbitrary bisubspace is of the form $S = (\bigcup_i \{f_i\})^c = \bigcap_i \{f_i\}^c$. Because the collection of d-sober subspaces is closed under arbitrary intersections, and because singletons are all d-sober, if each complement $\{f_i\}^c$ is d-sober then S is d-sober, too. The same argument also shows that (4) implies (1). Finally, let us show that (4) implies (5). Suppose that every bisubspace of the form $\mathsf{dpt}(\mathcal{L})$ is d-sober. Then, for every point $f \in \mathsf{dpt}(\mathcal{L})$ we cannot possibly have that the inclusion $i : \mathsf{dpt}(\mathcal{L}) \subseteq \mathsf{dpt}(\mathcal{L})$ when dualized becomes a d-frame isomorphism, or we would contradict that $\mathsf{d}\Omega$ is part of an equivalence of categories between d-sober bispaces and spatial d-frames (see Lemma 8.2.8). This means that the space $\mathsf{dpt}(\mathcal{L})$ is d-T_D, by comparing items (1) and (3) of Proposition 8.2.7. Then, (4) implies (5). Finally, let us show that (5) implies (1). Suppose that there is a bisubspace with inclusion $i_Y : Y \subseteq \mathsf{dpt}(\mathcal{L})$ which is not d-sober. Dualizing this we and precomposing with the spatialization map we get a d-frame surjection $\mathsf{d}\Omega(i_Y) \circ \varphi_{\mathcal{L}} : \mathcal{L} \to \mathsf{d}\Omega(Y)$. Let $q_R : \mathcal{L} \to \mathcal{L}/\mathcal{R}$ be the corresponding quotient. We know that $\mathsf{dpt}(\mathcal{L}/\mathcal{R})$ is a subspace of $\mathcal{L}$. It cannot be Y, as this is assumed to not be d-sober. By its construction it also induces on $\mathsf{d}\Omega(\mathsf{dpt}(\mathcal{L}))$ the same quotient as Y. By comparing items (1) and (2) of Proposition 8.2.7, we see that this means that $\mathsf{dpt}(\mathcal{L})$ is not d-T_D. □

Comparing the proposition above with the diagram we recalled from Chapter 3 after Proposition 8.2.7, we obtain the following d-frame version of Theorem 8.1.33.

Theorem 8.2.10. *For any d-frame $\mathcal{L}$, the following are equivalent.*

1. *The space* $\mathsf{dpt}(\mathcal{L})$ *is d-*T_D*.*

2. *All bisubspaces of* $\mathsf{dpt}(\mathcal{L})$ *are d-sober.*

3. *For every point* $f \in \mathsf{dpt}(\mathcal{L})$ *there is a relation structure* $\mathcal{R}$ *on* $\mathcal{L}$ *such that* $\{f\} = |\mathsf{dpt}(\mathcal{L}/\mathcal{R})|^c$*.*

4. *The map* $\mathsf{dpt} : \mathsf{sp}[\mathsf{S}(\mathcal{L})] \to \mathcal{P}(\mathsf{dpt}(\mathcal{L}))$ *is an isomorphism of complete lattices.*

We have then characterized those d-frames $\mathcal{L}$ such that the spatial d-quotients of $\mathcal{L}$ perfectly represent the subspaces of $\mathsf{dpt}(\mathcal{L})$. We will now refine this result, and ask ourselves when is it that spatial d-quotients of certain special kinds perfectly represent the subspaces of $\mathsf{dpt}(\mathcal{L})$. Let us see what each of the four disjuncts of the definition of a d-T_D space means when we are dealing with a bispace of the form $\mathsf{dpt}(\mathcal{L})$. In the following proposition, we use the usual convention of considering spectra of the form $\mathsf{dpt}(\mathcal{L}/\mathcal{R})$ as subspaces of $\mathsf{dpt}(\mathcal{L})$. Recall that in Chapter 3 (Proposition 8.2.12) we have proved that for every collection $\mathcal{R}_i$ of relation structures on a d-frame $\mathcal{L}$ we have that $\bigcap_i \mathsf{dpt}(\mathcal{R}_i) = \mathsf{dpt}(\mathcal{L}/\bigvee_i \mathcal{R}_i)$.

Proposition 8.2.11. *Let* $\mathcal{L}$ *be a d-frame, and let* $f \in \mathsf{dpt}(\mathcal{L})$*. The following facts hold.*

- *We have* $\{f\} = \varphi^+_{\mathcal{L}}(a^+) \cap \varphi^+_{\mathcal{L}}(b^+)^c$ *if and only if* $\mathsf{dpt}(\mathcal{L})\backslash\{f\} = \mathsf{dpt}(\mathcal{L}/C^+)$ *for some congruence* C^+ *on* L^+*.*

- *We have* $\{f\} = \varphi^-_{\mathcal{L}}(a^-) \cap \varphi^-_{\mathcal{L}}(b^-)^c$ *if and only if* $\mathsf{dpt}(\mathcal{L})\backslash\{f\} = \mathsf{dpt}(\mathcal{L}/C^-)$ *for some congruence* C^- *on* L^-*.*

- *We have* $\{f\} = \varphi^+_{\mathcal{L}}(a^+) \cap \varphi^-_{\mathcal{L}}(a^-)$ *if and only if* $\mathsf{dpt}(\mathcal{L})\backslash\{f\} = \mathsf{dpt}(\mathcal{L}/c)$ *for some* con *update* c*.*

- *We have* $\{f\} = \varphi^+_{\mathcal{L}}(a^+)^c \cap \varphi^-_{\mathcal{L}}(a^-)^c$ *if and only if* $\mathsf{dpt}(\mathcal{L})\backslash\{f\} = \mathsf{dpt}(\mathcal{L}/t)$ *for some* tot *update* t*.*

Proof. Let us prove each item in turn.

- Suppose that $\{f\} = \varphi^+_{\mathcal{L}}(a^+) \cap \varphi^+_{\mathcal{L}}(b^+)^c$. By definition of the open sets of $\mathsf{dpt}(\mathcal{L})$, this means that $\{f\} = \{g \in \mathsf{dpt}(\mathcal{L}) : g^+(a^+) = 1^+ \text{ and } g^+(b^+) = 0^+\}$. We claim that this is the complement of the subspace $\mathsf{dpt}(\mathcal{L}/\nabla(a^+) \cap \Delta(b^+))$. Since the congruence $\nabla(a^+) \cap \Delta(b^+)$ is that generated by the relation $\{(a^+, b^+)\}$, we have by Lemma 7.1.1 that this latter subspace is $\{g \in \mathsf{dpt}(\mathcal{L}) : g^+(a^+) = 0^+ \text{ or } g^+(b^+) = 1^+\}$. Indeed, this is the complement of the singleton $\{f\}$. For the converse, suppose that $\{f\}^c = \mathsf{dpt}(\mathcal{L}/C^+)$ for

some congruence C^+ on L^+. We may write $C^+ = \bigvee\{\nabla(a^+) \cap \Delta(b^+) : (a^+, b^+) \in C^+\}$. Then, by Lemma 3.1.8, we have that $\mathsf{dpt}(\mathcal{L})\backslash\{f\} = \bigcap\{\mathcal{L}/\nabla(a^+) \cap \Delta(b^+) : (a^+, b^+) \in C^+\}$. Taking complements of both sides, and using the fact that $\{f\}$ is a singleton, we obtain that there must be some pair $(a^+, b^+) \in C^+$ such that $\{f\} = (\mathsf{dpt}(\mathcal{L}/\nabla(a^+) \cap \Delta(b^+)))^c$. As observed above, this implies that $\{f\} = \varphi^+_{\mathcal{L}}(a^+) \cap \varphi^+_{\mathcal{L}}(b^+)^c$.

- The argument is similar to that for the first item.
- Suppose that $\{f\} = \varphi^+_{\mathcal{L}}(a^+) \cap \varphi^-_{\mathcal{L}}(a^-)$. By definition of the opens of $\mathsf{dpt}(\mathcal{L})$ we have that this is the same as $\{g \in \mathsf{dpt}(\mathcal{L}) : g^+(a^+) = 1^+ \text{and } g^+(b^+) = 1^+\}$. By Lemma 7.1.1, on the other hand, the subspace $\mathsf{dpt}(\mathcal{L}/\{(a^+, a^-)\})$, with $\{(a^+, a^-)\}$ treated as a con update, is $\{g \in \mathsf{dpt}(\mathcal{L}) : g^+(a^+) = 0^+ \text{ or } g^-(a^-) = 0^-\}$, and this is the complement of $\{f\}$. For the converse direction, suppose that $\{f\}^c = \mathsf{dpt}(\mathcal{L}/c)$ for some con update c. By Lemma 3.1.8, this means that $\{f\}^c = \bigcap\{\mathsf{dpt}(\mathcal{L}/\{(a^+, a^-)\}) : (a^+, a^-) \in c\}$, where each $\{(a^+, a^-)\}$ is treated as a con update. Taking complements of both sides of this equality, we obtain that there must be some $(a^+, a^-) \in c$ such that $\{f\} = (\mathcal{L}/\{(a^+, a^-)\})^c$. As we have observed above, this means that $\{f\} = \varphi^+_{\mathcal{L}}(a^+) \cap \varphi^-_{\mathcal{L}}(a^-)$.
- The argument is similar to that for the third item. □

For a d-frame $\mathcal{L}$, let us denote as $\mathsf{S}^+(\mathcal{L})$ the ordered subcollection of $\mathsf{S}(\mathcal{L})$ consisting of all quotients of the form $\mathcal{L}/C^+$ for some congruence C^+ on L^+. Let us first look at our first strengthening of the condition of the bispace $\mathsf{dpt}(\mathcal{L})$ being T_D. We will now characterize those d-frames $\mathcal{L}$ such that the first disjunct in the definition of the T_D property holds for all singletons, and show that this is equivalent to all bisubspaces of $\mathsf{dpt}(\mathcal{L})$ being the spectrum of some quotient in $\mathsf{S}^+(\mathcal{L})$.

Proposition 8.2.12. *D-sober subspaces of a d-sober space are closed under arbitrary intersections. In particular, for a collection of d-quotients $\mathcal{L}/\mathcal{R}_i$, we have $\mathsf{dpt}(\mathcal{L}/\bigvee_i \mathcal{R}_i) = \bigcap_i \mathsf{dpt}(\mathcal{L}/\mathcal{R}_i)$.*

Proposition 8.2.13. *The following are equivalent for a d-frame $\mathcal{L}$.*

1. *For each $f \in \mathsf{dpt}(\mathcal{L})$ we have $\{f\} = \varphi^+_{\mathcal{L}}(a^+) \cap \varphi^+_{\mathcal{L}}(b^+)^c$.*
2. *The bispace $\mathsf{dpt}(\mathcal{L})$ is such that its positive topology is T_D.*
3. *The positive topology of $Sk_{\pm}(\mathsf{dpt}(\mathcal{L})) \cong \mathsf{dpt}(\mathsf{A}_{\pm}(\mathcal{L}))$ is discrete.*
4. *All subspaces of $\mathsf{dpt}(\mathcal{L})$ are of the form $\mathsf{dpt}(\mathcal{L}/C^+)$ for some congruence C^+ on L^+.*

5. *The map* $\mathsf{dpt} : \mathsf{sp}[\mathsf{S}^+(\mathcal{L})] \to \mathcal{P}(|\mathsf{dpt}(\mathcal{L})|)$ *is an isomorphism.*

Proof. The equivalence between (1) and (2) is immediate from the fact that a topological space X is T_D if and only if for every $x \in X$ we have $\{x\} = U \cap V^c$ for opens $U, V \subseteq X$. The equivalence between these conditions and (3) is immediate because of the definition of the positive topology of $Sk_{\pm}(X)$. Let us show that (1) and (4) are equivalent. Suppose, then, that (1) holds. Every bisubspace of $\mathsf{dpt}(\mathcal{L})$ is of the form $(\bigcup_i \{f_i\})^c = \bigcap_i \{f_i\}^c$. By Lemma 8.2.11, our assumption implies that each subspace is of the form $\bigcap_i \mathsf{dpt}(\mathcal{L}/C_i^+)$ for some collection of congruences C_i^+ on L^+. By Proposition 8.2.12, this is the same as $\mathsf{dpt}(\mathcal{L}/\bigvee_i C_i^+)$. Here by $\bigvee_i C_i^+$ we mean the relation structure $\bigvee_i (C_i^+, \mathsf{id}^-, c_m, t_m) = (\bigvee_i C_i^+, \mathsf{id}^-, c_m, t_m)$. Indeed, then, (1) implies (4). For the converse, suppose that (4) holds, and let $f \in \mathsf{dpt}(\mathcal{L})$. This means that $\{f\}^c = \mathsf{dpt}(\mathcal{L}/C^+)$ for some congruence C^+ on L^+. By item (1) of Proposition 8.2.11, this implies that (1) holds. Finally, let us show that (4) and (5) are equivalent. We already know by Theorem 8.2.10 that $\mathsf{dpt} : \mathsf{sp}[\mathsf{S}(\mathcal{L})] \to \mathcal{P}(\mathsf{dpt}(\mathcal{L}))$ is an isomorphism if and only if all subspaces of $\mathsf{dpt}(\mathcal{L})$ are sober. Then, we have that the restriction of this map to $\mathsf{sp}[\mathsf{S}^+(\mathcal{L})]$ is an isomorphism (that is, condition (5) holds) if and only if all subspaces of $\mathsf{dpt}(\mathcal{L})$ are spectra of positive quotients, and this is condition (4). □

The proposition above can be adapted to the case where all singletons of $\mathsf{dpt}(\mathcal{L})$ satisfy the second disjunct of the definition of a d-T_D space. Let us call $\mathsf{S}^-(\mathcal{L})$ the ordered subcollection consisting of all quotients of the form $\mathcal{L}/C^-$ for some congruence C^- on L^-. By adapting the proof of Proposition 8.2.13 we obtain the following.

Proposition 8.2.14. *The following are equivalent for a d-frame* $\mathcal{L}$*.*

1. *For each* $f \in \mathsf{dpt}(\mathcal{L})$ *we have* $\{f\} = \varphi_{\mathcal{L}}^-(a^-) \cap \varphi_{\mathcal{L}}^-(b^-)^c$*.*
2. *The bispace* $\mathsf{dpt}(\mathcal{L})$ *is such that its negative topology is* T_D*.*
3. *The negative topology of* $Sk_{\pm}(\mathsf{dpt}(\mathcal{L})) \cong \mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L}))$ *is discrete.*
4. *All subspaces of* $\mathsf{dpt}(\mathcal{L})$ *are of the form* $\mathsf{dpt}(\mathcal{L}/C^-)$ *for some congruence* C^- *on* L^-*.*
5. *The map* $\mathsf{dpt} : \mathsf{sp}[\mathsf{S}^-(\mathcal{L})] \to \mathcal{P}(|\mathsf{dpt}(\mathcal{L})|)$ *is an isomorphism.*

Let us now look at the third disjunct. We call $\mathsf{S}^{\mathsf{con}}(\mathcal{L})$ the ordered subcollection of $\mathsf{S}(\mathcal{L})$ consisting of the quotients of the form $\mathcal{L}/c$ for some con update c. We call the elements of this subcollection the con *quotients* of the d-frame $\mathcal{L}$.

Proposition 8.2.15. *The following are equivalent for a d-frame* $\mathcal{L}$*.*

1. *For each* $f \in \mathsf{dpt}(\mathcal{L})$ *we have* $\{f\} = \varphi^+_{\mathcal{L}}(a^+) \cap \varphi^-_{\mathcal{L}}(a^-)$.

2. *The bispace* $\mathsf{dpt}(\mathcal{L})$ *is patch-discrete.*

3. *The positive topology of* $Sk_{cf}(\mathsf{dpt}(\mathcal{L})) \cong \mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L}))$ *is discrete.*

4. *All subspaces of* $\mathsf{dpt}(\mathcal{L})$ *are of the form* $\mathsf{dpt}(\mathcal{L}/c)$ *for some* con *update* $c \subseteq L^+ \times L^-$.

5. *The map* $\mathsf{dpt} : \mathsf{sp}[\mathsf{S}^{\mathsf{con}}(\mathcal{L})] \to \mathcal{P}(|\mathsf{dpt}(\mathcal{L})|)$ *is an isomorphism.*

Proof. It is immediate that (1) and (2) are equivalent, and by definition of the topology on $Sk_{cf}(\mathsf{dpt}(\mathcal{L}))$ it is also immediate that both these are equivalent to (3). Let us show that (1) implies (4). By an argument similar to that which we have used in the proof that (1) and (4) are equivalent in Proposition 8.2.13, we observe that every subspace is of the form $\bigcap_i \mathsf{dpt}(\mathcal{L}/c_i) = \mathsf{dpt}(\mathcal{L}/\bigvee_i c_i)$ for some collection c_i of con updates. Let us show that (4) implies (1). Suppose that (4) holds, and let $f \in \mathsf{dpt}(\mathcal{L})$. This means that $\{f\} = \mathsf{dpt}(\mathcal{L}/c)$ for some con update c. This by item (3) of Lemma 8.2.11 implies that $\{f\}$ is of the form described in item (1). To see that (4) and (5) are equivalent, finally, as in the proof of Proposition 8.2.13 we observe that the restriction to the spatializations of con quotients of the map $\mathsf{sp}[\mathsf{S}(\mathcal{L})] \to \mathcal{P}(\mathsf{dpt}(\mathcal{L}))$ is an isomorphism if and only if all bisubspaces of $\mathcal{P}(\mathsf{dpt}(\mathcal{L}))$ are spectra of con quotients. □

Finally, let us now look at the fourth disjunct. We call $\mathsf{S}^{\mathsf{tot}}(\mathcal{L})$ the ordered subcollection of $\mathsf{S}(\mathcal{L})$ consisting of the quotients of the form $\mathcal{L}/t$ for some tot update t. We call these quotients the tot *quotients* of the d-frame $\mathcal{L}$.

Proposition 8.2.16. *The following are equivalent for a d-frame* $\mathcal{L}$.

1. *For each* $f \in \mathsf{dpt}(\mathcal{L})$ *we have* $\{f\} = \varphi^+_{\mathcal{L}}(a^+)^c \cap \varphi^-_{\mathcal{L}}(a^-)^c$.

2. *The bispace* $\mathsf{dpt}(\mathcal{L})$ *is pairwise* T_1.

3. *The negative topology of* $Sk_{cf}(\mathsf{dpt}(\mathcal{L})) \cong \mathsf{dpt}(\mathsf{A}_{cf}(\mathcal{L}))$ *is discrete.*

4. *All subspaces of* $\mathsf{dpt}(\mathcal{L})$ *are of the form* $\mathsf{dpt}(\mathcal{L}/t)$ *for some* tot *update* $t \subseteq L^+ \times L^-$.

5. *The map* $\mathsf{dpt} : \mathsf{sp}[\mathsf{S}^{\mathsf{tot}}(\mathcal{L})] \to \mathcal{P}(|\mathsf{dpt}(\mathcal{L})|)$ *is an isomorphism.*

Proof. The equivalence between (1) and (2) is Lemma 8.2.1. The equivalence between (1) and (3) holds by definition of the topology $Sk_{cf}(\mathsf{dpt}(\mathcal{L}))$. The argument to show that (1) implies (4) is similar to that used in the proof of Proposition 8.2.13 to show the analogous

statement. We write an arbitrary bisubspace as $\bigcap_i \{f_i\}^c$ and use Lemma 8.2.11 to prove the desired claim. The fact that (4) implies (1), too, holds by a similar argument: we start from the fact that for each $f \in \mathsf{dpt}(\mathcal{L})$ we have $\{f\}^c = \mathsf{dpt}(\mathcal{L}/t)$ for some tot update t and deduce (1) from item (4) of Lemma 8.2.11. The argument to show that (4) and (5) are equivalent is analogous to that in the proof of Proposition 8.2.13, too. □

Bibliography

[1] "Completion" nLab webpage. `https://ncatlab.org/nlab/show/completion`.

[2] "Godement product" nLab webpage. `https://ncatlab.org/nlab/show/Godement+product`.

[3] Abe, J. M., Akama, S., and Nakamatsu, K. *Introduction to Annotated Logics: Foundations for Paracomplete and Paraconsistent Reasoning*. Springer International Publishing, 2015.

[4] Akama, S. *Introduction to Annotated Logics: Foundations for Paracomplete and Paraconsistent Reasoning*. Springer International Publishing, 2016.

[5] Anderson, A. R., Belnap, N. D., Dunn, J. M., Fine, K., Urquhart, A., Cohen, D., Giambrone, S., Grover, D. L., Gupta, A., Helman, G., Martin, E. P., McRobbie, M. A., Shapiro, S., and Wolf, R. G. *Entailment, Vol. II: The Logic of Relevance and Necessity*. Princeton University Press, 1992.

[6] Ball, R. N., Picado, J., and Pultr, A. Notes on exact meets and joins. *Applied Categorical Structures 22* (2014), 699–714.

[7] Banaschewski, B. The filter space of a lattice: Its role in general topology. available at "`https://www.math.uh.edu/~hjm/1973_Lattice/p00147-p00155.pdf`".

[8] Banaschewski, B., Brümmer, G. C., and Hardie, K. A. Biframes and bispaces. *Quaestiones Mathematicae 6*, 1-3 (1983), 13–25.

[9] Banaschewski, B., Frith, J. L., and Gilmour, C. R. A. On the congruence lattice of a frame. *Pacific J. Math. 130*, 2 (1987), 209–213.

[10] Banaschewski, B., and Pultr, A. Booleanization. *Cahiers de Topologie et Géométrie Différentielle Catégoriques 37*, 1 (1996), 41–60.

[11] Banaschewski, B., and Pultr, A. On covered prime elements and complete homomorphisms of frames. *Quaestiones Mathematicae 37*, 3 (2014), 451–454.

[12] Bezhanishvili, G., Bezhanishvili, N., Gabelaia, D., and Kurz, A. Bitopological duality for distributive lattices and Heyting algebras. *Mathematical Structures in Computer Science 20*, 3 (2010), 359–393.

[13] Bezhanishvili, N., and Holliday, W. H. Choice-free stone duality. *The Journal of Symbolic Logic 85*, 1 (2020), 109–148.

[14] Caramello, O. A topos-theoretic approach to stone-type dualities, 2011.

[15] da Costa, N. C. A., Subrahmanian, V. S., and Vago, C. The paraconsistent logics pj. *Mathematical Logic Quarterly 37*, 9-12 (1991), 139–148.

[16] Dacar, F. Closure operators on dcpos, 2017. arXiv:1709.06170.

[17] Ehresmann, C. Gattungen von lokalen strukturen. *Jahresbericht der Deutschen Mathematiker-Vereinigung 60* (1958), 49–77.

[18] Erné, M. General stone duality. *Topology and its Applications 137*, 1 (2004), 125 – 158. Topology and its Applications.

[19] Gehrke, M. Stone duality, topological algebra, and recognition. *Journal of Pure and Applied Algebra 220*, 7 (2016), 2711 – 2747.

[20] Goubault-Larrecq, J. *Non-Hausdorff Topology and Domain Theory: Selected Topics in Point-Set Topology*. New Mathematical Monographs. Cambridge University Press, 2013.

[21] Isbell, J. R. Atomless parts of spaces. *Mathematica Scandinavica 31* (Jun. 1972), 5–32.

[22] Jakl, T. *d-Frames as algebraic duals of bitopological spaces*. PhD thesis, Charles University and University of Birmingham, 2018.

[23] Jakl, T., and Jung, A. Free constructions and coproducts of d-frames.

[24] Jakl, T., Jung, A., and Pultr, A. Bitopology and four-valued logic. *Electronic Notes in Theoretical Computer Science 325* (2016), 201 – 219.

[25] Jakl, T., Jung, A., and Pultr, A. Quotients of d-frames. *Applied Categorical Structures 27* (2019), 261–275.

[26] Johnstone, P. T. *Stone Spaces*, vol. 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1982.

[27] Johnstone, P. T. Vietoris locales and localic semilattices. *Lecture Notes in Pure and Applied Mathematics 101* (1985). Proceedings of a conference held in the University of Bremen, July 2-3, 1982.

[28] Joyal, A., and Tierney, M. *An Extension of the Galois Theory of Grothendieck*. American Mathematical Society: Memoirs of the American Mathematical Society. American Mathematical Society, 1984.

[29] Jung, A., and Moshier, M. A. On the bitopological nature of Stone duality. Tech. Rep. CSR-06-13, University of Birmingham, 2006. 110 pages.

[30] Kamide, N., and Wansing, H. Proof theory of nelson's paraconsistent logic: A uniform perspective. *Theoretical Computer Science 415* (2012), 1 – 38.

[31] Klinke, O. A presentation of the assembly of a frame by generators and relations exhibits its bitopological structure. *Algebra Universalis 71* (2014), 55–64.

[32] MacLane, S. *Categories for the Working Mathematician*. Springer-Verlag, 1971. Graduate Texts in Mathematics, Vol. 5.

[33] McKinsey, J. C. C., and Tarski, A. The algebra of topology. *Annals of Mathematics 45*, 1 (1944), 141–191.

[34] Moshier, M., Mozo Carollo, I., and Walters-Wayland, J. On Isbell's density theorem for bitopological pointfree spaces. *Topology and its Applications* (2019).

[35] Moshier, M. A., Pultr, A., and A.L.S. Exact and strongly exact filters, 2020. to appear in *Applied Categorical Structures*.

[36] Papert, D., and Papert, S. Sur les treillis des ouverts et les paratopologies. *Séminaire Ehresmann. Topologie et géométrie différentielle 1* (1957-1958).

[37] Picado, J., and Pultr, A. *Frames and Locales: Topology without points*. Springer-Birkhäuser Basel, 2011.

[38] Picado, J., and Pultr, A. Notes on the product of locales. *Mathematica Slovaca 65* (04 2015), 247–264.

[39] Picado, J., and Pultr, A. Axiom T_D and the Simmons sublocale theorem, 2019.

[40] Priest, G. *Paraconsistent Logic*. Springer Netherlands, Dordrecht, 2002, pp. 287–393.

[41] Priest, G., Tanaka, K., and Weber, Z. Paraconsistent Logic. In *The Stanford Encyclopedia of Philosophy*, E. N. Zalta, Ed., summer 2018 ed. Metaphysics Research Lab, Stanford University, 2018.

[42] Priestley, H. A. Representation of Distributive Lattices by means of ordered Stone Spaces. *Bulletin of the London Mathematical Society 2*, 2 (1970), 186–190.

[43] Priestley, H. A. Ordered topological spaces and the representation of distributive lattices. *Proceedings of the London Mathematical Society s3-24*, 3 (1972), 507–530.

[44] Pultr, A. Pointless uniformities. i. complete regularity. *Commentationes Mathematicae Universitatis Carolinae 025*, 1 (1984), 91–104.

[45] Pultr, A. Pointless uniformities. ii: (dia)metrization. *Commentationes Mathematicae Universitatis Carolinae 025*, 1 (1984), 105–120.

[46] Reilly, I. L. Zero dimensional bitopological spaces. *Indagationes Mathematicae (Proceedings) 76*, 2 (1973), 127 – 131.

[47] Salbany, S. *Bitopological spaces, compactifications and completions*. University of Cape Town, Dept. of Mathematics, 1974.

[48] Schauerte, A. *Biframes*. PhD thesis, McMaster University, Hamilton, Ontario, 1992.

[49] Schauerte, A. Normality for biframes. *Applied Categorical Structures 3* (1995), 1–9.

[50] Simmons, H. The lattice theoretic part of topological separation properties. *Proceedings of the Edinburgh Mathematical Society 21*, 1 (1978), 41–48.

[51] Stone, M. H. The theory of representation for boolean algebras. *Transactions of the American Mathematical Society 40*, 1 (1936), 37–111.

[52] Subrahmanian, V. On the semantics of qualitative logic programs. In *Proc. 4th IEEE Symp. Logic Programming, San Francisco, CA* (1987), IEEE Computer Society Press, pp. 178–182.

[53] Vickers, S. *Topology via Logic*. Cambridge University Press, USA, 1989.

[54] Wigner, D. Two notes on frames. *Journal of the Australian Mathematical Society 28* (11 1979), 257 – 268.

[55] Wilson, J. T. *The Assembly Tower and Some Categorical and Algebraic Aspects of Frame Theory.* PhD thesis, Carnegie Mellon University, 1994.